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ON A CLASS OF CONVOLUTION TRANSFORMS

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1. Introduction. In 1947 D. V. Widder showed [6] that the convolution transform

$$f(x)=\int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt$$

can be inverted by a linear differential operator of infinite order for a restricted class of kernels $G(x)$. In 1949 I. I. Hirschman, Jr. and D. V. Widder [1], [2] greatly enlarged the class of admissible kernels. Further, in 1950 D. V. Widder [7] enlarged the class of admissible kernels so far as their bilateral Laplace transforms are meromorphic functions $F(x)$ with real zeros and poles only and obtained the inversion formula. This case, however, was so complicated that the properties of the transform was not explained clearly enough and it was necessary to assume an order condition for $\varphi(t)$.

Instead of the condition for $\varphi(t)$ if we restrict the kernel in some sense, then the properties shall be determined completly [5].

In this paper we shall study the inversion and representation theory for the class of convolution transforms

$$f(x)=\int_{-\infty}^{\infty} G(x-t)\,e^{ct}\,d\alpha(t) \quad (c:\text{real}), \tag{1}$$

for which the kernel $G(t)$ is of the form

$$G(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[F(s)]^{-1}e^{st}\,ds. \tag{2}$$

Here

$$F(s)=\frac{e^{bs}\prod_{k=1}^{\infty}(1-s/a_k)\,e^{s/a_k}\prod_{k=1}^{\infty}(1-s/d_k)\,e^{s/d_k}}{\prod_{k=1}^{\infty}(1-s/c_k)\,e^{s/c_k}}, \tag{3}$$

where b, $\{a_k\}_1^\infty$, $\{c_k\}_1^\infty$, $\{d_k\}_1^\infty$ are constants such that for all k

$$(4)\quad a_k c_k > 0,\ a_k d_k < 0,\ |a_k| \leqq |c_k|\,,\ \sum_1^\infty a_k^{-2} < \infty,\ \sum_1^\infty c_k^{-2} < \infty,\ \sum_1^\infty d_k^{-2} < \infty$$

and satisfies the condition: for some positive α and any positive number R,

$$(5)\qquad \frac{1}{|F(\sigma+i\tau)|} = O\left(\frac{1}{|\tau|^{2+\alpha}}\right) \qquad |\tau| \to \infty\,,\ s = \sigma + i\tau\,,$$

uniformly in the strip $|\sigma| \leqq R$.

This form is the one in which $E_1(s)$, $E_2(s)$ are reciprocals of the generating functions of kernels of class I and II [1], [2], respectively, and $F(s)=E_1(s)/E_2(s)$ satisfies the conditions (4) and (5).

However, when we assume that $E_2(s)$ be the reciprocal of the generating function of the kernel of class I or class III, we may similarly discuss.

In these cases we shall know that the behavior of the transform (1) is similar to the one of the convolution transform with the class I kernel.

2. Properties of the kernel. Let us define

$$g_k^{(1)}(t) = \begin{cases} \begin{cases} a_k e^{a_k t - 1} & (-\infty < t < 1/a_k) \\ 0 & (1/a_k < t < \infty) \end{cases} & \text{if} \quad a_k > 0 \\ \begin{cases} 0 & (-\infty < t < 1/a_k) \\ -a_k e^{a_k t - 1} & (1/a_k < t < \infty) \end{cases} & \text{if} \quad a_k < 0\,, \end{cases}$$

$g_k^{(2)}(t)$ as the function $g_k^{(1)}(t)$ in which we put d_k for a_k,

and

$$h_k^{(1)}(t) = \int_{-\infty}^{t} (1 - a_k/c_k)\, g_k(u + 1/c_k)\, du + \frac{a_k}{c_k} j[t - (1/a_k - 1/c_k)]\,,$$

$$h_k^{(2)}(t) = \int_{-\infty}^{t} g_k^{(2)}(u)\, du\,,$$

where $j(t)$ is the standard jump function, that is $j(t) = 1$ for $t > 0$, $1/2$ for $t=0$ and 0 for $t < 0$.

It is easily verified that $h_k^{(1)}(t)$ is a normalized distribution function with mean 0 and variance $a_k^{-2} - c_k^{-2}$, and

$$\int_{-\infty}^{\infty} e^{-st}\, dh_k^{(1)}(t) = \frac{(1-s/c_k)\, e^{s/c_k}}{(1-s/a_k)\, e^{s/a_k}},$$

the bilateral Laplace transform converging absolutely for $\Re s < a_k$ if $a_k > 0$ and for $\Re s > a_k$ if $a_k < 0$ and that $h_k^{(2)}(t)$ is a normalized distribution function with mean 0 and variance d_k^{-2} and

$$\int_{-\infty}^{\infty} e^{-st}\, dh_k^{(2)}(t) = \frac{1}{(1-s/d_k)\, e^{s/d_k}},$$

the bilateral Laplace transform converging absolutely for $\Re s < d_k$ if $d_k > 0$ and for $\Re s > d_k$ if $d_k < 0$.

Let us define the constants

$$\alpha_1 = \begin{cases} \max\limits_{d_k<0} d_k & \text{if} \quad a_k > 0 \\ \max\limits_{a_k<0} a_k & \text{if} \quad a_k < 0, \end{cases} \qquad \alpha_2 = \begin{cases} \min\limits_{a_k>0} a_k & \text{if} \quad a_k > 0 \\ \min\limits_{d_k>0} d_k & \text{if} \quad a_k < 0. \end{cases}$$

THEOREM 1. *If*

1. $F(s)$ *is defined by* (3) *and* (4) *of* §1 *and satisfies the condition* (5) *of* §1,
2. μ_1 *=multiplicities of* α_1 *as a zero of* $F(s)$,
   μ_2 *=multiplicities of* α_2 *as a zero of* $F(s)$,
3. $$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{1}{F(s)}\, e^{st}\, ds \qquad (-\infty < t < \infty),$$

then:

A. $G(t)$ *is a frequency function with mean* b *and variance*

$$\left(\sum_1^{\infty} a_k^{-2} - \sum_1^{\infty} c_k^{-2}\right) + \sum_1^{\infty} d_k^{-2};$$

B. $\int_{-\infty}^{\infty} G(t)\, e^{-st}\, dt = 1/F(s)$, *the bilateral Laplace transform converging absolutely in the strip* $\alpha_1 < \Re s < \alpha_2$;

C. $G(t) \in C^1(-\infty, \infty)$;

D. $G(t) = p(t)e^{\alpha_1 t} + R_+(t)$, $G(t) = q(t)e^{\alpha_2 t} + R_-(t)$,
where $p(t)$, $q(t)$ *are real polynomials of degree* $\mu_1 - 1$, $\mu_2 - 1$, *respectively and*

$$[R_+(t)]^{(n)} = O(e^{(\alpha_1-\varepsilon)t}) \quad t\to\infty \qquad (n=0,1)$$

$$[R_-(t)]^{(n)} = O(e^{(\alpha_2+\varepsilon)t}) \quad t\to-\infty \quad (n=0,1)$$

for some $\varepsilon>0$.

PROOF. If we set

$$H_n(t) = h_1^{(1)} \# h_1^{(2)} \# h_2^{(1)} \# h_2^{(2)} \# \cdots \# h_n^{(1)} \# h_n^{(2)}(t-b),$$

where operation $\#$ denotes the Stieltjes convolution for distribution functions, then by the convolution theorem [8] $H_n(t)$ is a distribution function with mean b and variance $\left(\sum_1^n a_k^{-2} - \sum_1^n c_k^{-2}\right) + \sum_1^n d_k^{-2}$ and with the bilateral Laplace transform

$$\int_{-\infty}^{\infty} e^{-st}\, dH_n(t) = \frac{\prod_1^n (1-s/c_k)\, e^{s/c_k}}{e^{bs}\prod_1^n (1-s/a_k)\, e^{s/a_k} \prod_1^n (1-s/d_k)\, e^{s/d_k}},$$

converging absolutely for $\alpha_{1,n} < \Re s < \alpha_{2,n}$, where

$$\alpha_{1,n} = \begin{cases} \max\limits_{1\le k\le n} d_k & \text{if } a_k>0 \\ \max\limits_{1\le k\le n} a_k & \text{if } a_k<0, \end{cases} \qquad \alpha_{2,n} = \begin{cases} \min\limits_{1\le k\le n} a_k & \text{if } a_k>0 \\ \min\limits_{1\le k\le n} d_k & \text{if } a_k<0. \end{cases}$$

We have

$$\lim_{n\to\infty} \frac{\prod_1^n (1-s/c_k)\, e^{s/c_k}}{e^{bs}\prod_1^n (1-s/a_k)\, e^{s/a_k} \prod_1^n (1-s/d_k)\, e^{s/d_k}} = 1/F(s),$$

uniformly for s in $\alpha_1 < \Re s < \alpha_2$.

By the well known theorem [2, p. 41] $1/F(i\tau)$ $(s=\sigma+i\tau)$ is the characteristic function of a distribution function $H(t)=\lim\limits_{n\to\infty} H_n(t)$. Moreover, by the Lévy's theorem [2, p. 51],

$$H(t_1) - H(t_2) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st_1}-e^{st_2}}{sF(s)}\, ds.$$

From the condition (5) of §1 for $F(s)$, it follows that $H(t)$ is twice differentiable and $G(t)=\frac{d}{dt}H(t)$, so $G(t)$ is a frequency function and $G(t) \in C^1(-\infty, \infty)$.

By use of a theorem of Hamburger [8, p. 265] we see that

$$\int_{-\infty}^{\infty} e^{-st}\, G(t)\, dt = 1/F(s)\,, \tag{1}$$

the integral converging absolutely for $\alpha_1 < \Re s < \alpha_2$.

Differentiating equation (1) with respect to s and setting $s=0$ we see that the mean of $G(t)$ is b and the variance is $\left(\sum_1^\infty a_k^{-2} - \sum_1^\infty c_k^{-2}\right) + \sum_1^\infty d_k^{-2}$.

Now, let us choose $\varepsilon > 0$ so small that no zeros of $F(s)$ lie in the interval $\alpha_1 - \varepsilon \leqq \sigma < \alpha_2$. Let $T > 0$ and define D as the rectangular contour with vertices at $\pm iT$, $\alpha_1 - \varepsilon \pm iT$. Integrals about D proceed counterclockwise.

The integral

$$\frac{1}{2\pi i}\int_D \frac{s^n e^{st}}{F(s)}\, ds$$

is by Cauchy's residue theorem equal to the nth derivative of the residue of $e^{st}/F(s)$ at $s=\alpha_1$.

Let $F(s)=(s-\alpha_1)^{\mu_1}F_1(s)$. The expansion

$$1/F_1(s) = \sum_{j=0}^{\infty} A_j (s-\alpha_1)^j\,, \quad A_0 \neq 0$$

is valid in some circle about α_1. Thus

$$\begin{aligned} e^{st}/F(s) &= (s-\alpha_1)^{-\mu_1}\left[\sum_{j=0}^{\infty} A_j(s-\alpha_1)^j\right]\left[e^{\alpha_1 t}\sum_{j=0}^{\infty}\frac{t^j}{j!}(s-\alpha_1)^j\right] \\ &= e^{\alpha_1 t}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} A_{j-k}\frac{t^k}{k!}\right)(s-\alpha_1)^{-(\mu_1-j)}\,. \end{aligned}$$

It follows that the residue of $e^{st}/F(s)$ at α_1 is

$$e^{\alpha_1 t}\sum_{k=0}^{\mu_1-1} A_{\mu_1-1-k}\frac{t^k}{k!} = e^{\alpha_1 t}p(t)\,.$$

We write

$$\int_D \frac{s^n e^{st}}{F(s)}\, ds = \int_{-iT}^{iT} + \int_{\alpha_1-\varepsilon+iT}^{\alpha_1-\varepsilon-iT} + \int_{iT}^{\alpha_1-\varepsilon+iT} + \int_{\alpha_1-\varepsilon-iT}^{-iT}$$
$$= I_1 + I_2 + I_3 + I_4 ,$$

By the condition (5) of §1 we have

$$\lim_{T\to\infty} I_3 = \lim_{T\to\infty} I_4 = 0 .$$

Hence

$$\left(\frac{d}{dt}\right)^n G(t) = [e^{\alpha_1 t} p(t)]^{(n)} + [R_+(t)]^{(n)} , \; R_+(t) = \frac{1}{2\pi i}\int_{\alpha_1-\varepsilon+iT}^{\alpha_1-\varepsilon-iT} \frac{e^{st}}{F(s)}\, ds .$$

If we apply the condition again, we obtain

$$[R_+(t)]^{(n)} = O(e^{(\alpha_1-\varepsilon)t}) \quad (t \to \infty) .$$

Similar arguments serve to establish the other part of conclusion D. Thus we complete the proof.

Let us define

$$E_1(s) = e^{bs} \prod_{k=1}^{\infty} (1-s/a_k)\, e^{s/a_k} \prod_{k=1}^{\infty} (1-s/d_k)\, e^{s/d_k} ,$$

$$E_2(s) = \prod_{k=1}^{\infty} (1-s/c_k) e^{s/c_k} ,$$

$$G_1(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{E_1(s)}\, ds \qquad (\in \text{class I}),$$

$$G_2(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{E_2(s)}\, ds \qquad (\in \text{class II}).$$

The following two theorems are well known. [2, p. 55, p. 107]

THEOREM 2. A. *$G_1(t)$ is a frequency function with mean b and variance* $\sum_1^\infty a_k^{-2} + \sum_1^\infty d_k^{-2}$;

B. $\int_{-\infty}^{\infty} e^{-st} G_1(t)\,dt = 1/E_1(s)$, *the bilateral Laplace transform converging absolutely in the strip* $\alpha_1 < \Re s < \alpha_2$;

C. $G_1(t) \in C^{\infty}(-\infty, \infty)$;

D. $\left(\frac{d}{dt}\right)^n G_1(t) = [p_1(t)\,e^{\alpha_1 t}]^{(n)} + O(e^{(\alpha_1 - \varepsilon_1)t}) \quad (t \to \infty) \quad (n=0, 1, 2, \cdots)$,

$\left(\frac{d}{dt}\right)^n G_2(t) = [q_1(t)\,e^{\alpha_2 t}]^{(n)} + O(e^{(\alpha_2 + \varepsilon_1)t}) \quad (t \to -\infty) \quad (n=0, 1, 2, \cdots)$,

for some $\varepsilon_1 > 0$, *where* $p_1(t)$, $q_1(t)$ *are real polinomials of degree* $\mu_1 - 1$, $\mu_2 - 1$, *respectively.*

THEOREM 3. A. $G_2(t)$ *is a frequeny function with mean* 0 *and variance* $\sum_1^{\infty} c_k^{-2}$;

B. $\int_{-\infty}^{\infty} e^{-st} G_2(t)\,dt = 1/E_2(s)$, *the bilateral Laplace transform converging absolutely in the half strip* $\max_{c_k<0}(c_k, -\infty) < \Re s < \min_{c_k>0}(c_k, +\infty)$;

C. $G_2(t) \in C^{\infty}(-\infty, \infty)$.

From Theorem 1 and Theorem 2 we see that the properties of kernel function $G(t)$ are similar to the one of $G_1(t)$ and if $G(t)$ satisfies the condition (5) of §1 for any positive α then $G(t) \in C^{\infty}(-\infty, \infty)$.

3. Convergence. We can now determine the convergence behavior of the transform $\int_{\infty-}^{\infty} G(x-t)\,e^{ct}\,d\alpha(t)$.

THEOREM 4. *If* $\alpha(t)$ *is of bounded variation in every finite interval and* $\int_{-\infty}^{\infty} G(x_0 - t)\,e^{ct}\,d\alpha(t)$ $\left(\int_{-\infty}^{\infty} G_1(x_0 - t)\,e^{ct}\,d\alpha(t)\right)$ *converges (conditionally), then*

$$\int_{-\infty}^{\infty} G(x-t)\,e^{ct}\,d\alpha(t) \qquad \left(\int_{-\infty}^{\infty} G_1(x-t)\,e^{ct}\,d\alpha(t)\right)$$

converges uniformly for x *in any finite interval.*

PROOF. It is well known that the theorem is valid for the transform in the parenthesis. [2, p. 124]

For the proof it is necessary to show that the function $G(t) \neq 0$ for t with sufficiently large absolute value.

If $t = t_0$ is a zero of $G(t)$, then from the conclusion D of Theorem 1, we have

$$(1)\qquad \begin{aligned} |p(t_0)e^{\alpha_1 t_0}| &= |R_+(t_0)| \quad \text{if} \quad t_0 > 0\,, \\ |q(t_0)e^{\alpha_2 t_0}| &= |R_-(t_0)| \quad \text{if} \quad t_0 < 0\,. \end{aligned}$$

However, by the orders of $R_+(t)$ and $R_-(t)$

$$|p(t)e^{\alpha_1 t}| > |R_+(t)| \quad \text{or} \quad |q(t)e^{\alpha_2 t}| > |R_-(t)|$$

for all t with sufficiently large absolute value.

Therefore, the equation (1) is valid only for t_0 in some finite interval. Now, we must show that

$$(2)\qquad \lim_{A,B\to+\infty}\int_A^B G(x-t)\,e^{ct}\,d\alpha(t) = 0\,,$$

$$(2)'\qquad \lim_{A,B\to-\infty}\int_A^B G(x-t)\,e^{ct}\,d\alpha(t) = 0\,,$$

uniformly for x in any finite interval. By Theorem 1 we have

$$(3)\qquad \begin{aligned} &\frac{G(x-t)}{G(x_0-t)} = O(1) \quad (|t|\to\infty)\,, \\ &\frac{d}{dt}\left[\frac{G(x-t)}{G(x_0-t)}\right] = O(1/t^2) \quad (|t|\to\infty)\,, \end{aligned}$$

uniformly for x in any finite interval. If we set

$$L(t) = \int_t^\infty G(x_0-u)\,e^{cu}\,d\alpha(u)\,,$$

then $L(t)$ is bounded and

$$(4)\qquad L(t) = o(1) \quad \text{as} \quad t\to\infty\,.$$

We have

$$\begin{aligned} \int_A^B G(x-t)\,e^{ct}\,d\alpha(t) &= -\int_A^B \frac{G(x-t)}{G(x_0-t)}\,dL(t) \\ &= \left[-\frac{G(x-t)}{G(x_0-t)}L(t)\right]_A^B + \int_A^B\left[\frac{d}{dt}\frac{G(x-t)}{G(x_0-t)}\right]L(t)\,dt. \end{aligned}$$

Using the equations (3) and (4) we see that the equation (2) holds uniformly for x in any finite interval. We may similarly establish (2)'.

From this theorem we know that the convergence behavior of our transform is similar to the one with a kernel of class I.

THEOREM 5. *The transform* $\int_{-\infty}^{\infty} G(x-t)\, e^{ct}\, d\alpha(t)$ *converges if, and only if, the transform* $\int_{-\infty}^{\infty} G_1(x-t)\, e^{ct}\, d\alpha(t)$ *converges.*

PROOF. If the first transform converges for $x=x_0$, by Theorem 1 and Theorem 2, we have

$$G_1(x_0-t)/G(x_0-t) = O(1) \quad (|t| \to \infty),$$

$$\frac{d}{dt}[G_1(x_0-t)/G(x_0-t)] = O(1/t^2) \quad (|t| \to \infty).$$

From this it follows by the arguments similar to Theorem 4 that the second transform converges. We may similarly establish the only-if part.

4. Inversion theorem. We suppose that we are given a sequence $\{b_n\}_0^\infty$ of real numbers such that $b_0=b$, $\lim_{n\to\infty} b_n=0$. We define, as usual,

$$E_{1,n}(D) = e^{(b-b_n)D} \prod_{k=1}^{n} (1-D/a_k)\, e^{D/a_k}(1-D/d_k)\, e^{D/a_k},$$

where D stands for differentiation and we interpret e^{lD} the operation of translation through distance l.

On the other hand, by virtue of the equation in Theorem 3 we define

$$\frac{1}{E_2(D)} f(x) = \int_{-\infty}^{\infty} f(x-t)\, G(t)\, dt,$$

whenever this integral coverges [7, p. 121].

THEOREM 6. *If*

1. $f(x) = \int_{-\infty}^{\infty} G(x-t)\, e^{ct}\, d\alpha(t)$ *converges,*

2. $\alpha(t)$ *is of bounded variation in every finite interval and continuous at* x_1, x_2, *then:*

A. $\alpha_1 < c < \alpha_2$ *implies that*

$$\lim_{n\to\infty}\int_{x_1}^{x_2} e^{-cx}E_{1,n}(D)([E_2(D)]^{-1}f(x))\,dx = \alpha(x_2)-\alpha(x_1);$$

B. $c \geqq \alpha_2$ *implies that* $\alpha(+\infty)$ *exists and that*

$$\lim_{n\to\infty}\int_{x_1}^{\infty} e^{-cx}E_{1,n}(D)([E_2(D)]^{-1}f(x))\,dx = \alpha(+\infty)-\alpha(x_1);$$

C. $c \leqq \alpha_1$ *implies that* $\alpha(-\infty)$ *exists and that*

$$\lim_{n\to\infty}\int_{-\infty}^{x_2} e^{-cx}E_{1,n}(D)([E_2(D)]^{-1}f(x))\,dx = \alpha(x_2)-\alpha(-\infty).$$

PROOF. From Theorem 1 and Theorem 3 it follows that the bilateral Laplace transform of $G(t)$ and $G_2(t)$ have a common region of absolute convergence $\alpha_1 < \Re s < \alpha_2$ and hence the product theorem [8] implies that

$$\frac{1}{F(s)}\frac{1}{E_2(s)} = \frac{1}{E_1(s)} = \int_{-\infty}^{\infty} e^{-st}G_1(t)\,dt\,,\ \alpha_1<\sigma<\alpha_2,\ s=\sigma+i\tau\,, \tag{1}$$

$$G_1(x) = \int_{-\infty}^{\infty} G(x-t)\,G_2(t)\,dt \quad -\infty < x < \infty\,,$$

both integrals converging absolutely.

From the equation (1) and the definition of operator $[E_2(D)]^{-1}$ we have

$$\begin{aligned}[E_2(D)]^{-1}f(x) &= \int_{-\infty}^{\infty} G_2(t)\,dt \int_{-\infty}^{\infty} G(x-t-u)\,e^{cu}\,d\alpha(u)\\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} G(x-u-t)\,G_2(t)\,dt\right)e^{cu}\,d\alpha(u) \qquad (2)\\ &= \int_{-\infty}^{\infty} G_1(x-u)\,e^{cu}\,d\alpha(u).\end{aligned}$$

The change of the order of integration here employed is justified by the following argument.

From Theorem 4 for arbitrary finite numbers A, B we have

$$(3)\quad \int_A^B \left(\int_{-\infty}^{\infty} G(x-t-u)\,e^{cu}\,d\alpha(u)\right) G_2(t)\,dt = \int_{-\infty}^{\infty}\left(\int_A^B G(x-t-u)\,G_2(t)\,dt\right) e^{cu}\,d\alpha(u).$$

Given $\varepsilon > 0$, by Theorem 5 we can choose $U_0 > 0$ so that $C < -U_0$, $D > U_0$ imply

$$(4)\qquad \left|\int_{-\infty}^{\infty} G_1(x-u)\,e^{cu}\,d\alpha(u) - \int_C^D G_1(x-u)\,e^{cu}\,d\alpha(u)\right| < \varepsilon/3\,.$$

If we choose T_0 such that $|t| \geqq T_0$ implies $e^{\alpha_1 t}|t|^{\mu_1 - 1} \leqq 1 (t > T_0)$, $e^{\alpha_2 t}|t|^{\mu_2 - 1} \leqq 1 (t < -T_0)$, then by the argument similar to Theorem 4 the integral $\int_{-\infty}^{\infty} G(x-t-u)e^{cu}\,d\alpha(u)$ converges for such $t(|t| \geqslant T_0)$.

Then, by Theorem 1 we can choose $U_1 > 0$ independent of A, B $(|A|, |B| \geqslant T_0)$ so that

$$(5)\quad \left|\int_A^B G_2(t)dt \int_{-\infty}^{\infty} G(x-t-u)e^{cu}\,d\alpha(u) - \int_A^B G_2(t)\,dt \int_C^D G(x-t-u)e^{cu}\,d\alpha(u)\right| \leqq \varepsilon/3\,.$$

Therefore, if we set $U_2 = \max(U_0, U_1)$, then (4), (5) are both hold for $C < -U_2$, $D > U_2$ independently of $A, B(|A|, |B| \geqq T_0)$.

On the other hand, by Theorem 1 and Theorem 3

$$\lim_{\substack{A\to-\infty\\ B\to+\infty}} \int_A^B G(x-t-u)\,G_2(t)\,dt = \int_A^B G(x-t-u)\,G_2(t)\,dt = G_1(x-u)\,,$$

uniformly for $u \in (-\infty, \infty)$. Then, there exists T_1 $(\geqq T_0)$ such that $A < -T_1$, $B > T_1$ implies

$$(6)\quad \left|\int_C^D G_1(x-u)e^{cu}\,d\alpha(u) - \int_C^D\left(\int_A^B G(x-t-u)\,G_2(t)\,dt\right) e^{cu}\,d\alpha(u)\right| < \varepsilon/3\,.$$

Combining (4), (5), (6) we find that $A < -T_1$, $B > T_1$ imply

$$\left|\int_{-\infty}^{\infty} G_1(x-u)e^{cu}\,d\alpha(u) - \int_A^B G_2(t)\,dt \int_{-\infty}^{\infty} G(x-t-u)e^{cu}\,d\alpha(u)\right| < \varepsilon .$$

In other words, (2) holds.

Thus obtained (2) is the convolution transform with class I kernel $G_1(t)$ and appealing the familiar theorem [2, p. 135], we obtain our desired result.

The proof of the following theorem goes exactly as preceding theorem.

COROLLARY. *If*

1. $\varphi(t)$ *is integrable on every finite interval,*
2. $f(x) = \int_{-\infty}^{\infty} G(x-t)\,\varphi(t)\,dt$ *converges,*
3. $\varphi(t)$ *is continuous at* x,

then

$$\lim_{n\to\infty} E_{1,n}(D)([E_2(D)]^{-1} f(x)) = \varphi(x) .$$

5. Applications. (a) J. M. C. Joshi [3] discussed a generalized Stieltjes transform

$$\varphi(s) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \frac{\Gamma(\beta+1)}{s} \int_0^{\infty} (y/s)^{\beta} \times F\Big(\beta+\eta+1,\ \beta+1;\ \alpha+\beta+\eta+1;\ -\frac{y}{s}\Big) f(y)\,dy$$

where $f(y) \in L(0, \infty)$, $\beta \geqq 0$, $\eta > 0$, $0 < s < \infty$.

That is, after some calculation and after an exponential change of variable, this becomes putting $\xi(s) = -e^s\varphi'(e^s)$, $\zeta(s) = f(e^s)$,

$$\xi(s) = \frac{\Gamma(\beta+\eta+1)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+\eta+1)} \int_{-\infty}^{\infty} e^{-(s-y)(\beta+1)} \times F(\beta+\eta+1, \beta+2; \alpha+\beta+\eta+1; -e^{-(s-y)})\,\zeta(y)\,dy .$$

The inversion function of this convolution transform is

$$E(x) = \frac{\Gamma(\beta+\eta+1)\Gamma(\beta+2)}{\Gamma(\alpha+\beta+\eta+1)} \int_{-\infty}^{\infty} e^{-y(\beta+x+1)} F(\beta+\eta+1, \beta+2; \alpha+\beta+\eta+1;\ -e^{-y})dy$$

$$= \frac{\Gamma(\eta-x)\Gamma(\beta+x+1)\Gamma(1-x)}{\Gamma(\alpha+\eta-x)},$$

provided that $b \neq 0, -1, -2, \cdots$, $\Re(1-x) > 0$, $\Re(\eta - x) > 0$ and $\Re(\beta + x + 1) > 0$. From this fact, he obtained the inversion theorem by infinite product expansion of $E(x)$.

In this case, $E(x)$ has zeros at the points $\eta, \eta+1, \eta+2, \cdots; 1, 2, 3, \cdots;$ $-(\beta+1), -(\beta+2), -(\beta+3), \cdots$ and poles at $\alpha+\eta, \alpha+\eta+1, \alpha+\eta+2, \cdots$.

If $\alpha = 0$, then $E(x)$ has only zeros and the kernel function is the one of class I originally. Now, we assume $\alpha > 0$ and we take $\{\eta+k-1\}_1^\infty$ as $\{a_k\}_1^\infty$, $\{\alpha+\eta+k-1\}_1^\infty$ as $\{c_k\}$, the other zeros as $\{d_k\}_1^\infty$.

By Stirling's formula

$$\Gamma(\sigma+i\tau) \sim \sqrt{2\pi}\, e^{-\pi|\tau|/2} |\tau|^{\sigma-1/2} \quad (|\tau| \to \infty),$$

we see that

$$E(\sigma+i\tau) \sim 2\pi\, e^{-\pi|\tau|} |\tau|^{\beta-\alpha-1} \quad (|\tau| \to \infty) \text{ for all } \sigma,$$

so $E(x)$ satisfies the condition (5) of §1.

Hence, our general theorem is applicable to this transform.

In this case, we can interpret the meaning of $[E_2(D)]^{-1}$ as

$$[E_2(D)]^{-1} f(x) = \int_{-\infty}^{\infty} f(x-t) e^{-e^t} e^{(\alpha+\eta)t}\, dt,$$

because

$$\Gamma(\alpha+\eta-s) = \int_{-\infty}^{\infty} e^{-e^t} e^{(\alpha+\eta)t} e^{-st}\, dt \qquad \Re s < \alpha+\eta.$$

(b) D. V. Sumner [4] pointed out that for the inversion formula of integro-differential type the operating order of the integral and the differentiation is essential using the convolution transform

$$f(x) = \int_{-\infty}^{\infty} H(x-t)\, \varphi(t)\, dt \quad (x: \text{complex}),$$

where

$$H(x) = [e^{-2x} + 2e^{-x} \cos \pi\beta + 1]^{-1}, \quad 0 < \beta < 1.$$

After the differentiation this becomes the convolution transform with the kernel

$$H'(x) = \frac{2(e^{-2x} + e^{-x} \cos \pi\beta)}{(e^{-2x} + 2e^{-x} \cos \pi\beta + 1)^2}$$

and the inversion function $-\dfrac{4\cos\pi\beta s}{\sin\pi s}$, so this case is some modified one of ours.

(c) The integral transform

$$F(X)=\pi^{-1/2}\int_0^\infty e^{-XT/2}I_\nu\left(\frac{1}{2}XT\right)\Phi(T)\,dT$$

(ν : real, $>1/2$, I_ν : modified Bessel function of order ν) becomes, after an exponential change of variables, the convolution transform with the kernel $G(t)=\pi^{-1/2}\exp\left\{-\frac{1}{2}e^{-t}\right\}I_\nu(e^{-t}/2)$ and the inversion function $F(s)=\dfrac{\Gamma(1+\nu-s)}{\Gamma(1/2-s)\Gamma(s+\nu)}$.

(d) The integral transform

$$F(X)=\int_0^\infty e^{-XT/2}M_{k,\mu}(XT)(XT)^{k-1/2}\,\Phi(T)\,dT$$

(k,μ : real, $\mu+1/2>k>0$, $M_{k,\mu}$: Whittaker's function) becomes, after an exponential change of variables, the convolution transform with the kernel $G(t)=\exp\left\{-\frac{1}{2}e^{-t}\right\}M_{k,\mu}(e^{-t})$ and the inversion function

$$F(s)=\frac{\Gamma\left(\mu+\frac{1}{2}+k\right)\Gamma\left(\mu+\frac{1}{2}-s\right)}{\Gamma(2\mu+1)\Gamma\left(\mu+\frac{1}{2}+s\right)\Gamma(k-s)}.$$

(e) The modified Meijer transform

$$F(X)=\int_0^\infty e^{-XY/2}\,W_{k+1/2,m}(XY)(XY)^{-k-1/2}\,\Phi(Y)\,dY$$

(k,m : real, $3k-m-1>0$) becomes, after an expontial change of variables, the convolution transform with the kernel $G(t)=\exp\left\{-\frac{1}{2}e^{t}\right\}W_{k+1/2,m}(e^{t})\,e^{-kt}$ and with the inversion function

$$F(s)=\frac{\Gamma\left(-s-2k+\frac{1}{2}\right)}{\Gamma\left(m-s-k+\frac{1}{2}\right)\Gamma\left(-m-s-k+\frac{1}{2}\right)}.$$

These integral transforms (c), (d), (e) are discussed by our general theorems.

6. Representstion theorem. In this section we shall have necessary and sufficient conditions for $f(x)$ to be represented as convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$$

with $\alpha(t)$ in some class, where the kernel $G(t)$ is defined as in §1.

If $f(t)$ is integrable on every finite interval, then from the definition of operator $[E_2(D)]^{-1}$,

$$[E_2(D)]^{-1} f(x) = \int_{-\infty}^{\infty} G_2(x-t) f(t)\, dt,$$

whenever the integral converges.

This transform is a convolution transform of class II (or III) [2], so if we interpret the operator $E_2(D)$ as $\lim_{n\to\infty} \prod_{1}^{n} (1-D/c_k)\, e^{D/c_k}$, then from the familiar theorem [2, p. 131, 132] $E_2(D)[E_2(D)]^{-1} f(x) = f(x)$ at all continuity point x of $f(x)$.

THEOREM 7. *Necessary and sufficient conditions that*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t),$$

where $\alpha(t) \in \uparrow$ *are*:

A. $[E_2(D)]^{-1} f(x) \in C^{\infty}(-\infty, \infty)$;

B. $[E_2(D)]^{-1} f(x) = o(e^{\alpha_2 x}) \qquad x \to +\infty$
$\qquad = o(e^{\alpha_1 x}) \qquad x \to -\infty$;

C. $E_{1,n}(D)([E_2(D)]^{-1} f(x)) \geqq 0 \qquad (-\infty < x < \infty;\ n = 0, 1, 2, \cdots)$.

PROOF. By the well known theorem [2, p. 158] the necessity of A, B, C is obvious, noticing

$$[E_2(D)]^{-1} f(x) = \int_{-\infty}^{\infty} G_1(x-t)\, d\alpha(t), \qquad \alpha(t) \in \uparrow$$

and $G_1(t) \in$ class I.

Conversely, we have to establish their sufficiency. Again, by the same theorem there exists $\alpha(t)$ non-decreasing

$$[E_2(D)]^{-1} f(x) = \int_{-\infty}^{\infty} G_1(x-t)\, d\alpha(t).$$

It is familiar [2, p. 52] that $G_1(x) \in C^{\infty}(-\infty < x < \infty)$ and that for any positive number p and R

$$\frac{1}{|E_1(\sigma + i\tau)|} = O\Big(\frac{1}{|\tau|^p}\Big)$$

uniformly in the strip $|\sigma| \leqq R$.

Hence, for all positive integer n the integral

$$(1) \qquad \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{k=1}^{n} (1-s/c_k)\, e^{s/c_k}}{E_1(s)}\, e^{sx}\, ds$$

coverges uniformly for x.

If D stands for differentiation with respect to x and we define $E_{2,n}(D) = \prod_{k=1}^{n} (1-D/c_k) e^{D/c_k}$, then, using the equation $E_{2,n}(D)\, e^{sx} = e^{sx}\, E_{2,n}(s)$, we have

$$E_{2,n}(D)\, G_1(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{k=1}^{n} (1-s/c_k)\, e^{s/c_k}}{E_1(s)}\, e^{sx}\, ds = K_n(x), \quad \text{say.}$$

Differentiation under the integral sign is justified by the uniform conergence of the integral (1).

By the condition (5) of §1, we have

$$|G(x) - K_n(x)| \leqq \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{|F(i\tau)|}\, d\tau < \infty\,.$$

Hence, by Lebesgue's dominated convergence theorem we obtain

$$\lim_{n\to\infty} K_n(x) = G(x)$$

uniformly for $x \in (-\infty, \infty)$.

As the proof of Theorem 5 it is shown that for all n the transform $\int_{-\infty}^{\infty} K_n(x-t)\, d\alpha(t)$ converges, if and only if the transform $\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$ converges and that the transform converges uniformly for x in any finite interval.

From this fact and Theorem 5 we have

$$f(x) = \lim_{n\to\infty} E_{2,n}(D)([E_2(D)]^{-1} f(x)) = \lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(x-t)\, d\alpha(t)$$

$$(*) \qquad = \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t)$$

as desired.

It was shown in this proof that

$$\lim_{n\to\infty} \|G(t) - K_n(t)\|_\infty = 0\,.$$

It is evident that

$$\int_{-\infty}^{\infty} G(t)\, dt = 1\,, \qquad \int_{\to\infty}^{\infty} K_n(t)\, dt = 1\,,$$

so by the Lebesgue's theorem we obtain

$$\lim_{n\to\infty} \|G(t) - K_n(t)\|_1 = 0\,.$$

Since

$$\|G(t) - K_n(t)\|_p \leqq \|G(t) - K_n(t)\|_1^{1/p} \|G(t) - K_n(t)\|_\infty^{1-1/p} \quad (1 \leqq p \leqq \infty)$$

we have

$$\lim_{n\to\infty} \|G(t) - K_n(t)\|_p = 0 \quad (1 \leqq p \leqq \infty)\,.$$

THEOREM 8. *Necessary and sufficient conditions that*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\, \varphi(t)\, dt$$

with $\|\varphi(t)\|_p \leqq M$, $1 < p \leqq \infty$, *are*:

A. $[E_2(D)]^{-1} f(x) \in C^\infty \ (-\infty < x < \infty)$;
B. $\|E_{1,n}(D)([E_2(D)]^{-1} f(x))\|_p \leqq M \qquad n = 0, 1, 2, \cdots$.

PROOF. This theorem may be established just as in the preceding theorem except the relation (*). However, by the preceding result, we have

$$\left|\int_{-\infty}^{\infty}\{G(x-t)-K_n(x-t)\}\,\varphi(t)\,dt\right| \leqq \|G-K_n\|_q\cdot\|\varphi\|_p \leqq M\|G-K_n\|_q \quad \text{as } n\to\infty .$$

Thus we have our desired resut.

The proof of the following theorem goes exactly as the preceding theorem.

THEOREM 9. *Necessary and sufficient conditions that*

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\,d\alpha(t)$$

with $\alpha(t)$ *of total variation not exceeding* M *are*:

A. $[E_2(D)]^{-1}f(x) \in C^{\infty} \qquad (-\infty < x < \infty)$;
B. $\|E_{1,n}(D)([E_2(D)^{-1}f(x))\|_1 \leqq M \qquad n=0,1,2,\cdots$.

REFERENCES

[1] I. I. HIRSCHMAN, JR. AND D. V. WIDDER, The inversion of a general class of convolution transforms, Trans. Amer. Math. Soc., 66(1949), 135-201.
[2] I. I. HIRSCHMAN, JR. AND D. V. WIDDER, The convolution transform, Princeton (1955).
[3] J. M. C. JOSHI, On a generalized Stieltjes transform, Pacific Journ. Math., 14(1964), 969-976.
[4] D. B. SUMNER, A convolution transform admitting an inversion formula of integro-differential type, Canadian Journ. Math., 5(1953), 114-117.
[5] Y. TANNO, On the convolution transform IV, Sci. Rep. Hirosaki Univ., in press.
[6] D. V. WIDDER, Inversion formulas for convolution transforms, Duke Math. Journ., 14 (1947), 217-251.
[7] D. V. WIDDER, Symbolic inversions of the Fourier sine transform and of related transforms, Journ. Indian Math. Soc., 14(1950), 119-128.
[8] D. V. WIDDER, The Laplace transform, Princeton (1946).

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