CENTRAL LIMIT THEOREM OF TRIGONOMETRIC SERIES

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(Received May 16, 1963 and revised August 20, 1963)

1. The purpose of the present note is to prove the central limit theorem of trigonometric series. In [1] R. Salem and A. Zygmund have proved this theorem for lacunary trigonometric series. From now on let us put

(1. 1) $$S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi k(t+\alpha_k) \quad \text{and } B_N=\left(\frac{1}{2}\sum_{K=1}^{N} a_k^2\right)^{1/2}.$$

In §§ 2−5 we prove the following

THEOREM. *Let $S_N(t)$ be the N-th partial sum* (1. 1) *of a trigonometric series for which*

(1. 2) $$B_N \uparrow +\infty, \qquad \textit{as } N\to+\infty,$$

and let $\{n_k\}$ be a sequence of positive integers such that

(1. 3) $$n_{k+1}/n_k > q > 1.$$

We set

(1. 4) $$R_1(t)=S_{n_1}(t) \quad \textit{and } R_k(t)=S_{n_k}(t)-S_{n_{k-1}}(t), \qquad \textit{for } k>1,$$

and suppose that

(1. 5) $$B_{n_k}^2-B_{n_{k-1}}^2=o(B_{n_k}^2), \qquad \textit{as } k\to+\infty,$$

(1. 6) $$\sup_t |R_k(t)|^2=O(B_{n_k}^2-B_{n_{k-1}}^2), \qquad \textit{as } k\to+\infty,$$

and, for some function $g(t)$

(1. 7) $$\lim_{k\to\infty}\int_0^1\left|\frac{1}{B_{n_k}^2}\sum_{m=1}^{k}\{R_m^2(t)+2R_m(t)R_{m+1}(t)\}-g(t)\right|dt=0.$$

Then $g(t)$ is bounded and non-negative and we have, for any set $E\subset[0,1]$ of positive measure and any real number $\omega\neq 0$,

(1. 8) $$\lim_{N\to\infty}\frac{1}{|E|}\left|\{t\,;t\in E,\ S_N(t)/B_N\leqq\omega\}\right|=\frac{1}{\sqrt{2\pi}|E|}\int_E dt\int_{-\infty}^{\omega/g(t)^{\frac{1}{2}}}\exp\{-\frac{u^2}{2}\}\,du,$$

where $\omega/0$ denotes $+\infty$ or $-\infty$ according as $\omega>0$ or $\omega<0$.

(1. 6) implies that $g(t)$ is bounded and $\int_0^1 g(t)\,dt=1$. (1. 7) and (1. 8) show

that the "*interrelations*" of $R_k(t)$ and $R_n(t)$ have no "*influence*" on the limit distribution of $S_N(t)$ whenever $|n-k|\geqq 2$. However, we can construct an $S_N(t)$ such that $\lim_{k\to\infty}\frac{1}{B_{n_k}^2}\sum_{m=1}^{k} R_m(t)\,R_{m+1}(t)$ exists and does not vanish identically.

Especially, if $S_N(t)$ is lacunary, then we can take a sequence $\{n_k\}$, satisfying (1. 3), such that $R_{2k-1}(t)=0$ and $R_{2k}(t)$ contains only one term for $k>k_0$. In this case under the conditions (1. 2) and (1. 5), (1. 6) and (1. 7) always hold for $g(t)=1$. Therefore $\{S_N(t)\}$ obeys the ordinary central limit theorem.

Salem and Zygmund have also proved that if $S_N(t)$ is lacunary, then (1. 2) and (1. 5) are necessary for (1. 8) (c. f. [1]).

Let $F(\omega)$ be the distribution function on the right hand side of (1. 8), then we have, for any real number λ

$$\int_{-\infty}^{\infty} e^{i\lambda\omega}dF(\omega)=\frac{1}{|E|}\int_E \exp\left\{-\frac{\lambda^2}{2}g(t)\right\}dt.$$

$F(\omega)$ is called "*pseudo Gaussian*" owing to the form of its characteristic function. $F(\omega)$ is continuous except zero and is discontinuous at zero if and only if the set $\{t; t\in E,\ g(t)=0\}$ is a set of positive measure.

In the same way we can prove the central limit theorem for the remainder terms of the Fourier series of a square integrable function (c. f. § 6).

In §§ 2–5 we prove that for any fixed real λ, we have

$$\lim_{N\to\infty}\frac{1}{|E|}\int_E \exp\left\{\frac{i\lambda}{B_N}S_N(t)\right\}dt=\frac{1}{|E|}\int_E \exp\left\{-\frac{\lambda^2}{2}g(t)\right\}dt, \tag{1. 9}$$

which is equivalent to (1. 8).

2. Hereafter let us assume that the conditions of the theorem are satisfied. From (1. 3) there exists a positive integer r such that

$$q^r(1-q^{-1})>6. \tag{2. 1}$$

Using this r, let us put

$$\Delta_l(t)=\sum_{k=(l-1)r+1}^{lr} R_k(t)=S_{n_{lr}}(t)-S_{n_{(l-1)r}}(t), \tag{2. 2}$$

$$D_l^2=B_{n_{lr}}^2-B_{n_{(l-1)r}}^2, \tag{2. 3}$$

$$A_l=\sup_t|\Delta_l(t)|, \tag{2. 4}$$

and

$$C_N^2=\sum_{l=1}^{N} D_l^2=B_{n_{Nr}}^2 \tag{2. 5}$$

Then by the conditions of the theorem, we have

(2. 6) $$C_N \uparrow +\infty \quad \text{and } D_N = o(C_N), \qquad \text{as } N \to +\infty,$$

(2. 7) $$A_l \leqq \sum_{k=(l-1)r+1}^{lr}{\!\!\rightarrow} \sup_t |R_k(t)| \leqq r^{1/2}\left\{\sum_{k=(l-1)r+1}^{lr}{\!\!\rightarrow} \sup_t |R_k(t)|^2\right\}^{1/2} = O(D_l), \qquad \text{as } l \to +\infty,$$

and, for any n such that $n_{(N-1)r} < n \leqq n_{Nr}$,

(2. 8) $$\int_0^1 \left| S_n(t) - \sum_{l=1}^N \Delta_l(t)\right|^2 dt \leqq D_N^2 = o(C_N^2), \qquad \text{as } N \to +\infty.$$

LEMMA 1. *We have*

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{C_N^2} \sum_{l=1}^N \left\{ \Delta_l^2(t) + 2\Delta(t)\Delta_{l+1}(t)\right\} - g(t) \right| dt = 0.$$

PROOF. We have

$$\Delta_l^2(t) = \sum_{k=(l-1)r+1}^{lr} \{R_k^2(t) + 2R_k(t)R_{k+1}(t)\} - 2R_{lr}(t)R_{lr+1}(t) + X_l(t),$$

and

$$\Delta_l(t)\Delta_{l+1}(t) = R_{lr}(t)R_{lr+1}(t) + Y_l(t),$$

where

(2. 9) $$X_l(t) = 2\sum_{k=(l-1)r+3}^{lr} R_k(t) \sum_{j=(l-1)r+1}^{k-2} R_j(t),$$

(2. 9′) $$Y_l(t) = \left\{\sum_{k=lr+1}^{(l+1)r} R_k(t) \sum_{j=(l-1)r+1}^{lr} R_j(t)\right\} - R_{lr}(t)R_{lr+1}(t).$$

By (1. 6) we have

(2.10) $$|X_l(t)| \leqq \left(\sum_{k=(l-1)r+1}^{lr} |R_k(t)|\right)^2 = O(D_l^2), \qquad \text{as } l \to +\infty,$$

and

(2.10′) $$|Y_l(t)| = O(D_l D_{l+1}), \qquad \text{as } l \to +\infty.$$

Let w_l (or w_l') denotes the maximum (or minimum) frequency of a trigometric polynomial $X_l(t)$, then we have, by (1. 4) and (2.9),

$$w_l \leqq n_{lr} + n_{lr-2} < 2n_{lr},$$

and

$$w_l' \geqq \min_{(l-1)r+3 \leqq k \leqq lr} \{n_{(k-1)} + 1 - n_{(k-2)}\} \geqq n_{(l-1)r+2}(1 - q^{-1}).$$

From (2. 1), we can see that

$$w'_{l+2}/w_l > q^{r+2}(1-q^{-1})2^{-1} > 3.$$

This shows that $X_l(t)$ and $X_k(t)$ are orthogonal if $|k-l| \geqq 2$, and we have

$$\int_0^1 \left\{\sum_{l=1}^N X_l(t)\right\}^2 dt \leqq 2\left[\int_0^1 \left\{\sum_{2l\leqq N} X_{2l}(t)\right\}^2 dt + \int_0^1 \left\{\sum_{2l+1\leqq N} X_{2l+1}(t)\right\}^2 dt\right]$$
$$= 2\sum_{l=1}^N \int_0^1 X_l^2(t)\,dt.$$

By (2. 10) and (2. 6), we have

$$\int_0^1 \left\{\sum_{l=1}^N X_l(t)\right\}^2 dt = O\left(\sum_{l=1}^N D_l^4\right) = o(C_N^4), \qquad \text{as } N \to +\infty. \tag{2.11}$$

In the same way $Y_l(t)$ and $Y_k(t)$ are orthogonal if $|l-k| \geqq 2$ and we have, by (2. 10′) and (2. 6),

$$\int_0^1 \left\{\sum_{l=1}^N Y_l(t)\right\}^2 dt \leqq 2\sum_{l=1}^N \int_0^1 Y_l^2(t)\,dt = O\left(\sum_{l=1}^N D_l^2 D_{l+1}^2\right) \tag{2.11′}$$
$$= O\left(\sum_{l=1}^{N+1} D_l^4\right) = o\,(C_N^4), \qquad \text{as } N \to +\infty.$$

On the other hand we have

$$\sum_{l=1}^N \{\Delta_l^2(t) + 2\Delta_l(t)\Delta_{l+1}(t)\} = \sum_{k=1}^{Nr} \{R_k^2(t) + 2R_k(t)R_{k+1}(t)\} + \sum_{l=1}^N \{X_l(t) + 2Y_l(t)\}.$$

Hence by (1. 7), (2. 11) and (2. 11′), we can prove the lemma.

3. From (2. 6) and (2. 7) there exists a sequence of integers $\{\phi(N)\}$ such that

$$\phi(1) > 1,\ \phi(N) \uparrow +\infty \quad \text{and} \quad \phi(N) \operatorname*{Max}_{l\leqq N} A_l = o(C_N), \text{ as } N \to \infty. \tag{3. 1}$$

Putting $M_k = \sum_{l=1}^k \phi(l)$, we can choose a sequence of integers $\{N_k\}$ satisfying the following conditions;

$$\begin{cases} N_0 = 1, & \text{and for } k \geqq 1 \\ D_{N_k-1}^2 \leqq (\phi(k))^{-1} \sum_{l=M_{2k-1}}^{M_{2k}-1} D_l^2 & \text{and } M_{2k-1} < N_k \leqq M_{2k}. \end{cases} \tag{3. 2}$$

Since $N_{k-1} < M_{2k-1} < N_k < M_{k2} < N_{k+1}$, we have

(3. 3) $$\sum_{k=1}^{m} D^2_{N_k-1} \leqq \sum_{k=1}^{m-1} \{\phi(k)\}^{-1} \sum_{l=N_{k-1}}^{N_{k+1}-1} D_l^2 + D^2_{N_m-1} = o(C^2_{N_m}), \qquad \text{as } m \to +\infty.$$

If we put

(3. 4) $$T_k(t) = \sum_{l=N_{k-1}}^{N_k-2} \Delta_l(t),$$

then we have, by (3. 3)

(3. 5) $$\int_0^1 \left| \sum_{k=1}^{m} T_k(t) - \sum_{l=1}^{N_m} \Delta_l(t) \right|^2 dt = \sum_{k=1}^{m} D^2_{N_k-1} + D^2_{N_m} = o(C^2_{N_m}), \text{ as } m \to +\infty.$$

and, by (3. 1) and (3. 2)

(3. 6) $$|T_k(t)| \leqq \sum_{l=N_{k-1}}^{N_k-1} |\Delta_l(t)| \leqq (N_k - N_{k-1}) \operatorname*{Max}_{l \leqq N_k} A_l$$

$$\leqq (M_{2k} - M_{2k-3}) \operatorname*{Max}_{l \leqq N_k} A_l \leqq 3\phi(2k) \operatorname*{Max}_{l \leqq N_k} A_l \leqq 3\phi(N_k) \operatorname*{Max}_{l \leqq N_k} A_l = o(C_{N_k})^{*)}, \text{ as } k \to \infty.$$

From (2. 8), (3. 5) and (3. 6) we have, for any n satisfying $n_{rN_{m-1}} < n < n_{rN_m}$,

$$\int_0^1 \left| S_n(t) - \sum_{k=1}^{m} T_k(t) \right|^2 dt = o(C^2_{N_m}), \qquad \text{as } m \to +\infty.$$

and

$$B_n^2 = C^2_{N_m}(1 + o(1)), \qquad \text{as } m \to +\infty.$$

Hence for the proof of the theorem it is sufficient to show that for any fixed real number λ, we have (c. f. (1. 9))

(3. 7) $$\lim_{m \to \infty} \frac{1}{|E|} \int_E \exp\left\{ \frac{i\lambda}{C_{N_m}} \sum_{k=1}^{m} T_k(t) \right\} dt = \frac{1}{|E|} \int_E \exp\left\{ \frac{-\lambda^2}{2} g(t) \right\} dt.$$

LEMMA 2. *We have*

$$\lim_{m \to \infty} \int_0^1 \left| \frac{1}{C^2_{N_m}} \sum_{k=1}^{m} T_k^2(t) - g(t) \right| dt = 0.$$

REMARK. From this lemma it is seen that $g(t) \geqq 0$.

PROOF. We have

(3. 8) $$T_k^2(t) = \sum_{l=N_{k-1}}^{N_k-2} \{\Delta_l^2(t) + 2\Delta_l(t)\Delta_{l+1}(t)\} - 2\Delta_{N_k-2}(t)\Delta_{N_k-1}(t) + Z_k(t),$$

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*) $N_k \geqq M_{2k-1} \geqq (2k-1)\phi(1) > (2k-1)$.

where

$$Z_k(t) = 2 \sum_{l=N_{k-1}+2}^{N_k-2} \Delta_l(t) \sum_{j=N_{k-1}}^{l-2} \Delta_j(t). \tag{3.8'}$$

If u_k (or u_k') denotes the maximum (or minimum) frequency of terms of $Z_k(t)$, then $u_k < 2n_{r(N_k-2)}$ and $u_k' \geqq \mathrm{Min}\{n_{(l-1)r} + 1 - n_{(l-2)r};\ N_{k-1} + 2 \leqq l \leqq N_k - 2\}$. Hence we have, by (2.1)

$$\frac{u'_{k+1}}{u_k} > \frac{n_{(N_k+1)r}(1-q^{-r})}{2n_{r(N_k-2)}} > \frac{q^{3r}(1-q^{-r})}{2} > 3. \tag{3.9}$$

This implies that $\{Z_k(t)\}$ is orthogonal and we have, by (3.8')

$$\int_0^1 \left\{\sum_{k=1}^m Z_k(t)\right\}^2 dt = \sum_{k=1}^m \int_0^1 Z_k^2(t)dt = 4\sum_{k=1}^m \int_0^1 \left\{ \sum_{l=N_{k-1}+2}^{N_k-2} \Delta_l(t) \sum_{j=N_{k-1}}^{l-2} \Delta_j(t)\right\}^2 dt.$$

In the same way $\left\{ \Delta_l(t) \sum_{j=N_{k-1}}^{l-2} \Delta_j(t)\right\}$ and $\left\{\Delta_s(t) \sum_{j=N_{k-1}}^{s-2} \Delta_j(t)\right\}$ is orthogonal if $|l-s| \geqq 2$. Therefore we have

$$\int_0^1 \left\{\sum_{k=1}^m Z_k(t)\right\}^2 dt \leqq 8 \sum_{k=1}^m \sum_{l=N_{k-1}+2}^{N_k-2} \int_0^1 \Delta_l^2(t) \left\{\sum_{j=N_{k-1}}^{l-2} \Delta_j(t)\right\}^2 dt.$$

By (3.6), we obtain

$$\int_0^1 \left\{\sum_{k=1}^m Z_k(t)\right\}^2 dt \leqq 8 \sum_{k=1}^m \sup_t \left\{\sum_{j=N_{k-1}}^{N_k-2} |\Delta_j(t)|^2\right\} \sum_{l=N_{k-1}+2}^{N_k-2} \int_0^1 \Delta_l^2(t)\, dt$$

$$= \sum_{k=1}^m o(C_{N_k}^2) \sum_{l=N_{k-1}+2}^{N_k-2} D_l^2 = o(C_{N_m}^4), \qquad \text{as } m \to +\infty.$$

On the other hand we have, by (3.8)

$$\sum_{k=1}^m T_k^2(t) = \sum_{l=1}^{N_m} \left\{\Delta_l^2(t) + 2\Delta_l(t)\ \Delta_{l+1}(t)\right\} + \sum_{k=1}^m Z_k(t) - \Delta_{N_m}^2(t)$$

$$- 2\Delta_{N_m}(t)\Delta_{N_m+1}(t) - \sum_{k=1}^m [\Delta_{N_k-1}^2(t) + 2\Delta_{N_k-1}(t)\{\Delta_{N_k-2}(t) + \Delta_{N_k}(t)\}].$$

From (3.3) it is seen that

$$\int_0^1 \left|\sum_{k=1}^m \left[\Delta_{N_k-1}^2(t) + 2\Delta_{N_k-1}(t)\{\Delta_{N_k-2}(t) + \Delta_{N_k}(t)\}\right]\right| dt$$

$$\leqq \sum_{k=1}^m D_{N_k-1}^2 + 2\sum_{k=1}^m \left\{\int_0^1 \Delta_{N_k-1}^2(t)\, dt\right\}^{1/2} \left\{\int_0^1 |\Delta_{N_k-2}(t) + \Delta_{N_k}(t)|^2 dt\right\}^{1/2}$$

$$\leqq \sum_{k=1}^{m} D_{N_{k}-1}^{2} + 2\left\{\sum_{k=1}^{m} D_{N_{k}-1}^{2}\right\}^{1/2}\left\{\sum_{k=1}^{m} (D_{N_{k}-2}^{2} + D_{N_{k}}^{2})\right\}^{1/2}$$

$$= o(C_{N_m}^2), \qquad \text{as } m \to +\infty,$$

and

$$\int_0^1 \left\{\Delta_{N_m}^2(t) + 2|\Delta_{N_m}(t)\Delta_{N_m+1}(t)|\right\} dt \leqq D_{N_m}^2 + 2D_{N_m} D_{N_m+1} = o(C_{N_m}^2), \text{ as } m \to +\infty.$$

Therefore we obtain

$$\int_0^1 \left|\sum_{k=1}^{m} T_k^2(t) - \left\{\sum_{l=1}^{N_m} \{\Delta_l^2(t) + 2\Delta_l(t)\Delta_{l+1}(t)\}\right\}\right| dt = o(C_{N_m}^2), \qquad \text{as } m \to +\infty.$$

By the above relation and Lemma 1, we can prove this lemma.

4. LEMMA 3. *We have, for any real number* λ,

$$\int_0^1 \left|\prod_{k=1}^{m}\left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right|^2 dt \leqq e^{\lambda^2 K},$$

where K *is a constant.*

PROOF. By (2.7), we have

$$\sum_{l=N_{k-1}}^{N_k-2} \left\{\Delta_l^2(t) + 2\Delta_l(t)\Delta_{l+1}(t)\right\} - 2\Delta_{N_k-2}(t)\Delta_{N_k-1}(t)$$

$$\leqq 3\sum_{l=N_{k-1}}^{N_k-1} \Delta_l^2(t) \leqq K\sum_{l=N_{k-1}}^{N_k-1} D_l^2, \qquad \text{for some constant } K.$$

Hence we have, by (3.8)

$$\left|\prod_{k=1}^{m}\left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right|^2 \leqq \prod_{k=1}^{m}\left\{1 + \frac{K\lambda^2 \sum_{l=N_{k-1}}^{N_k-1} D_l^2 + \lambda^2 Z_k(t)}{C_{N_m}^2}\right\}. \tag{4.1}$$

If we put

$$\prod_{k=1}^{m}\left\{1 + \frac{K\lambda^2 \sum_{l=N_{k-1}}^{N_k-1} D_l^2 + \lambda^2 Z_k(t)}{C_{N_m}^2}\right\} = 1 + \Psi_m(t), \tag{4.2}$$

then $\Psi_m(t)$ is a sum of terms in the following form;

$$(\text{constant}) \times \prod_{j=1}^{s} \cos 2\pi v_{m_j}(t + \beta_{m_j}),$$

$$1 \leqq m_1 < m_2 < \cdots < m_s \leqq m \quad \text{and} \quad u'_{m_j} \leqq v_{m_j} \leqq u_{m_j},$$

where u_j (or u_j') denotes the maximum (or minimum) frequency of terms of $Z_j(t)$. Further we have

$$(4.3)\quad \prod_{j=1}^{s} \cos 2\pi v_{m_j}(t+\beta_{m_j}) = \frac{1}{2^{s-1}} \sum \cos 2\pi \{v_{m_s}(t+\beta_{m_s}) + \sum_{j=1}^{s-1} \delta_j v_{m_j}(t+\beta_{m_j})\}$$

where δ_j denotes $+1$ or -1 and $\sum$ denotes the summation over all combinations of $(\delta_{s-1}, \delta_{s-2}, \cdots, \delta_1)$. From (3.9) it is seen that

$$v_{m_s} + \sum_{j=1}^{s-1} \delta_j v_{m_j} \geqq v_{m_s} - \sum_{j=1}^{s-1} v_{m_j} \geqq u'_{m_s} - \sum_{j=1}^{s-1} u_{m_j}$$
$$\geqq u'_{m_s} - \sum_{j=1}^{m_s-1} u_j \geqq u'_{m_s}\left(1 - \sum_{j=1}^{m_s-1} 3^{j-m_s}\right) \geqq \frac{1}{2} u'_{m_s} > 0.$$

Since v_{m_j}'s are integers, (4.3) and the above relation imply

$$\int_0^1 \prod_{j=1}^{s} \cos 2\pi v_{m_j}(t+\beta_{m_j})\, dt = 0,$$

and this implies $\int_0^1 \Psi_m(t)\, dt = 0$. Therefore we obtain, from (4.1) and (4.2)

$$\int_0^1 \left| \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} \right|^2 dt \leqq \prod_{k=1}^{m} \left\{1 + \frac{\lambda^2 K \sum_{l=N_{k-1}}^{N_k-1} D_l^2}{C_{N_m}^2}\right\}$$
$$\leqq \exp\left\{\frac{\lambda^2 K}{C_{N_m}^2} \sum_{k=1}^{m} \sum_{l=N_{k-1}}^{N_k-1} D_l^2\right\} \leqq e^{\lambda^2 K}.$$

LEMMA 4. *We have, for any measurable set E and any real number λ*

$$\lim_{m\to\infty} \int_E \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} dt = |E|.$$

PROOF. Let $f_m(t)$ (or $h_m(t)$) be the real (or imaginary) parts of

$$\prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\}.$$

Further let $e(t)$ denotes the indicator of the set E, that is, $e(t) = 1$ if $t \in E$ and $e(t) = 0$ if $t \notin E$ and let us put

$$f_m(t) \sim c_{0,m} + \sum_{k=1}^{\infty} c_{k,m} \cos 2\pi k(t + \gamma_{k,m}),$$

$$h_m(t) \sim d_{0,m} + \sum_{k=1}^{\infty} d_{k,m}\cos 2\pi k(t+\delta_{k,m}),$$

and

$$e(t) \sim e_0 + \sum_{k=1}^{\infty} e_k \cos 2\pi k(t+\varepsilon_k).$$

Since $f_m(t)$, $h_m(t)$ and $e(t)$ belong to $L_2(0,1)$, we have by Parseval's relation

$$\int_E \prod_{k=1}^{m} \left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\} dt = e_0(c_{0,m}+id_{0,m})$$

$$+\frac{1}{2}\sum_{k=1}^{\infty} e_k\{c_{k,m}\cos 2\pi k(\gamma_{k,m}-\varepsilon_k)+id_{k,m}\cos 2\pi k(\delta_{k,m}-\varepsilon_k)\}.$$

Hence we have

$$\left|\int_E \prod_{k=1}^{m} \left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\} dt - e_0(c_{0,m}+id_{0,m})\right| \leqq \sum_{k=1}^{\infty} |e_k|\{|c_{k,m}|+|d_{k,m}|\}. \tag{4. 4}$$

On the other hand it is easily seen that

$$\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\} = 1+\frac{i\lambda T_1(t)}{C_{N_m}}+\sum_{k=2}^{m}\frac{i\lambda T_k(t)}{C_{N_m}}\prod_{j=1}^{k-1}\left\{1+\frac{i\lambda T_j(t)}{C_{N_m}}\right\}.$$

If z_k(or z_k') denotes the maximum (or minimum) frequency of terms of $T_k(t)$, then we have $z_k \leqq n_{r(N_k-2)}$ and $z_k' \geqq n_{r(N_{k-1}-1)}+1$. Hence we have, by (1. 3) and (2. 1),

$$z_{k+1}'/z_k > q^{\tau} > 6. \tag{4. 5}$$

If $z_j' \leqq v_j \leqq z_j$, then we have, for any $(m_1, m_2, \cdots, m_s)$ such that $1 \leqq m_1 < m_2 < \cdots < m_s < k$,

$$v_k + v_{m_s} + \cdots + v_{m_1} \leqq \sum_{j=1}^{k} z_j \leqq z_k \sum_{j=0}^{k} 6^{j-k} \leqq \frac{6}{5} z_k$$

and

$$v_k - v_{m_s} - \cdots - v_{m_1} \geqq z_k' - \sum_{j=1}^{k-1} z_j \geqq z_k' - z_k' \sum_{j=1}^{k-1} 6^{j-k} > \frac{4}{5} z_k'.$$

Therefore the frequencies of terms of

$$T_k(t)\prod_{j=1}^{k-1}\left\{1+\frac{i\lambda T_j(t)}{C_{N_m}}\right\}$$

lie in the interval $\left[\frac{4}{5}z_k', \frac{6}{5}z_k\right]$ and by (4. 5) these intervals are disjoint.

This implies that

$$\frac{i\lambda T_k(t)}{C_{N_m}} \prod_{j=1}^{k-1} \left\{1 + \frac{i\lambda T_j(t)}{C_{N_m}}\right\}$$
$$= \sum_{n=4z_k'/5}^{6z_k/5} \{c_{n,m}\cos 2\pi n(t + \gamma_{n,m}) + id_{n,m}\cos 2\pi n(t + \delta_{n,m})\},$$

and

(4. 6) $$c_{0,m} = 1 \text{ and } d_{0,m} = 0 \qquad \text{for all } m.$$

Let us put

$$L_k = \sum_{l=N_{k-1}}^{N_k - 2} \sum_p{}' |a_p|$$

where $\sum_p{}'$ denotes the summation over all p such that $n_{(l-1)r-1} < p \leqq n_{lr}$, that is, the double summation runs over all p such that $a_p \cos 2\pi p(t + \alpha_p)$, a term of $S_N(t)$, belongs to $T_k(t)$. Then we have

(4. 7) $$\frac{|\lambda| L_k}{C_{N_m}} \prod_{j=1}^{k-1} \left\{1 + \frac{|\lambda| L_j}{C_{N_m}}\right\} \geqq \sum_{n=4z_k'/5}^{6z_k/5} \{|c_{n.m}| + |d_{n,m}|\}.$$

For each m let us define $p(m)$ as follows

(4. 9) $$p(m) = \text{Max}\left\{6z_k/5\,;\ \sum_{l=1}^{k} L_l \leqq C_{N_m}^{1/2}\right\}.^{*)}$$

Since $C_{N_m} \to +\infty$ as $m \to \infty$, $\text{Max}\left\{k; \sum_{l=1}^{k} L_l \leqq C_{N_m}^{1/2}\right\}$ increases to $+\infty$, as $m \to +\infty$. Therefore we have

(4. 9) $$p(m) \to +\infty, \qquad \text{as } m \to +\infty.$$

By (4. 4), (4. 6) and the fact that $e_0 = |E|$, we have

$$\left|\int_E \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} dt - |E|\right| \leqq \sum_{k=1}^{\infty} |e_k| \{|c_{k,m}| + |d_{k,m}|\}$$
$$\leqq \underset{k \leqq p(m)}{\text{Max}} |e_k| \sum_{k=1}^{p(m)} \left\{|c_{k,m}| + |d_{k,m}|\right\} + \left\{\sum_{k>p(m)} e_k^2\right\}^{1/2} \left\{\left(\sum_{k>p(m)} c_{k,m}^2\right)^{1/2} + \left(\sum_{k>p(m)} d_{k,m}^2\right)^{1/2}\right\}$$

By Lemma 3 the last term of the above formula is less than

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*) Since $L_k^2 \sim \int_0^1 T_k^2(t)\,dt$, $\left(\sum_{k=1}^{m} L_k\right)^2 \geqq \sum_{k=1}^{m} L_k^2 \geqq \sum_{k=1}^{m} \int_0^1 T_k^2(t)\,dt \geqq \frac{1}{2} C_{N_m}^2$ for $m > m_0$.

Therefore we can always define $p(m)$ for $m > m_0$.

$$\sqrt{2}\left\{\sum_{k>p(m)} e_k^2\right\}^{1/2} e^{K\lambda^2/2}=o(1), \qquad \text{as } m\to+\infty.$$

By (4.8) we have, for $k(m)=\operatorname{Max}\left\{k\,;\sum_{l=1}^{k} L_l \leqq C_{N_m}^{1/2}\right\}$,

$$\sum_{k=1}^{p(m)}\left\{|c_{k,m}|+|d_{k,m}|\right\}\leqq\sum_{k=1}^{k(m)}\frac{|\lambda|L_k}{C_{N_m}}\prod_{j=1}^{k-1}\left(1+\frac{|\lambda|L_j}{C_{N_m}}\right)=o(1), \qquad \text{as } m\to\infty.$$

LEMMA 5. *For any $f(t)\in L_2(0,1)$, we have*

$$\lim_{m\to\infty}\int_0^1\left[\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right]f(t)\,dt=\int_0^1 f(t)\,dt.$$

PROOF. If $f(t)$ is a simple function, that is, $f(t)$ assumes only finite number of values, then the lemma follows from Lemma 4. If $f(t)\in L_2(0,1)$, we can take a simple function $f_\varepsilon(t)$ such that

$$\left\{\int_0^1 |f(t)-f_\varepsilon(t)|^2dt\right\}^{1/2}<\varepsilon, \qquad \text{for any given } \varepsilon>0.$$

By Lemma 3, we have

$$\left|\int_0^1\left[\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right]\{f(t)-f_\varepsilon(t)\}\,dt\right|$$
$$\leqq\left[\int_0^1\left|\prod_{k=1}^{n}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right|^2dt\right]^{1/2}\left\{\int_0^1 |f(t)-f_\varepsilon(t)|^2dt\right\}^{1/2}\leqq e^{\lambda^2K/2}\varepsilon.$$

Therefore we have

$$\left|\int_0^1\left[\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right]f(t)-\int_0^1 f(t)\,dt\right|$$
$$\leqq\left|\int_0^1\left[\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\}\right]f_\varepsilon(t)-\int_0^1 f_\varepsilon(t)\,dt\right|+e^{\lambda^2K/2}\varepsilon+\int_0^1|f_\varepsilon(t)-f(t)|\,dt$$
$$\leqq\varepsilon+e^{\lambda^2K/2}\varepsilon+\varepsilon, \qquad \text{for } m>m_0.$$

5. LEMMA 6. *We have, for any fixed real λ,*

$$\lim_{m\to\infty}\frac{1}{|E|}\int_E\exp\left\{\frac{i\lambda}{C_{N_m}}\sum_{k=1}^{m}T_k(t)\right\}dt=\frac{1}{|E|}\int_E\exp\left\{\frac{-\lambda^2}{2}g(t)\right\}dt.$$

PROOF. Let us put

$$E_m=\left\{t\,;0\leqq t\leqq 1,\left|\frac{1}{C_{N_m}^2}\sum_{k=1}^{m}T_k^2(t)-g(t)\right|<1\right\}.$$

Then Lemma 2 implies that $|E_m^c| \to 0$, as $m \to \infty$.*) Hence we have

$$(5.\ 1) \qquad \int_{E \cap E_m^c} \exp\left\{\frac{i\lambda}{C_{N_m}} \sum_{k=1}^{m} T_k(t)\right\} dt = o(1), \qquad \text{as } m \to \infty.$$

By (3. 6) we have for $k \leqq m$

$$\sup_t |T_k(t)| = o(C_{N_m}), \quad \text{for } k \leqq m, \qquad \text{as } m \to \infty.$$

By (2. 7) and (2. 6) it follows that for some constant K

$$\frac{1}{C_N^2} \sum_{l=1}^{N} \{\Delta_l^2(t) + 2\Delta_l(t)\Delta_{l+1}(t)\} \leqq \frac{3}{C_N^2} \sum_{l=1}^{N+1} \Delta_l^2(t) \leqq \frac{K}{2C_N^2} \sum_{l=1}^{N+1} D_l^2(t) \leqq K.$$

and by Lemma 1 this implies $|g(t)| \leqq K$. Therefore if $t \in E_m$, we have

$$\frac{1}{C_{N_m}^3} \sum_{k=1}^{m} |T_k(t)|^3 \leqq K \operatorname*{Max}_{k \leqq m} \left| \frac{T_k(t)}{C_{N_m}} \right| = o(1), \text{ for all } t, \qquad \text{as } m \to \infty.$$

By the above relations and the fact that $e^z = (1+z)\exp\{\frac{z^2}{2} + O(|z|^3)\}$, as $z \to 0$, we have, for $t \in E_m$,

$$\exp\left\{\frac{i\lambda}{C_{N_m}} \sum_{k=1}^{m} T_k(t)\right\} = \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} \exp\left\{\frac{-\lambda^2}{2C_{N_m}^2} \sum_{k=1}^{m} T_k^2(t) + o(1)\right\}, \text{ as } m \to \infty.$$

Since

$$\left| \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} \exp\left\{\frac{-\lambda^2}{2C_{N_m}^2} \sum_{k=1}^{m} T_k^2(t)\right\} \right| \leqq 1,$$

it is seen that for $t \in E_m$

$$\exp\left\{\frac{i\lambda}{C_{N_m}} \sum_{k=1}^{m} T_k(t)\right\} = \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} \exp\left\{\frac{-\lambda^2}{2C_{N_m}^2} \sum_{k=1}^{m} T_k^2(t)\right\} + o(1), \text{ as } m \to \infty$$

Further for $t \in E_m$

$$\left| \prod_{k=1}^{m} \left\{1 + \frac{i\lambda T_k(t)}{C_{N_m}}\right\} \left[\exp\left\{\frac{-\lambda^2}{2C_{N_m}^2} \sum_{1}^{N} T_k^2(t)\right\} - \exp\left\{\frac{-\lambda^2}{2} g(t)\right\}\right] \right|$$

$$\leqq \left| \exp\left\{\frac{\lambda^2}{2C_{N_m}^2} \sum_{k=1}^{m} T_k^2(t) - \frac{\lambda^2}{2} g(t)\right\} - 1 \right|$$

*) F^c is the complement of a set F with respect to the interval [0, 1].

$$< K'\left|\frac{1}{C^2_{N_m}}\sum_{k=1}^{m} T_k^2(t) - g(t)\right|, \qquad \text{for some constant } K'.$$

Hence we have, by (5.1) and Lemma 2,

$$\left|\int_E \exp\left\{\frac{i\lambda}{C_{N_m}}\sum_{k=1}^{m} T_k(t)\right\}dt - \int_{E_m}\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\exp\left\{\frac{-\lambda^2}{2}g(t)\right\}dt\right|$$

$$\leqq K'\int_0^1\left|\frac{1}{C^2_{N_m}}\sum_{k=1}^{m} T_k^2(t) - g(t)\right|dt + o(1) = o(1), \qquad \text{as } m\to\infty.$$

Since $g(t)\geqq 0$, by Lemma3 and Lemma 5 it is seen that

$$\int_E \exp\left\{\frac{i\lambda}{C_{N_m}}\sum_{k=1}^{m} T_k(t)\right\}dt = \int_E\prod_{k=1}^{m}\left\{1+\frac{i\lambda T_k(t)}{C_{N_m}}\right\}\exp\left\{-\frac{\lambda^2}{2}g(t)\right\}dt + o(1)$$

$$= \int_E \exp\left\{\frac{-\lambda^2}{2}g(t)\right\}dt + o(1), \qquad \text{as } m\to\infty.$$

By Lemma 6 and (3. 7) we can prove the theorem.

6. In this paragraph let $f(t) \in L_2(0, 1)$ and

$$f(t) \sim a_0 + \sum_{k=1}^{\infty} a_k\cos 2\pi k(t+\alpha_k),$$

$$S_N(t) = a_0 + \sum_{k=1}^{N} a_k \cos 2\pi k(t+\alpha_k) \text{ and } R_N = \left(\frac{1}{2}\sum_{k>N} a_k^2\right)^{1/2}$$

On the remainder $f(t) - S_N(t)$ we can prove the following

THEOREM 2. *Let $\{ n_k \}$ be a sequence of positive integers satisfying the Hadamard gap condition $n_{k+1} / n_k > q > 1$. We put*

$$U_k(t) = S_{n_{k+1}}(t) - S_{n_k}(t) \quad \textit{and} \quad E_k^2 = R^2_{n/k} - R^2_{n_{k+1}},$$

and suppose that

$$\sup_t |U_k(t)| = O(E_k), \qquad E = o(R^2{}_{n_k}), \qquad \textit{as } k\to+\infty,$$

and, for some function $g(t)$

$$\lim_{k\to\infty}\int_0^1\left|\frac{1}{R^2_{n_k}}\sum_{m=k}^{\infty}\{U_m^2(t) + 2U_m(t)U_{m+1}(t)\} - g(t)\right|dt = 0.$$

Then $g(t)$ is bounded and non-negative and we have, for any set $E \subset [0, 1)$ of positive measure and any real number $\omega \neq 0$,

$$\lim_{N\to\infty}\frac{1}{|E|}\,|\{t\,;t\in E, \{f(t) - S_N(t)\}/R_N \leqq\omega\}| = \frac{1}{\sqrt{2\pi|E|}}\int_E dt\int_{-\infty}^{\omega/(g(t))^{\frac{1}{2}}} e^{-u^2/2}du,$$

where $\omega/0$ *denotes* $+\infty$ *or* $-\infty$ *according as* $\omega>0$ *or* $\omega<0$.

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