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ON THE ABSOLUTE SUMMABILITY FACTORS OF POWER SERIES

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1. Let

$$f(z)=\sum_{n=0}^{\infty} c_n z^n=\sum_{n=0}^{\infty} c_n r^n e^{in\theta}$$

be a function regular for $r=|z|<1$. If for some $p>0$, the integral

$$\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta$$

remains bounded when $r\to 1-0$, the function $f(z)$ is said to belong to the class H^p. It is well known that, if $f(z)$ belongs to the class H^p, then $f(z)$ has a boundary value $f(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ for almost all θ $(0\leqq\theta\leqq 2\pi)$ and $f(e^{i\theta})$ is integrable L^p. Moreover if $p\geqq 1$, a necessary and sufficient condition for the function $f(z)$ to belong to the class H^p is that

$$\sum_{n=0}^{\infty} c_n e^{in\theta}$$

is the Fourier series of its boundary function $f(e^{i\theta})$.

Let us denote

$$s_n(\theta)=s_n(\theta\,;f)=\sum_{u=0}^{n} c_\nu e^{i\nu\theta},$$

$$t_n(\theta)=t_n(\theta\,;f)=nc_n e^{in\theta},$$

$$\sigma_n^\alpha(\theta)=\sigma_n^\alpha(\theta\,;f)=(C,\alpha)\text{ mean of the sequence }\{s_n(\theta)\}$$

$$=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}s_\nu(\theta)\qquad \textit{for }\alpha>-1,$$

and

$$\tau_n^\alpha(\theta) = \tau_n^\alpha(\theta\,; f) = (C, \alpha) \text{ mean of the sequence } \{t_n(\theta)\}$$

$$= \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} t_\nu(\theta) \quad \text{for} \quad \alpha > 0\,,$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)} \quad \text{as} \quad n \to \infty\,.$$

Then we have

$$\tau_n^\alpha(\theta) = n\{\sigma_n^\alpha(\theta) - \sigma_{n-1}^\alpha(\theta)\} = \alpha\{\sigma_n^{\alpha-1}(\theta) - \sigma_n^\alpha(\theta)\}\,.$$

Further we put and

$$h_\alpha(\theta) = h_\alpha(\theta\,; f) = \left[\sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(\theta)|^2}{n}\right]^{\frac{1}{2}}$$

and

$$g_\alpha^*(\theta) = g_\alpha^*(\theta; f) = \left[\int_0^1 (1-r)^{2\alpha} dr \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}} d\varphi\right]^{\frac{1}{2}}.$$

Then for $f(z) \in H^p$ $(0 < p < \infty)$, we have the following relation;

$$A_\alpha \leqq \frac{g_\alpha^*(\theta)}{h_\alpha(\theta)} \leqq B_\alpha\,,$$

where A_α and B_α are positive constants depending only on α.

If $\alpha = 1$, $g_\alpha^*(\theta)$ reduces to the function $g^*(\theta)$ of Littlewood and Paley excepting constant factor. It is known that $g^*(\theta)$ is a majorant of many important functions in the theory of Fourier series.

T. M. Flett [3], G. Sunouchi [5], [6] and A. Zygmund [8] have proved the following theorems.

THEOREM (1. 1). *If $f(z) \in H^p$ $(0 < p \leqq 2)$, then for $\alpha > \frac{1}{p}$,*

(i) $$\int_0^{2\pi} [g_\alpha^*(\theta)]^p\, d\theta \leqq A_{p,\alpha} \int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta\,,$$

(ii) $$\int_0^{2\pi} [h_\alpha(\theta)]^p\, d\theta \leqq A_{p,\alpha} \int_0^{2\pi} |f(e^{i\theta})|^p\, d\theta\,.$$

THEOREM (1. 2). *If $f(z) \in H^p$ $(0 < p \leqq 2)$, then for $\alpha = \frac{1}{p}$,*

(i) $$\int_0^{2\pi}\left[\int_0^1 \frac{r^{2\alpha}(1-r)^{2\alpha}}{|\log(1-r)|^{2\alpha}}dr\int_0^{2\pi}\frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}d\varphi\right]^{\frac{p}{2}}d\theta \leqq A_p\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta.$$

(ii) $$\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2\alpha}}\right]^{\frac{p}{2}}d\theta \leqq A_p\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta,$$

where

$$A_\alpha \leqq \frac{\displaystyle\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2\alpha}}}{\displaystyle\int_0^1\frac{r^{2\alpha}(1-r)^{2\alpha}}{|\log(1-r)|^{2\alpha}}dr\int_0^{2\pi}\frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}d\varphi} \leqq B_\alpha,$$

for $\alpha > 0$ (see, G. Sunouchi [5]).

THEOREM (1. 3). *If* $f(z) \in H^p$ $(0 < p \leqq 1)$, *then for* $\alpha = \dfrac{1}{p}$,

(i) $$\left\{\int_0^{2\pi}[g_\alpha^*(\theta)]^{p\mu}\,d\theta\right\}^{\frac{1}{p\mu}} \leqq A_{p,\mu}\left\{\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\frac{1}{p}},\quad 0<\mu<1,$$

(ii) $$\left\{\int_0^{2\pi}[h_\alpha(\theta)]^{p\mu}\,d\theta\right\}^{\frac{1}{p\mu}} \leqq A_{p,\mu}\left\{\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\frac{1}{p}},\quad 0<\mu<1.$$

A. Zygmund [8] proposes the problem whether Theorem (1.3) holds for $1<p<2$. For $p=2$, it fails.

Concerning this conjecture, we shall prove the following theorem.

THEOREM I. *If* $f(z) \in H^p$ $(1<p<2)$, *then for* $\alpha = \dfrac{1}{p}$,

$$\left\{\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|^2}{n\{\log(n+1)\}^{2(1-1/p)}}\right]^{\frac{p\mu}{2}}d\theta\right\}^{\frac{1}{p\mu}} \leqq A^{p,\mu}\left\{\int_0^{2\pi}|f(e^{i\theta})|^p\,d\theta\right\}^{\frac{1}{p}},\quad 0<\mu<1.$$

Next, we pass to application to the absolute Cesàro summability factors of power series of the class H^p. If the series

$$\sum_{n=1}^{\infty}|\sigma_n^\alpha(\theta)-\sigma_{n-1}^\alpha(\theta)| = \sum_{n=1}^{\infty}\frac{|\tau_n^\alpha(\theta)|}{n}$$

converges, we say the series

$$\sum_{n=0}^{\infty}c_ne^{in\theta}$$

absolutely Cesàro summable with order α and write

$$\sum_{n=0}^{\infty} c_n e^{in\theta} \in |C, \alpha|.$$

H. C. Chow [1] [2] has proved the following theorem.

THEOREM (1.4). (i) *If* $f(z) \in H^p$ $(0 < p \leqq 1)$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{\{\log(n+1)\}^{1/2+\delta}} c_n e^{in\theta} \in \left| C, \frac{1}{p} \right| \quad a.e., \quad \delta > 0.$$

(ii) *If* $f(z) \in H^p$ $(1 < p \leqq 2)$, *then for* $\alpha > \dfrac{1}{p}$,

$$\sum_{n=1}^{\infty} \frac{1}{\{\log(n+1)\}^{1/2+\delta}} c_n e^{in\theta} \in |C, \alpha| \quad a.e., \quad \delta > 0.$$

(iii) *If* $f(z) \in H^p$ $(1 < p \leqq 2)$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{\{\log(n+1)\}^{1+\delta}} c_n e^{in\theta} \in \left| C, \frac{1}{p} \right| \quad a.e., \quad \delta > 0.$$

REMARK. In (i) and (ii), the $\delta > 0$ cannot be cancelled, and moreover in (ii) when $p=2$ the inequality $\alpha > \dfrac{1}{p}$ cannot be replaced by $\alpha = \dfrac{1}{p}$. However for (iii), the only case when $p = 2$ is best possible by T. Tsuchikura [7]. It is remarkable that if Theorem (1.3) holds for $1 < p < 2$, then

$$\sum_{n=1}^{\infty} \frac{1}{\{\log(n+1)\}^{1/2+\delta}} c_n e^{in\theta} \in \left| C, \frac{1}{p} \right| \quad \text{a.e.}$$

H. C. Chow [1] proposes the following problem; if $f(z) \in H^p$ $(1 < p \leqq 2)$, then

$$\sum_{n=1}^{\infty} \frac{1}{\{\log(n+1)\}^{1/p+\delta}} c_n e^{in\theta} \in \left| C, \frac{1}{p} \right| \quad \text{a.e. ?}$$

In the case $p = 1$, this problem is not well-proposed because of Theorem (1.4) (i), and when $p = 2$, it does not hold by Theorem (1.4) (ii).

Taking use of Theorem I, we can prove the following theorem.

THEOREM II. *If* $f(z) \in H^p$ $(1 \leqq p \leqq 2)$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{\{\log(n+1)\}^{(1-1/p)+1/2+\delta}} c_n e^{in\theta} \in \left| C, \frac{1}{p} \right| \quad a.e.$$

This result is better than Theorem (1.4) (iii). Moreover, when $p = 1$, it is best possible, and for $p = 2$, it is so, too. However we could not decide whether for $1 < p < 2$ it is best possible or not.

2. To prove Theorem I, we need the following well-known interpolation theorem.

LEMMA (2.1). *Let* $\mathfrak{P}$ *be the class of all polynomials and* (M, μ) *a measure space, where* M *is the point set and* μ *the measure. Suppose that the family of linear operators* T_z *depending on the complex parameter* z *satisfies the following properties;*
(*a*) *for each* z $(0 \leqq \Re(z) \leqq 1)$, T_z *is a linear transformation mapping* $\mathfrak{P}$ *into* $L^1(M, \mu)$,
(*b*) *for each* $P \in \mathfrak{P}$ *and* $g \in L^\infty(M, \mu)$,

$$G(z) = \int_M T_z(P)\, g\, d\mu$$

is analytic in $0 < \Re(z) < 1$ *and continuous on the closed strip* $0 \leqq \Re(z) \leqq 1$, *and*
(*c*) $\sup_{0 \leqq x \leqq 1} \log|G(x+iy)| \leqq Ae^{a|y|}$ *for* $-\infty < y < \infty$, *where* $a < \pi$.

Now let p_0, p_1, q_0 *and* q_1 *be positive numbers and assume that for all* y $(-\infty < y < \infty)$,

$$\|T_{iy}(P)\|_{q_0} \leqq A_0(y)\|P\|_{p_0}$$

$$\|T_{1+iy}(P)\|_{q_1} \leqq A_1(y)\|P\|_{p_1}$$

for all $P \in \mathfrak{P}$, *where* $\log A_j(y) \leqq B_j e^{C_j|y|}$, $B_j > 0$ *and* $0 < C_j < \pi$ *for* $j = 0, 1$. *Then, for each* t *satisfying* $0 \leqq t \leqq 1$, *we have*

$$\|T_t(P)\|_{q_t} \leqq A_t\|P\|_{p_t}$$

for all $P \in \mathfrak{P}$, *where* p_t *and* q_t *are given by the relations*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

and A_t *depends only on* t, p_j, q_j, B_j, C_j, $(j=0,1)$, *but not on* P.

For the proof of this lemma, we shall refer to E. M. Stein and G. Weiss [4].

We next define the Cesàro mean $\sigma_n^\lambda(\theta)$ of complex order $\lambda=\alpha+i\beta$ of the series

$$\sum_{n=0}^{\infty} c_n e^{in\theta}.$$

Let us denote

$$s_n(\theta)=\sum_{\nu=0}^{n} c_\nu e^{i\nu\theta},$$

$$\sigma_n^\lambda(\theta)=\frac{1}{A_n^\lambda}\sum_{\nu=0}^{n} A_{n-\nu}^{\lambda-1} s_\nu(\theta)=\frac{1}{A_n^\lambda}\sum_{\nu=0}^{n} A_{n-\nu}^{\lambda} c_\nu e^{i\nu\theta},$$

and

$$\tau_n^\lambda(\theta)=\lambda\{\sigma_n^{\lambda-1}(\theta)-\sigma_n^\lambda(\theta)\},$$

where

$$A_n^\lambda=\frac{(\lambda+1)\cdots(\lambda+n)}{n!}.$$

LEMMA (2.2). (i) *If* $\alpha>-1$, *then there exists a constant* B_α *such that*

$$\frac{1}{B_\alpha}(n+1)^\alpha \leqq A_n^\alpha \leqq B_\alpha(n+1)^\alpha \quad for \quad n\geqq 0.$$

(ii) *If* $\alpha>-1$ *and* $-\infty<\beta<\infty$, *then*

$$1\leqq\left|\frac{A_n^{\alpha+i\beta}}{A_n^\alpha}\right|\leqq C_\alpha e^{2\beta^2} \quad for \quad n\geqq 0,$$

where C_α *depends only on* α.

For the proof, we refer to E. M. Stein and G. Weiss [4].

Now we shall prove the following inequalities;

$$\text{(2. 3)}\quad \left\{\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\frac{|\tau_n^{1/2+i\beta}(\theta)|^2}{n\log(n+1)}\right]d\theta\right\}^{\frac{1}{2}} \leqq Ae^{2\beta^2}\left\{\int_0^{2\pi}|f(e^{i\theta})|^2\,d\theta\right\}^{\frac{1}{2}} \quad \text{for } f(z)\in H^2,$$

$$(2.4)\quad \left\{\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\frac{|\tau_n^{1+i\beta}(\theta)|^2}{n}\right]^{\frac{\mu}{2}}d\theta\right\}^{\frac{1}{\mu}} \leqq A_\mu e^{2\pi\beta}\int_0^{2\pi}|f(e^{i\theta})|\,d\theta$$

for $f(z)\in H^1$ and $0<\mu<1$.

We shall begin with (2.3). Since

$$\sigma_n^{-1/2+i\beta}(\theta)-\sigma_n^{1/2+i\beta}(\theta)=\frac{1}{\left(\frac{1}{2}+i\beta\right)A_n^{1/2+i\beta}}\sum_{\nu=0}^{n}A_{n-\nu}^{-1/2+i\beta}\,\nu c_\nu e^{i\nu\theta},$$

that is

$$\tau_n^{1/2+i\beta}(\theta)=\frac{1}{A^{1/2+i\beta}}\sum_{\nu=0}^{n}A_{n-\nu}^{-1/2+i\beta}\,\nu c_\nu e^{i\nu\theta},$$

using Parseval's identity and Lemma (2.2) we obtain

$$\begin{aligned}\int_0^{2\pi}|\tau_n^{1/2+i\beta}(\theta)|^2\,d\theta &= \frac{1}{|A_n^{1/2+i\beta}|^2}\sum_{\nu=0}^{n}|A_{n-\nu}^{-1/2+i\beta}|^2\,\nu^2|c_\nu|^2\\ &\leqq A\frac{1}{n+1}e^{4\beta^2}\sum_{\nu=0}^{n}\frac{1}{n+1-\nu}\,\nu^2|c_\nu|^2.\end{aligned}$$

Hence,

$$\begin{aligned}(2.5)\quad \sum_{n=0}^{\infty}\int_0^{2\pi}\frac{|\tau_n^{1/2+i\beta}(\theta)|^2}{n\log(n+1)}\,d\theta &\leqq Ae^{4\beta^2}\sum_{n=1}^{\infty}\frac{1}{n(n+1)\log(n+1)}\sum_{\nu=1}^{n}\frac{1}{n+1-\nu}\,\nu^2\,|c_\nu|^2\\ &\leqq Ae^{4\beta^2}\sum_{n=1}^{\infty}\frac{1}{n^2\log(n+1)}\sum_{\nu=1}^{n}\frac{1}{n+1-\nu}\,\nu^2\,|c_\nu|^2\\ &= Ae^{4\beta^2}\sum_{\nu=1}^{\infty}\nu^2\,|c_\nu|^2\sum_{n=\nu}^{\infty}\frac{1}{n^2\log(n+1)}\cdot\frac{1}{n+1-\nu}\\ &= Ae^{4\beta^2}\sum_{\nu=1}^{\infty}\nu^2\,|c_\nu|^2\left[\sum_{n=\nu}^{2\nu}+\sum_{n=2\nu+1}^{\infty}\right]=Ae^{4\beta^2}\sum_{\nu=1}^{\infty}\nu^2|c_\nu|^2(S_1+S_2)\quad \textit{say}.\end{aligned}$$

Since

$$\begin{aligned}S_1 &= \sum_{n=\nu}^{2\nu}\frac{1}{n^2\log(n+1)}\cdot\frac{1}{n+1-\nu}\leqq\frac{1}{\nu^2\log(\nu+1)}\sum_{n=\nu}^{2\nu}\frac{1}{n+1-\nu}\\ &\leqq B_1\frac{1}{\nu^2\log(\nu+1)}\log(\nu+1)=\frac{B_1}{\nu^2},\end{aligned}$$

$$S_2 = \sum_{n=2\nu+1}^{\infty} \frac{1}{n^2 \log(n+1)} \frac{1}{n+1-\nu} \leqq \frac{1}{(\nu+1)\log(2\nu+2)} \sum_{n=2\nu+1}^{\infty} \frac{1}{n^2}$$

$$\leqq \frac{1}{\log 4} \frac{1}{\nu} \sum_{n=2\nu+1}^{\infty} \frac{1}{n^2} \leqq B_2 \frac{1}{\nu} \frac{1}{2\nu+1} \leqq \frac{B_2}{2} \frac{1}{\nu^2},$$

the last term (2. 5) is majorized by

$$Be^{4\beta^2} \sum_{\nu=1}^{\infty} |c_\nu|^2 = Be^{4\beta^2} \int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta .$$

Therefore by exchanging the order of summation and integration in the first term of (2. 5), we get the estimate (2. 3).

Next we pass to (2.4). If we can prove

$$\sum_{n=1}^{\infty} \frac{|\tau_n^{\alpha+i\beta}(\theta)|^2}{n} \leqq Ae^{4\pi\beta} \{g_\alpha^*(\theta)\}^2 \quad \text{for} \quad \alpha > 0, \tag{2. 6}$$

where A does not depend on α and β, then the estimate (2. 4) follows immediately from Theorem (1.3) (i) for $p = 1$. For the sake of the proof of (2. 6) we write

$$\Phi_{\alpha+i\beta}(r, \theta) = \sum_{n=1}^{\infty} |A_n^{\alpha+i\beta}|^2 |\tau_n^{\alpha+i\beta}(\theta)|^2 r^{2n} .$$

Then

$$\int_0^1 (1-r)^{2\alpha} \Phi_{\alpha+i\beta}(r, \theta) \, dr = \sum_{n=1}^{\infty} |A_n^{\alpha+i\beta}|^2 |\tau_n^{\alpha+i\beta}(\theta)|^2 \int_0^1 (1-r)^{2\alpha} r^{2n} \, dr \tag{2. 7}$$

$$= \sum_{n=1}^{\infty} |A_n^{\alpha+i\beta}|^2 |\tau_n^{\alpha+i\beta}(\theta)|^2 \frac{1}{(2n+2\alpha+1) A_{2n}^{2\alpha}} .$$

On the other hand, since

$$\sum_{n=0}^{\infty} A_n^{\alpha+i\beta} \tau_n^{\alpha+i\beta}(\theta) z^n = \frac{ze^{i\theta} f'(ze^{i\theta})}{(1-z)^{\alpha+i\beta}},$$

we have by Parseval's theorem,

$$\Phi_{\alpha+i\beta}(r, \theta) \leqq \frac{1}{2\pi} e^{4\pi\beta} \int_0^{2\pi} \frac{|rf'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}} \, d\varphi \leqq \frac{1}{2\pi} e^{4\pi\beta} \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}} \, d\varphi. \tag{2. 8}$$

Therefore by Lemma (2. 2), (2. 7), and (2. 8), we get

$$\sum_{n=1}^{\infty} |\tau_n^{\alpha+i\beta}(\theta)|^2 \frac{1}{n} \leqq A \sum_{n=1}^{\infty} \frac{(A_n^{\alpha})^2}{(2n+2\alpha+1)A_{2n}^{2\alpha}} |\tau_n^{\alpha+i\beta}(\theta)|^2$$

$$\leqq A \sum_{n=1}^{\infty} |A_n^{\alpha+i\beta}|^2 |\tau_n^{\alpha+i\beta}(\theta)|^2 \frac{1}{(2n+2\alpha+1)A_{2n}^{2\alpha}} = A\int_0^1 (1-r)^{2\alpha}\Phi_{\alpha+i\beta}(r,\theta)\,dr$$

$$\leqq Be^{4\pi\beta}\int_0^1 (1-r)^{2\alpha}\,dr \int_0^{2\pi} \frac{|f'(re^{i(\theta+\varphi)})|^2}{|1-re^{i\varphi}|^{2\alpha}}\,d\varphi = Be^{4\pi\beta}\{g_\alpha^*(\theta)\}^2 .$$

Thus (2. 6) and consequently (2. 4) are proved.

Now we consider the family of operators defined by

$$T_z(f) = \sum_{n=1}^{\infty} \frac{\tau_n^{\delta(z)}(\theta)}{\sqrt{n}\,\{\log(n+1)\}^{1-\delta(z)}}\,\phi_n(\theta),$$

where $\delta(z) = \frac{1}{2}z + \frac{1}{2}$ and $\{\phi_n(\theta)\}$ $n = 1, 2, \cdots$ is a sequence such that $\left[\sum_{n=1}^{\infty} |\phi_n(\theta)|^2\right]^{\frac{1}{2}} \leqq 1$ for all θ, but is arbitary otherwise. Since by Schwarz's inequality,

$$|T_z(f)| \leqq \left[\sum_{n=1}^{\infty} \frac{|\tau_n^{\delta(z)}(\theta)|^2}{n|\{\log(n+1)\}^{2(1-\delta(z))}|}\right]^{\frac{1}{2}},$$

we have by (2. 3) and (2. 4), for each $P \in \mathfrak{P}$,

$$\|T_{iy}(P)\|_2 \leqq Ae^{\frac{1}{2}y^2}\|P\|_2$$

$$\|T_{1+iy}(P)\|_\mu \leqq Ae^{\pi|y|}\|P\|_1, \quad 0 < \mu < 1 .$$

For any given p $(1 < p < 2)$, we first choose t such that

$$\frac{1}{p} = \frac{1-t}{2} + \frac{t}{1} \quad \textit{i.e.} \quad t = \frac{2}{p} - 1,$$

and then for any given η $(0 < \eta < 1)$, we define μ such that

$$\frac{1}{p\eta} = \frac{1-t}{2} + \frac{t}{\mu} \quad \text{i.e.} \quad \mu = \frac{1}{1 + \dfrac{1-\eta}{\eta(2-p)}} .$$

Therefore by Lemma (2.1), for each $P \in \mathfrak{P}$, we have

$$\|T_t(P)\|_{p\eta} \leqq A_{p,\eta}\|P\|_p$$

that is,

$$\left[\int_0^{2\pi}\left|\sum_{n=1}^{\infty}\frac{\tau_n^{\delta(t)}(\theta)}{\sqrt{n}\,\{\log(n+1)\}^{1-\delta(t)}}\phi_n(\theta)\right|^{p\eta}d\theta\right]^{\frac{1}{p\eta}} \leqq A_{p,\eta}\|P\|_p\,.$$

Since now

$$\delta(t) = \frac{1}{2}t + \frac{1}{2} = \frac{1}{2}\left(\frac{2}{p}-1\right) + \frac{1}{2} = \frac{1}{p}\,,$$

and

$$\sup_{\phi}\left|\sum_{n=1}^{\infty}\frac{\tau_n^{1/p}(\theta)}{\sqrt{n}\,\{\log(n+1)\}^{1-1/p}}\phi_n(\theta)\right| = \left[\sum_{n=1}^{\infty}\frac{|\tau_n^{1/p}(\theta)|^2}{n\,\{\log(n+1)^{2(1-1/p)}}\right]^{\frac{1}{2}}\,,$$

we get Theorem I in the case when $f(z)$ is a polynomial.

The general case follows by a standard limiting process.

3. To prove Theorem II, we need the following lemma.

LEEMA (3.1). *Let $\alpha > 0$ and $\{\lambda_n\}$ be a sequence of positive numbers such that*

(*a*) $\dfrac{\lambda_n}{n}$ *is non-increasing,*

(*b*) $\Delta\lambda_n = \lambda_n - \lambda_{n+1} = O\left(\dfrac{\lambda_n}{n}\right)$

$\Delta^2\Delta_n = \Delta\Delta\lambda_n = O\left(\dfrac{\lambda_n}{n^2}\right)$

$\cdots\cdots\cdots\cdots\cdots$

$\Delta^{h+1}\lambda_n = \Delta\Delta^h\lambda_n = O\left(\dfrac{\lambda_n}{n^{h+1}}\right)$,

where h is the integral part of α when α is fractional and $h = \alpha - 1$ when α is an integer, and

(*c*) $\displaystyle\sum_{n=1}^{\infty}\frac{|\tau_n^{\alpha}(\theta)|}{n}\lambda_n$ *is convergent.*

Then

$$\sum_{n=1}^{\infty}\lambda_n c_n e^{in\theta} \in |C, \alpha|\,.$$

For the proof, we refer to H. C. Chow [2].

We can now prove Theorem II. The cases when $p=1$ and $p=2$ are well-known. If $f(z) \in H^p$ $(1<p<2)$, then from Theorem I, it follows that

$$\sum_{n=1}^{\infty} \frac{|\tau_n^{1/p}(\theta)|^2}{n\{\log(n+1)\}^{2(1-1/p)}}$$

converes almost everywhere. On the other hand, if we put

$$\lambda_n = \frac{1}{\{\log(n+1)\}^{\lambda}}, \quad \lambda = \left(1-\frac{1}{p}\right)+\frac{1}{2}+\delta,\ \delta>0,$$

then by Schwarz's inequality, we have

$$\sum_{n=1}^{\infty} \frac{|\tau_n^{1/p}(\theta)|}{n}\lambda_n \leqq \left[\sum_{n=1}^{\infty} \frac{|\tau_n^{1/p}(\theta)|^2}{n\{\log(n+1)\}^{2(1-1/p)}}\right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} \frac{1}{n\{\log(n+1)\}^{2\lambda-2(1-1/p)}}\right]^{\frac{1}{2}}. \tag{3.2}$$

The second term of the right hand side converges, since

$$2\lambda - 2\left(1-\frac{1}{p}\right) = 1+2\delta,\ \ \delta>0.$$

Hence the left hand side converges almost everywhere. Therefore from Lemma (3.1) we get Theorem II.

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