ON IYENGAR'S TAUBERIAN THEOREM FOR NÖRLUND SUMMABILITY

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1. Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of real, non-negative constants, and let us write

$$P_n = p_0 + p_1 + \cdots + p_n.$$

The sequence-to-sequence transformation:

$$t_n = \frac{p_0 s_n + p_1 s_{n-1} + \cdots + p_n s_0}{P_n} \quad (P_n \neq 0) \tag{1. 1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The sequence $\{s_n\}$ is said to be summable (N, p_n) to the sum s, if $\lim_{n\to\infty} t_n$ exists and equals s [1, p. 64].

In the special case in which p_n is defined by

$$\sum_{n=0}^{\infty} p_n x^n = \left(\sum_{0}^{\infty} \frac{x^n}{n+1}\right)^k = \left\{\frac{\log(1-x)^{-1}}{x}\right\}^k, \ |x| < 1, \ (k = 1, 2, \cdots), \tag{1. 2}$$

t_n reduces to the familiar 'Harmonic mean of order k.' In this case we say that the sequence $\{s_n\}$ is summable (H, k) to the sum s. The case $k = 1$ is due to Riesz [6] and the general case is due to Iyengar [2]. It should be noted that Iyengar writes (N, k) in place of (H, k). We do not use his notation to avoid confusion with (N, p_n) where $p_n = k$.

The main interest of this method of summation lies in the following Tauberian theorem:

THEOREM 1 [2]. *If s_n $(n = 0, 1, \cdots)$ is a sequence summable (H, k) to* 0, *and if*

$$s_n - s_{n-1} \equiv a_n \leqq A n^{-\mu} \ (A > 0, \ 1 > \mu > 0), \tag{1. 3}$$

then $s_n \to 0$.

Rajagopal [4] has given a simple proof of this theorem in the case $k = 1$ which has been adapted by Rangachari [5] to the general case.

Iyengar [3], after proving the case $k = 1$ of Theorem 1, states without proof the following extension which is interesting in view of the application which he has made of it, to obtain a scale of convergence tests for Fourier series of which the simplest is a test due to Hardy and Littlewood [7, p. 35].

THEOREM 2. *Let* $p_n(n = 0, 1, \cdots)$ *be a strictly positive sequence such that* $\{p_{n+1}/p_n\}$ *monotonically increases to the limit* 1, *and* $P_n \to \infty$. *Suppose that* $\{s_n\}$ *is summable* (N, p_n) *to* 0 *and*

$$s_n - s_{n-1} = a_n \leqq \frac{A}{N} (A > 0), \tag{1. 4}$$

where $N = N(n)$ *is a positive integer defined by*

$$P_N \leqq \mu P_n < P_{N+1}, \ 0 < \mu < 1. \tag{1. 5}$$

Then $s_n \to 0$.

In connection with this theorem Rangachari [5] remarks, "Unfortunately, however, it does not seem possible to prove it like Theorem 1, and Iyengar's proof of it is not known since he died apparenly before he could fulfil his promise to publish the proof $\cdots$".

The object of this paper is to give a simple proof of Theorem 2.

2. We shall neeed the following lemmas for proving Theorem 2.

LEMMA 1 [1, p. 68]. *If* $p(x) = \sum p_n x^n$ *is convergent for* $|x| < 1$, *and*

$$p_0 = 1, \ p_n > 0, \ \frac{p_{n+1}}{p_n} \geqq \frac{p_n}{p_{n-1}} \quad (n > 0), \tag{2. 1}$$

then

$$\{p(x)\}^{-1} = 1 - c_1 x - c_2 x^2 - \cdots, \tag{2. 2}$$

where

(2. 3) (i) $c_n \geqq 0 \ (n > 0)$; (ii) $p_{n+1} \leqq p_n \ (n \geqq 0)$.

If $\sum p_n = \infty$, *then*

$$\sum c_n = 1. \tag{2. 4}$$

We note that under the conditions of Theorem 2, $\frac{p_{n+1}}{p_n} \to 1$ and this implies that $p(x)$ is convergent for $|x| < 1$. Further, we assume, without any loss of generality, that $p_0 = 1$ in Theorem 2. Hence all the conditions of this lemma are satisfied. Clearly

$$p_n = c_1 p_{n-1} + c_2 p_{n-2} + \cdots + c_n p_0 \ (n > 0). \tag{2. 5}$$

LEMMA 2. *If* $\{p_n\}$ *satisfies the conditions of Lemma* 1, *and if* $P_n \to \infty$, *then*

$$(2.\ 6)\qquad \begin{cases} d_n \equiv c_{n+1} + c_{n+2} + \cdots = 1 - c_1 - c_2 - \cdots - c_n \leqq \dfrac{1}{P_n} & (n > 0) \\ 1 + d_1 + d_2 + \cdots + d_n \leqq \dfrac{1}{p_n} & (n > 0). \end{cases}$$

PROOF. Using (2. 5) repeatedly we have

$$\begin{aligned} p_n &= c_1 p_{n-1} + c_2 p_{n-2} + \cdots + c_n p_0, \\ p_{n-1} &= \qquad\quad c_1 p_{n-2} + \cdots + c_{n-1} p_0, \\ &\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots \\ p_1 &= \qquad\qquad\qquad\qquad c_1 p_0. \end{aligned}$$

Adding we obtain

$$p_n + d_1 p_{n-1} + \cdots + d_{n-1} p_1 = (c_1 + c_2 + \cdots + c_n) p_0,$$

which reduces to

$$(2.\ 7)\qquad p_n + d_1 p_{n-1} + \cdots + d_n p_0 = p_0 = 1.$$

Now using (2. 3) we have

$$(2.\ 8)\qquad 1 \geqq d_n \geqq d_{n+1}\ (n > 0); \qquad 1 \geqq p_n \geqq p_{n+1} (n \geqq 0).$$

Finally using (2. 7) and (2. 8) we prove (2. 6).

LEMMA 3. *If* $\{s_n\}$ *is summable* (N, p_n) *to* 0, *then*

$$(2.\ 9)\qquad \lim_{n\to\infty} \left[\frac{\sum_{r=0}^{n} p_{r+m}\ s_{n-r}}{P_{n+m}} \right] = 0,$$

where $\{p_n\}$ *and* $N = N(n)$ *are defined as in Theorem* 2 *and* $m = m(n) = [\lambda N]$, $0 < \lambda < 1$.

PROOF:—Making use of the definition of t_n and (2. 5) we write

$$s_n = P_n t_n - (c_1 P_{n-1} t_{n-1} + c_2 P_{n-2} t_{n-2} + \cdots + c_n P_0 t_0).$$

Now inserting the expression for $s_{n-r} (r = 0, 1, \cdots n)$, we have

$$\begin{aligned} \sum_{r=0}^{n} p_{r+m} s_{n-r} &= \sum_{r=0}^{n} (p_{r+m} - c_1 p_{r-1+m} - \cdots - c_r p_m) P_{n-r} t_{n-r} \\ &= \sum_{r=0}^{n} A_{n-r} P_r t_r, \end{aligned}$$

where

$$(2.\ 10)\qquad A_{n-r} = \begin{cases} p_{n-r+m} - c_1 p_{n-r-1+m} - \cdots - c_{n-r} p_m (r < n),\ p_m (r = n) \\ c_{n-r+1} p_{m-1} + c_{n-r+2}\ p_{m-2} + \cdots + c_{n-r+m} p_0\ (r \leqq n), \end{cases}$$

the second expression for A_{n-r} in (2. 10) being derived from the first by (2. 5). Clearly $A_{n-r} \geqq 0$ $(r \leqq n)$.
Now

$$(2.\ 11)\qquad \sum_{r=0}^{n} A_{n-r}P_r t_r = \sum_{r=0}^{n-m} + \sum_{r=n-m+1}^{n} = \sum\nolimits_1 + \sum\nolimits_2, \quad \text{say.}$$

Since $t_n \to 0$, we have by (2. 10) and by Lemma 2,

$$\begin{aligned}
\left|\sum\nolimits_1\right| &\leqq o(P_n) \sum_{r=0}^{n-m} |A_{n-r}| \\
&= o(P_n) \sum_{r=0}^{n-m} \sum_{k=0}^{m-1} p_k c_{n-r+m-k} \\
&= o(P_n) \sum_{k=0}^{m-1} p_k (c_{2m-k} + c_{2m-k+1} + \cdots + c_{n+m-k}) \\
&\leqq o(P_n) \sum_{k=0}^{m-1} p_k d_{2m-k-1} \\
&\leqq o(P_n) \sum_{k=0}^{m-1} p_k / P_{2m-k-1} \\
&\leqq o(P_n)\ P_{m-1}/P_m \\
(2.\ 12)\qquad &= o(P_n).
\end{aligned}$$

Further, by the first expression for A_{n-r} in (2. 10) and by (2. 3), (i), (ii) and (2. 6), we have

$$\begin{aligned}
\left|\sum\nolimits_2\right| &\leqq o(P_n) \sum_{n-m+1 \leqq r \leqq n} |A_{n-r}| = o(P_n) \sum_{k=0}^{m-1} d_k p_{m+k} \\
&\leqq o(P_n)\ p_m\ (1 + d_1 + d_2 + \cdots + d_{m-1}) \\
(2.\ 13)\qquad &\leqq o(P_n).
\end{aligned}$$

The lemma now follows from (2.11), (2. 12) and (2. 13).

3. Proof of Theorem 2. The proof follows the usual argument and it is only for the sake of completeness that we give it here. We suppose that

$$(3.\ 1)\qquad \liminf s_n = -h,\ h > 0.$$

Then there exists a sequence of positive integers $n_0 < n_1 < \cdots < n_k \to \infty$, such that

$$(3.\ 2)\qquad s_{n_k} \leqq -h_1,\ 0 < h_1 < h.$$

Let us assume, without any loss of generality, that $A > h_1/2$. Now choose $\lambda = h_1/(2A)$ in Lemma 3. Then by (1. 4)

$$s_n = s_{n_k} + a_{n_k+1} + \cdots + a_n \leqq -h_1 + a_1/2, \quad n_k \leqq n \leqq n_k + m_k \tag{3. 3}$$

where $m_k = m(n_k)$.

Now

$$\sum_{r=0}^{n+m} p_{n+m-r} s_r = \sum_{r=0}^{n} + \sum_{r=n+1}^{n+m} = S_1 + S_2, \text{ say.} \tag{3. 4}$$

From Lemma 3,

$$S_1 = o\ (P_{n+m}). \tag{3. 5}$$

Choosing $n = n_k$, we have by (3. 3)

$$\begin{aligned} S_2 &\leqq -\frac{h_1}{2} \sum_{r=n+1}^{n+m} P_{n+m-r} \\ &= -\frac{h_1}{2} P_{m-1}. \end{aligned} \tag{3. 6}$$

Now in view of (1. 5) and (2. 3) (ii), we can show that there exists a $\nu > 0$ such that

$$P_{m-1} = P_{[\lambda N]-1} \geqq \nu P_{N+1} > \mu\nu P_n \geqq \frac{\mu\nu}{2} P_{n+m}, \quad k > k_0.$$

Hence

$$S_2/P_{n_k+m_k} \leqq -\frac{h_1}{2} \cdot \frac{\mu\nu}{2} \quad (k > k_0),$$

which contradicts the assumption that the sequence $\{s_n\}$ is summable (N, p_n) to 0.

We can easily see that lim inf $s_n \neq -\infty$.

Hence lim inf $s_n = 0$.

Similarly lim sup $s_n = 0$.

This completes the proof of Theorem 2.

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