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THE DE RHAM THEOREM FOR GENERAL SPACES*

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It is known that the differential forms on a differentiable manifold X may be defined as a species of singular real-valued cochains[1] on X. Now let X be an arbitrary topological space and $\mathfrak{F}$ a set of continuous real-valued functions on X. As will be seen in the sequel, one can again single out a species of real singular cochains on X by letting $\mathfrak{F}$ play the role of a differentiable structure[2], and obtain thus a graded differential exterior algebra G associated with the pair $(X, \mathfrak{F})$. Moreover, such pairs can be regarded as objects of a local category[3] $\mathfrak{D}$, in which case G becomes a contravariant functor on $\mathfrak{D}$ with values in the category $\mathfrak{A}$ of graded differential algebras. By a sheaf-theoretic process, G generates a functor F from $\mathfrak{D}$ to $\mathfrak{A}$ of the kind previously referred to as a sheaf[4] on $\mathfrak{D}$. This sheaf F constitutes an extension of the classical differential forms (regarded as a functor on the local category of differentiable manifolds). In the present paper we shall be concerned with the question under what conditions the cohomology of the complex $F(X)$ reduces to the real sheaf cohomology[5] of the underlying space X. It will be seen that this holds for objects X lying in a certain subcategory $\mathfrak{C}$ of $\mathfrak{D}$, which however is considerably larger than the category of differentiable manifolds. One has obtained in this way a generalized version of the de Rham Theorem.

Nonclassical objects in $\mathfrak{D}$ arise in various ways, e.g., as quotients of a differentiable manifold M. More precisely, every quotient space X of M carries a natural differentiable structure $\mathfrak{F}$ (in the sense referred to above).

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1) From this point of view, the theory of differential forms was extended to Lipschitzian manifolds by Whitney (see [7]).

2) Strictly speaking, we shall find it convenient to deal only with sets $\mathfrak{F}$ satisfying an appropriate closure condition.

3) For basic definitions regarding local categories we refer to Eilenberg [2].

4) See Clifton and Smith [1], p. 446.

5) This cohomology is defined in terms of the *canonical resolution* of the *simple sheaf with fibre* R (the group of real numbers). See Godement [3], p. 173. We shall not be concerned with general *families of support* Φ, but will always suppose Φ to be the family containing X itself.

In the last Section we shall examine the general de Rham Theorem with reference to spaces $\boldsymbol{X}=(X, \mathfrak{F})$ arising in this manner. It will be shown that $F(\boldsymbol{X})$ can now be identified with a subclass A of the (classical) differential forms $F(M)$. When X belongs to $\mathfrak{C}$ one may thus conclude that the cohomology of the complex A gives the real cohomology of the quotient space X. In the special case where X is the orbit space determined by the action of a Lie group, this result is closely related (but not equivalent) to the theorem of J. L. Koszul[6] regarding the cohomology of basic forms on a regular G-variety.

1. The category $\mathfrak{D}$.

Differentiable spaces. We let R denote the set of real numbers and $\mathfrak{R}$ the set of all open subsets $\Omega\subset R^n$, n ranging over the positive integers. Let a topological space X and a set $\mathfrak{F}$ of continuous real functions on X be given. For every $\Omega\in\mathfrak{R}$, $\mathfrak{F}(\Omega)$ shall denote the set of all (continuous) maps $g:\Omega\rightarrow X$ such that $f\circ g\in C^\infty(\Omega)$ for all $f\in\mathfrak{F}$, where $\circ$ indicates composition and $C^\infty(\Omega)$ the class of real C^∞-functions on Ω. The set $\mathfrak{F}$ will be called a *differentiable structure* on X provided it satisfies the following closure condition: Given a continuous function $f:X\rightarrow R$ such that $f\circ g\in C^\infty(\Omega)$ for all $\Omega\in\mathfrak{R}$ and $g\in\mathfrak{F}(\Omega)$, then $f\in\mathfrak{F}$. The term *differentiable space* will henceforth be used to denote pairs $(X, \mathfrak{F})$, $\mathfrak{F}$ being a differentiable structure on X.

This definition naturally leads to a number of simple observations, the first being

PROPOSITION 1.1. *Given a topological space X and arbitrary set $\mathfrak{F}$ of continuous real-valued functions on X, there exists a unique differentiable structure $\mathfrak{F}^*$ on X such that $\mathfrak{F}^*(\Omega)=\mathfrak{F}(\Omega)$*[7].

To prove existence one takes $\mathfrak{F}^*$ to be the set of all maps $f:X\rightarrow R$ such that $f\circ g\in C^\infty(\Omega)$ for all $\Omega\in\mathfrak{R}$ and $g\in\mathfrak{F}(\Omega)$. We note that $\mathfrak{F}(\Omega)\subset\mathfrak{F}^*(\Omega)$ for all $\Omega\in\mathfrak{R}$. Consequently, if $f:X\rightarrow R$ is a map such that $f\circ g\in C^\infty(\Omega)$ for all $\Omega\in\mathfrak{R}$ and $g\in\mathfrak{F}^*(\Omega)$, then $f\in\mathfrak{F}^*$. Hence $\mathfrak{F}^*$ is a differentiable structure on X. Now suppose $\Omega\in\mathfrak{R}$ and $g\in\mathfrak{F}(\Omega)$. Then by definition of $\mathfrak{F}^*$, $g\in\mathfrak{F}^*(\Omega)$, proving that $\mathfrak{F}^*(\Omega)=\mathfrak{F}(\Omega)$. One verifies immediately that $\mathfrak{F}^*$ is unique.

We shall say that the differentiable structure $\mathfrak{F}^*$ is *generated* by $\mathfrak{F}$. We also note that $\mathfrak{F}^*$ is closed under addition and multiplication of functions, and that it contains the constant functions. Thus $\mathfrak{F}^*$ constitutes a ring with unit element.

6) See Koszul [4].
7) This clearly implies $\mathfrak{F}\subset\mathfrak{F}^*$.

PROPOSITION 1.2. *Let X be a C^∞-manifold and $\mathfrak{F}$ the set of all real C^∞-functions on X. Then $\mathfrak{F}$ is a differentiable structure* (in our sense).

For suppose $f: X \to R$ is given such that $f \circ g \in C^\infty(\Omega)$ for all $\Omega \in \mathfrak{R}$ and $g \in \mathfrak{F}(\Omega)$. Given $x \in X$, let $\sigma: U \to \Omega$ be a chart defined on a neighborhood U of x. Then $\sigma^{-1} \in \mathfrak{F}(\Omega)$, and consequently the restriction $f|U$ is of class C^∞. We may conclude that f is a C^∞-function, proving that $\mathfrak{F}$ satisfies the required closure axiom.

General examples. Differentiable spaces come up in several different contexts, three of which will now be briefly considered.

(i) Let M be a differentiable[8] manifold, X a topological space and $\phi: X \to M$ a continuous map. Given $\Omega \in \mathfrak{R}$, let $[\Omega]$ denote the set of all maps $g: \Omega \to X$ such that $\phi \circ g$ is a C^∞-map. Let $\mathfrak{F}$ denote the set of all continuous functions $f: X \to R$ such that $f \circ g \in C^\infty(\Omega)$ for all $\Omega \in \mathfrak{R}$ and $g \in [\Omega]$. We assert now that $\mathfrak{F}(\Omega) = [\Omega]$ for all $\Omega \in \mathfrak{R}$, which would imply that $\mathfrak{F}$ is a differentiable structure on X.

To prove this assertion, we observe in the first place that if $g \in [\Omega]$, then $f \circ g \in C^\infty(\Omega)$ for all $f \in \mathfrak{F}$, which implies that $[\Omega] \subset \mathfrak{F}(\Omega)$. Conversely, let $g \in \mathfrak{F}(\Omega)$ be given. For every C^∞-function $h: M \to R$, $h \circ \phi \in \mathfrak{F}$, and consequently $h \circ \phi \circ g \in C^\infty(\Omega)$. But this clearly implies that $\phi \circ g$ is a C^∞-map, and therefore $g \in [\Omega]$.

It is easy to verify that in the special case where X is a submanifold of M (ϕ being the inclusion map), $\mathfrak{F}$ reduces to the class of C^∞-functions on X.

(ii) Next we consider a differentiable manifold M, a topological space X and a continuous map $\pi: M \to X$. Let $\mathfrak{F}$ denote the set of all continuous maps $f: X \to R$ such that $f \circ \pi$ is a C^∞-function. Then $\mathfrak{F}$ is a differentiable structure on X.

To show this, one first observes that if $\Omega \in \mathfrak{R}$ and $h: \Omega \to M$ is a C^∞-map, then $\pi \circ h \in \mathfrak{F}(\Omega)$. Given a function $f: X \to R$ such that $f \circ g \in C^\infty(\Omega)$ for all $\Omega \in \mathfrak{R}$ and $g \in \mathfrak{F}(\Omega)$, it follows now that $f \circ \pi \circ h \in C^\infty(\Omega)$ for all C^∞-maps $h: \Omega \to M$. But this implies that $f \circ \pi$ is a C^∞-map, and consequently that $f \in \mathfrak{F}$.

Let us suppose next that X is itself a differentiable manifold, and that π is an open, differentiable surjection. We now assert that $\mathfrak{F}$ is precisely the class of C^∞-functions on X.

It is obvious, in the first place, that every C^∞-function on X belongs to $\mathfrak{F}$. Conversely, let $f \in \mathfrak{F}$ be given, and let $x \in X$. Since π is surjective, there exists $y \in M$ such that $\pi(y) = x$, and since π is open and differentiable, it must have maximal rank at y. It follows now by the implicit function theorem that there exists a neighborhood U of x and a differentiable function $\rho: U \to M$ such that $\pi \circ \rho$ gives the identity map of U. Consequently the restriction $f|U$

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8) The term differentiable will always be used in the sense of C^∞.

is precisely $f \circ \pi \circ \rho$. But $f \circ \pi$ being differentiable, it follows that f is differentiable at x, which proves our assertion.

(iii) Lastly, let Y be a topological space and X the space of continuous real-valued functions on Y, endowed with the compact-open topology. Every $y \in Y$ determines a function $\psi_y : X \to R$ defined by

$$\psi_y(x) = x(y), \qquad x \in X;$$

and moreover, ψ_y is continuous. For every $\Omega \in \mathfrak{R}$, we denote by (Ω) the set of all continuous maps $g : \Omega \to X$ such that $\psi_y \circ g \in C^\infty(\Omega)$ for all $y \in Y$. Now let $\mathfrak{F}$ denote the set of all continuous functions $f : X \to R$ such that $f \circ g \in C^\infty(\Omega)$ for all $\Omega \in \mathfrak{R}$ and $g \in (\Omega)$. It is again a simple matter to verify that $\mathfrak{F}$ is a differentiable structure on X.

Differentiable maps. Let $\boldsymbol{X} = (X, \mathfrak{F})$ and $\boldsymbol{X}' = (X', \mathfrak{F}')$ be differentiable spaces. By a *map* $h : \boldsymbol{X} \to \boldsymbol{X}'$ we shall understand a continuous map $h : X \to X'$ such that $f' \circ h \in \mathfrak{F}$ whenever $f' \in \mathfrak{F}'$.

Two observations should be made in regard to this definition:

PROPOSITION 1.3. *Let* $\boldsymbol{X} = (X, \mathfrak{F})$ *be a differentiable space and let* $\boldsymbol{R}$ *denote the differentiable space of real numbers* (see Proposition 1.2). *A function* $f : X \to R$ *is then a map* $f : \boldsymbol{X} \to \boldsymbol{R}$ *if and only if* $f \in \mathfrak{F}$.

PROPOSITION 1.4. *For differentiable spaces which are* C^∞*-manifolds, the notion of map* (as defined above) *reduces to the ordinary notion of a* C^∞*-map.*

These facts are easily ascertained, and we will omit the proofs.

It is clear that the totality of differentiable spaces and their maps give rise to a category under ordinary composition of functions. We denote this category by $\mathfrak{D}$. One sees (in virtue of Proposition 1.2 and 1.4) that the category $\mathfrak{C}^\infty$ of differentiable manifolds and differentiable maps constitutes a full subcategory of $\mathfrak{D}$.

We now define a functor F^0 on $\mathfrak{D}$ as follows: Given $\boldsymbol{X} = (X, \mathfrak{F})$ in $\mathfrak{D}$, we take $F^0(\boldsymbol{X}) = \mathfrak{F}$. Given a map $h : \boldsymbol{X} \to \boldsymbol{X}'$ in $\mathfrak{D}$, $F^0(h)$ shall be the induced map $h^* : F^0(\boldsymbol{X}') \to F^0(\boldsymbol{X})$. This gives a contravariant functor with values in the category $\mathfrak{R}$ of rings with unit element. In order to describe certain good properties of F^0, it will now be convenient to avail ourselves of terminology pertaining to local categories. Let C denote the category of topological spaces and continuous maps, and let L denote the (covariant) functor from $\mathfrak{D}$ to C which to every differentiable space asisgns its underlying topological space, and to every map in $\mathfrak{D}$ the corresponding map in C. If $\boldsymbol{X}$ is an object in $\mathfrak{D}$ and U an open subset of $L(X)$, we let $\mathfrak{F}_U$ denote the differentiable structure on U generated by the set of all restrictions $f|U$ as f ranges over $F^o(X)$. We define $X|U$ to be the differentiable object $(U, \mathfrak{F}_U)$, and $i_X|U$ to be the inclusion map from $\boldsymbol{X}|U$ to $\boldsymbol{X}$. It is trivial to verify that this structure defines a local category (which we shall likewise denote by $\mathfrak{D}$). Given an object X in C, we shall denote by $c(X)$ the category of open subsets of X and all inclusion maps. With every object $\boldsymbol{X}$ in $\mathfrak{D}$ one can now associate the covariant functor $T_X : c(L(X)) \to \mathfrak{D}$ defined by

$$T_X(U) = \boldsymbol{X}|U\,, \qquad U \subset L(\boldsymbol{X});$$

$$T_X(i_U|V) = i_{X|U}|V\,, \qquad V \subset U \subset L(\boldsymbol{X})\,.$$

It follows readily that for every $\boldsymbol{X}$ in $\mathfrak{D}$, $F^o \circ T_X$ is a sheaf[9] on $L(\boldsymbol{X})$ with values in $\mathfrak{R}$. In accordance with the terminology of [1], F^o is thus a $\mathfrak{R}$-valued sheaf on the local category $\mathfrak{D}$.

2. Differential forms on $\mathfrak{D}$.

The functors G^p. We now consider a differentiable space $\boldsymbol{X}$, and our first task shall be to single out a preferred class of singular cochains on the underlying topological space X. By a *p-dimensional cube* ($p > 0$) we shall understand a closed subset of R^p, bounded by $2p$ axis-parallel hyperplanes. Let J^p be such a cube. A map $\sigma : J^p \to X$ will be called a *singular p-cube* in X provided there exists an $\Omega \in \mathfrak{N}$ and a map $f : \Omega \to X$ (f belonging to $\mathfrak{D}$) such that $\sigma = f|J^p$ [10]. Let $K_p(\boldsymbol{X})$ denote the set of all singular p-cubes in $\boldsymbol{X}$, and let $C^p(\boldsymbol{X})$ denote the set of all functions $\alpha : K_p(\boldsymbol{X}) \to R$. Clearly $C^p(\boldsymbol{X})$ is a vector space over R. For every $p > 0$ we now define a function $\lambda_p : \mathfrak{F}^{p+1} \to C^p(\boldsymbol{X})$, where $\mathfrak{F}$ denotes the given differentiable structure[11] on X. Thus, given $p+1$ functions $f_0, \cdots, f_p \in \mathfrak{F}$ and a singular p-cube $\sigma : J^p \to X$ in $K_p(\boldsymbol{X})$, we will define an inner product of σ by $\lambda_p(f_0, \cdots, f_p)$. This is done as follows: Let $\Omega \in \mathfrak{N}$ and $f : \Omega \to \boldsymbol{X}$ be a map in $\mathfrak{D}$ such that $\sigma = f|J^p$, and let $g_i = f_i \circ f$, $0 \leqq i \leqq p$. Each g_i is simply a real-valued differentiable function of p real variables, which we will denote by $(t_1, \cdots, t_p)$. The value of $\lambda_p(f_0, \cdots, f_p)$ on σ is now defined by

$$\text{(2.1)} \qquad \lambda_p(f_0, \cdots, f_p)(\sigma) = \int \underset{J^p}{\cdots} \int g_0 \frac{\partial(g_1, \cdots, g_p)}{\partial(t_1, \cdots, t_p)} dt_1 \cdots dt_p\,.$$

9) See Godement [3], p. 109.
10) Strictly speaking, we should say $\sigma = L(f)\,|J^p$.
11) In other words, $\mathfrak{F} = F^o(\boldsymbol{X})$.

Since the right side depends only on σ (and not on its extension f), $\lambda_p(f_0, \cdots, f_p)$ is well-defined as an element of $C^p(\boldsymbol{X})$. We now let $G^p(\boldsymbol{X})$ denote the linear subspace of $C^p(\boldsymbol{X})$ generated by the image of λ_p. For $p=0$ we define $G^0(X)$ to be precisely $\mathfrak{F}$ and take λ_0 to be the identity map of $\mathfrak{F}$.

It is clear that a map $f: \boldsymbol{X} \to \boldsymbol{X}'$ in $\mathfrak{D}$ induces (linear) maps $f^*: C^p(\boldsymbol{X}') \to C^p(\boldsymbol{X})$. Given $f_0', \cdots, f_p' \in G^0(X')$, let $f_i = f_i' \circ f$. Then

$$f^*[\lambda_p'(f_0', \cdots, f_p')] = \lambda_p(f_0, \cdots, f_p), \tag{2.2}$$

as may be verified by letting both sides operate on an element $\sigma \in K_p(\boldsymbol{X})$. Consequently the spaces $G^p(\boldsymbol{X})$ are seen to be *functorial* in the sense that they derive from a (contravariant) functor G^p on $\mathfrak{D}$.

We now observe that a space $\Omega \in \mathfrak{R}$ is certainly a C^∞-manifold, and consequently belongs to $\mathfrak{D}$. Let $F^p(\Omega)$ denote the space of exterior differential p-forms on Ω. It is useful at this point to make the following observation:

PROPOSITION 2.1. *There exists a canonical isomorphism*[12] $\phi: G^p(\Omega) \to F^p(\Omega)$, *such that*

$$\phi[\lambda_p(g_0, \cdots, g_p)] = g_0 \, dg_1 \wedge \cdots \wedge dg_p, \quad g_i \in C^\infty(\Omega). \tag{2.3}$$

To show that a linear map ϕ satisfying Equation (2.3) exists, we must verify that given

$$\alpha = \sum_i \lambda_p(g_0^i, \cdots, g_p^i), \quad g_j^i \in C^\infty(\Omega); \tag{2.4}$$

the differential form

$$\omega = \sum_i g_0^i \, dg_1^i \wedge \cdots \wedge dg_p^i \tag{2.5}$$

is uniquely determined by α, i.e., is independent of the representation (2.4). To this end we observe that

$$\int_\sigma \omega = \alpha(\sigma) \qquad \text{for all } \sigma \in K_p(\Omega).$$

But a differential form is uniquely determined by its action as an integral, and therefore ϕ exists as a linear map. The same observation proves that ϕ is injective. Moreover, since every $\omega \in F^p(\Omega)$ may be expressed in the form

12) *qua* linear spaces.

(2.5), it follows that ϕ is also surjective, and thus Proposition 2.1 is established.

Exterior product. We assert the existence of an exterior product:

PROPOSITION 2.2. *There exists a unique bilinear map*

$$\wedge : G^p(X)\times G^q(X)\to G^{p+q}(X),$$

defined for X *in* $\mathfrak{D}$ *and* $p, q > 0$, *such that*

$$\wedge(\alpha, \beta) = \lambda_{p+q}(f_0\bar{f}_0, f_1, \cdots, f_p, \bar{f}_1, \cdots, \bar{f}_q) \tag{2.6}$$

when

$$\alpha = \lambda_p(f_0, \cdots, f_p)$$

$$\beta = \lambda_q(\bar{f}_0, \cdots, \bar{f}_q), \quad f_i, \bar{f}_j \in G^0(X).$$

It is of course obvious that $\wedge$ is uniquely determined by the condition of bilinearity together with (2.6). To prove existence, one must verify that given

$$\alpha = \sum_i \lambda_p(f_0^i, \cdots, f_p^i)$$

$$\beta = \sum_j \lambda_q(\bar{f}_0^j, \cdots, \bar{f}_q^j), \quad f_k^i, \bar{f}_e^j \in G^0(X);$$

the cochain

$$\wedge(\alpha, \beta) = \sum_{i,j} \lambda_{p+q}(f_0^i\bar{f}_0^j, f_1^i, \cdots, f_p^i, \bar{f}_1^j, \cdots, \bar{f}_q^j)$$

is uniquely determined by α and β. To see this, consider an element $\sigma \in K_{p+q}(X)$ and let $f:\Omega\to X$ be an extension of σ. In virtue of Proposition 2.1, we may regard $f^*(\alpha)$ and $f^*(\beta)$ as ordinary differential forms on Ω. It follows now by a simple direct calculation that

$$\wedge(\alpha, \beta)(\sigma) = \int_\sigma f^*(\alpha)\wedge f^*(\beta). \tag{2.8}$$

Since the right side of Equation (2.8) depends only on α, β and σ, it follows that $\wedge$ is well-defined as a bilinear map.

It may be of interest from an expositional viewpoint to note that Proposition 2.2 can also be established by a direct argument, without referring the matter back to the classical theory of differential forms. The requisite calculations, however, are not entirely trivial.

We also observe that the $\wedge$-product, as defined by Proposition 2.2, is clearly associative and anticommutative in the sense that

$$\wedge(\alpha, \beta) = (-1)^{pq} \wedge (\beta, \alpha).$$

The usual notation $\wedge(\alpha, \beta)=\alpha\wedge\beta$ will henceforth be employed.

Exterior derivative. The existence of exterior derivatives may be established by the same approach.

PROPOSITION 2.3. *There exists a unique linear map*

$$d : G^p(\boldsymbol{X}) \to G^{p+1}(\boldsymbol{X}),$$

defined for $\boldsymbol{X}$ in $\mathfrak{D}$ and $p \geqq 0$, such that[13]

$$d\lambda_p(f_0, \cdots, f_p) = \lambda_{p+1}(\mathbf{1}, f_0, \cdots, f_p), \quad f_i \in G^0(X). \tag{2.9}$$

The proof is entirely analogous to the preceding argument and will be omitted. One can verify without difficulty that d is precisely the coboundary operator (Stokes theorem), and moreover one recovers the usual formula

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p \alpha\wedge d\beta,$$

where p denotes the degree (dimension) of α.

The sheaf F. For $\boldsymbol{X}$ in $\mathfrak{D}$ let

$$G(\boldsymbol{X}) = \sum_{p \geqq 0} G^p(\boldsymbol{X}),$$

the right side being understood as a direct sum. The $\wedge$-product and d-operator make $G(\boldsymbol{X})$ into a differential exterior algebra. If $f : \boldsymbol{X} \to \boldsymbol{X}'$ is a map in $\mathfrak{D}$, it follows by Equation (2.2) that the induced map $f^* : G(\boldsymbol{X}') \to G(\boldsymbol{X})$ constitutes a homomorphism of the differential exterior algebras. One obtains therefore a (contravariant) functor G on $\mathfrak{D}$ with values in the category $\mathfrak{A}$ of graded differential algebras. By the usual process[14] of "passing to the sheaf", G

13) Here $\mathbf{1}$ denotes the constant function with value 1.
14) See Clifton and Smith [1], p. 447, as well as Godement [3], p. 110.

generates an $\mathfrak{A}$-valued sheaf F. It follows by Proposition 2.1 that the restriction of F to $\mathfrak{C}^\infty$ may be identified with the classical sheaf of differential forms. We also note that on dimension 0, F reduces to the functor F^0 defined in Section 1.

The chain rule. In the preceding paragraphs the notion of *differential forms* was extended to arbitrary differentiable spaces. It is important to note that the computational aspects of the classical theory, in short, the calculus of differential forms, carries over to this more general setting with practically no modification. It is true that the notion of local coordinate systems has disappeared completely, so that we are no longer dealing with skew-symmetric tensors. But we are dealing with the elements of a differential exterior algebra, and this is the essential fact.

To develop the Cartan calculus on $\mathfrak{D}$, it will be convenient to drop the rather cumbersome λ_p-notation (which was used simply to clarify the basic definitions) by setting $\lambda_1(1,f) = df$. Since $\lambda_p(f_0, \cdots, f_p) = \lambda_0(f_0) \wedge \lambda_1(1,f_1) \wedge \cdots \wedge \lambda_1(1,f_p)$, one may now write[15]

$$\lambda_p(f_0, \cdots, f_p) = f_0 \quad df_1 \wedge \cdots \wedge df_p .$$

Given X in $\mathfrak{D}$, let $g_1, \cdots, g_n$ be functions in $G^0(X)$ and $\phi: R^n \to R$ a C^∞-map. A function $f: L(X) \to R$ may now be defined by setting

$$f(x) = \phi(g_1(x), \cdots, g_n(x)), \quad x \in L(X).$$

We claim that f belongs to $G^0(X)$, and that

$$df = \sum_{i=1}^{n} \frac{\partial \phi}{\partial g_i} dg_i . \tag{2.10}$$

To show that $f \in G^0(X)$, we set $\mathfrak{F} = G^0(X)$ as in Section 1 and consider $\Omega \in \mathfrak{R}$ and $h: \Omega \to L(X)$ in $\mathfrak{F}(\Omega)$. Then $g_i \circ h \in C^\infty(\Omega)$ and therefore $f \circ h \in C^\infty(\Omega)$. Since this holds for arbitrary Ω and h, one may conclude by the closure condition for differentiable structures that $f \in \mathfrak{F}$. Equation (2.10) follows now from the ordinary chain rule of calculus by letting both sides operate on a singular 1-cube σ in X.

Product spaces. In Section 1 we confined ourselves to consider only the most basic aspects of the category $\mathfrak{D}$. Among other matters which may be of interest, we neglected to examine how two differentiable spaces X' and X'' give rise to a product $X' \times X''$ in $\mathfrak{D}$. Since this idea will now become

15) We recall that $\lambda_0(f_0) = f_0$. Moreover, the $\wedge$-symbol following an element of dimension 0 may be suppressed without causing ambiguity.

relevant, we interpose the following consideration: Let $\mathfrak{F}'$ and $\mathfrak{F}''$ denote the differentiable structures of $\boldsymbol{X}'$ and $\boldsymbol{X}''$, respectively, and set $X=L(\boldsymbol{X}')\times L(\boldsymbol{X}'')$. Let $\mathfrak{F}$ denote the set of all continuous functions $f: X\to R$ such that $f(g', g'')\in C^\infty(\Omega)$ for all $\Omega\in\mathfrak{R}$, $g'\in\mathfrak{F}'(\Omega)$ and $g''\in\mathfrak{F}''(\Omega)$. We claim that $\mathfrak{F}$ is a differentiable structure on X.

For suppose $f: X\to R$ is a continuous function such that $f\circ h\in C^\infty(\Omega)$ for all $\Omega\in\mathfrak{R}$ and $h\in\mathfrak{F}(\Omega)$. Now consider $\Omega\in\mathfrak{R}$ and $h=(h', h'')$, with $h'\in\mathfrak{F}'(\Omega)$ and $h''\in\mathfrak{F}''(\Omega)$. Then for every $g\in\mathfrak{F}$, $g\circ h\in C^\infty(\Omega)$. It follows that $h\in\mathfrak{F}(\Omega)$, and therefore $f\circ h\in C^\infty(\Omega)$. By the definition of $\mathfrak{F}$, this implies that $f\in\mathfrak{F}$.

We define the product $\boldsymbol{X}'\times\boldsymbol{X}''$ to be the differentiable space $(X,\mathfrak{F})$. Let us suppose that $g_1',\cdots,g_n'\in\mathfrak{F}'$ and $g_1'',\cdots,g_m''\in\mathfrak{F}''$ are given, and that $\phi: R^{n+m}\to R$ is a C^∞-function. We let π' and π'' denote the natural projections of $X'\times X''$ onto X' and X'', respectively, and note that this gives $\mathfrak{D}$-maps of the corresponding objects. Let $f_i'=g_i'\circ\pi'$ and $f_j''=g_j''\circ\pi''$ for all relevant i and j. It is immediately verified that the function $f: X\to R$ given by

$$f=\phi(f_1',\cdots,f_n', f_1'',\cdots,f_m'') \tag{2.11}$$

belongs to $F^0(\boldsymbol{X}'\times\boldsymbol{X}'')$[16]. We let $\overline{F^0}$ denote the class of all $f\in F^0(\boldsymbol{X}'\times\boldsymbol{X}'')$ which admit a representation of the form (2.11). A map $h:\boldsymbol{X}'\times\boldsymbol{X}''\to\boldsymbol{Y}$ will be called *proper* if $g\circ h\in\overline{F^0}$ whenever $g\in F^0(\boldsymbol{Y})$. Clearly π' and π'' are proper. Moreover, when $\boldsymbol{X}'$ and $\boldsymbol{X}''$ lie in $\mathfrak{C}^\infty$, h is always proper, and thus this distinction does not arise in the classical context. It will be needed, however, to define a useful notion of homotopy on $\mathfrak{D}$.

$\overline{F^0}$, being a linear subspace of $G^0(\boldsymbol{X}'\times\boldsymbol{X}'')$, generates a differential subalgebra in $G(\boldsymbol{X}'\times\boldsymbol{X}'')$, which we denote by $\overline{G}(\boldsymbol{X}'\times\boldsymbol{X}'')$[17]. It follows that every proper map $h:\boldsymbol{X}'\times\boldsymbol{X}''\to\boldsymbol{Y}$ in $\mathfrak{D}$ induces a differential algebra homomorphism $h^*: G(\boldsymbol{Y})\to\overline{G}(\boldsymbol{X}'\times\boldsymbol{X}'')$.

Converse of the Poincaré lemma. We will consider the unit interval $I\subset R$ as an object in $\mathfrak{D}$ by way of general example (i), Section 1[18]. An object $\boldsymbol{X}$ in $\mathfrak{D}$ shall be called *differentiably contractible* (d.c.) if there exists a proper map $h: I\times\boldsymbol{X}\to\boldsymbol{X}$ such that $h(1,x)=x$ and $h(0,x)=x_0$ for all $x\in L(\boldsymbol{X})$, x_0 being a point in $L(\boldsymbol{X})$. The map h itself may be referred to as a *contraction* of X.

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16) To simplify the notation, we will henceforth avoid notational distinction between f_i' and g_i', f_j'' and g_j''. In other words, it will be understood that a function defined on one of the factor spaces may also be regarded as a function defined on the product.

17) As usual, a superscript p will denote the vector space of homogeneous elements of degree p.

18) The inclusion map $i: I\to R$ plays now the role of $\phi: X\to M$.

PROPOSITION 2.4. *Let* X *be d.c. and* $\alpha \in G^{p+1}(\boldsymbol{X})$ *such that* $d\alpha=0$. *Then* $\alpha=d\beta$ *for some* $\beta \in G^{p}(\boldsymbol{X})$.

On the present level of generality it would not be feasible to prove this result by a formal calculation involving the differential forms. Our proof will still proceed along classical lines, but now some care must be taken to handle the topological aspects of the problem by an essentially topological argument. This will entail that we introduce the singular chain groups $C_p(\boldsymbol{X})$, the boundary operators $\partial: C_{p+1}(\boldsymbol{X}) \to C_p(\boldsymbol{X})$ and cone operators $k: C_p(\boldsymbol{X}) \to C_{p+1}(I\times \boldsymbol{X})$.

For all $p \geqq 0$ $C_p(\boldsymbol{X})$ may be defined as the free abelian group generated by $K_p(\boldsymbol{X})$, where now $K_0(X)$ is understood to be precisely X, the underlying space of $\boldsymbol{X}$. Let $\sigma: J^{p+1} \to X$ be an element of $K_{p+1}(\boldsymbol{X})$. The region $J^{p+1} \subset R^{p+1}$ is defined by inequalities

$$A_i^- \leqq t_i \leqq A_i^+, \quad 1 \leqq i \leqq p+1;$$

where $t_1, \cdots, t_{p+1}$ denote as before the canonical coordinates on R^{p+1}. For an arbitrary value of the index i, let J_i^{p+1} denote the p-cube given by

$$\begin{aligned} A_j^- \leqq t_j \leqq A_j^+, &\qquad 1 \leqq j < i; \\ A_{j+1}^- \leqq t_j \leqq A_{j+1}^+, &\qquad i \leqq j \leqq p; \end{aligned}$$

and let the maps $\phi_i^\pm : J^{p+1} \to J^{p+1}$ be defined by

$$\phi_i^\pm(t_1, \cdots, t_p) = (t_1, \cdots, t_{i-1}, A_i^\pm, t_{i+1}, \cdots, t_p).$$

We define the boundary operator $\partial: C_{p+1}(\boldsymbol{X}) \to C_p(\boldsymbol{X})$ by the formula[19]

$$\partial\sigma = \sum_{i=1}^{p+1} (-1)^{i+1} \{\sigma\circ\phi_i^+ - \sigma\circ\phi_i^-\}. \tag{2.12}$$

When $p=0$, $\sigma\circ\phi_1^\pm$ must be interpreted as the endpoints of σ in the ordinary sense.

Let $\sigma: J^p \to X$ be an element of $K_p(\boldsymbol{X})$ for $p>0$. The product $I\times J^p$ may be regarded as a $(p+1)$-cube and one can define a singular $(p+1)$-cube $k\sigma: I\times J^p \to I\times X$ in $K_{p+1}(I\times \boldsymbol{X})$ by the formula

$$k\sigma(t_1, \cdots, t_{p+1}) = (t_1, \sigma(t_2, \cdots, t_{p+1})),$$

19) Equation (2.12) defines the action of ∂ on the generators. Thus ∂ is well-defined as a homomorphism.

(where t_1 varies over I and $(t_2, \cdots, t_{p+1})$ over J^p). This defines a homomorphism $k: C_p(\boldsymbol{X}) \to C_{p+1}(I \times \boldsymbol{X})$ for $p > 0$. On dimension 0, k is of course defined by the corresponding formula:

$$k\sigma(t_1) = (t_1, \sigma), \quad \sigma \in L(\boldsymbol{X}).$$

Lastly, we define two maps $u_0, u_1 : X \to I \times X$ by setting $u_i(x) = (i, x)$ for all $x \in X$. Our definition of the differentiable structure for product spaces guarantees that u_0 and u_1 are actually maps from $\boldsymbol{X}$ to $I \times \boldsymbol{X}$. One may now verify by a simple direct calculation that the classical formula[20]

$$k\partial + \partial k = u_{1*} - u_{0*} \tag{2.13}$$

holds on all dimensions $p > 0$.

Up to this point only some rudimentary notions of singular homology on $\mathfrak{D}$ have been involved. The next step will be to establish the following

LEMMA. *The dual operators $k^* : C^{p+1}(I \times \boldsymbol{X}) \to C^p(\boldsymbol{X})$ map $\overline{G}^{p+1}(I \times \boldsymbol{X})$ to $G^p(\boldsymbol{X})$.*

It should be noted, in the first place, that every $f \in \overline{G}^0(I \times \boldsymbol{X})$ admits a representation of the form

$$f = \phi(t_1, f_1, \cdots, f_n),$$

where t_1 denotes the canonical variable on I and $f_i \in G^0(\boldsymbol{X})$ (see footnote 15). One may therefore conclude by the chain rule (previously established) that every element of $\overline{G}^{p+1}(I \times \boldsymbol{X})$ can be represented as a sum of terms of the form

$$\omega = \begin{cases} \text{(i)} \quad a \; dt \wedge df_1 \wedge \cdots \wedge df_p, & \text{or} \\ \text{(ii)} \quad a \; df_1 \wedge \cdots \wedge df_{p+1}; \end{cases}$$

where $a \in \overline{G}^0(I \times \boldsymbol{X})$ and $f_j \in G^0(\boldsymbol{X})$. To examine the action of k^* on ω, we consider a singular p-cube $\sigma : J^p \to X$, let $g : \Omega \to X$ be an extension of σ and set $b = a \circ g$, $g_j = f_j \circ g$. One now sees, in the first place, that $k^*\omega = 0$ when ω is given by (ii). This is due to the fact that each g_j is independent of t_1, and therefore the resultant jacobian under the integral must vanish. For case (i), on the other hand, one obtains

20) The subscript $*$ on u_i indicates the associated chain map.

$$k^*\omega(\sigma) = \int \cdots_{I\times J^p} \int b \frac{\partial(g_1, \cdots, g_p)}{\partial(t_2, \cdots, t_{p+1})} dt_1 \cdots dt_{p+1}. \tag{2.14}$$

To see what this gives, we consider $\int_0^1 a\,dt_1$ as a function $f_0 : X \to R$. Since $a \in \overline{G}^0(I\times X)$, it follows that f_0 must be of the form

$$f_0 = \psi(f_1', \cdots, f_m'), \quad f_j' \in G^0(\boldsymbol{X});$$

where $\psi : R^m \to R$ is a C^∞-function. But this implies by the closure condition for differentiable structures that $f_0 \in G^0(\boldsymbol{X})$. Since the jacobian in Equation (2.14) is independent of t_1, one sees that now

$$k^*\omega = f_0\, df_1 \wedge \cdots \wedge df_p.$$

This establishes the Lemma.

As previously noted, the coboundary operator ∂^* reduces to the exterior derivative d on spaces $G(\boldsymbol{Y})$, $\boldsymbol{Y}$ being an object in $\mathfrak{D}$. Since $\overline{G}(I\times \boldsymbol{X})$ is clearly invariant under d, it follows that the operator $(k\partial+\partial k)^*$ maps $\overline{G}^{p+1}(I\times\boldsymbol{X})$ to $G^{p+1}(\boldsymbol{X})$. Therefore, by Equation (2.13),

$$dk^* + k^*d = u_1^* - u_0^* \quad \text{on} \quad \overline{G}^{p+1}(I\times\boldsymbol{X}). \tag{2.15}$$

Now let $h : I\times\boldsymbol{X}$ be a contraction of $\boldsymbol{X}$ and set $\omega = h^*\alpha$. Since h is proper, $\omega \in \overline{G}^{p+1}(I\times\boldsymbol{X})$. Moreover, $d\omega=0$ since d commutes with h^*. From the fact that $h\circ u_1$ is the identity map of $\boldsymbol{X}$ one concludes that $u_1^*\omega=\alpha$. Similarly $u_0^*\omega=0$ since $h\circ u_0$ is a constant map. It follows by Equation (2.15) that $\alpha=d\beta$ for $\beta=k^*\omega$. Proposition 2.4 is thus established.

We now observe that on dimension 0, Equation (2.13) should read

$$\partial k = u_{1*} - u_{0*},$$

so that

$$k^*d = u_1^* - u_0^* \quad \text{on} \quad G^0(I\times\boldsymbol{X}).$$

As an immediate consequence, one has

PROPOSITION 2.5. *Let $\boldsymbol{X}$ be d.c. and $f \in F^0(\boldsymbol{X})$ such that $df=0$. Then f is a constant function.*

3. The de Rham theorem for *D*-spaces. It is rather obvious that the notion of differentiable spaces, as defined in Section 1, is not sufficiently restrictive to yield a de Rham theorem, despite the fact that the theory of differential forms does carry over to the full category $\mathfrak{D}$. Our task now will be to specify appropriate conditions on differentiable spaces, as unrestrictive as possible, under which the assertion of the de Rham theorem may be guaranteed. In virtue of the Leray theory of cohomology with coefficients in a sheaf[21] one comes to see quite easily what these conditions should be.

Let $\boldsymbol{X}$ be an object in $\mathfrak{D}$ and X its underlying topological space. In accordance with the definitions of the preceding Sections, $F \circ T_X$ constitutes an $\mathfrak{A}$-valued sheaf on X. We will denote this sheaf by $\mathfrak{F}$, and for each $p \geqq 0$, $\mathfrak{F}^p$ will denote the subsheaf of homogeneous elements of dimension p. We will also let $\boldsymbol{R}$ denote the simple sheaf on X with fiber R.[22] Since the constant functions on X belong to every differentiable structure on X, there exists an inclusion homomorphism $j : \boldsymbol{R} \rightarrow \mathfrak{F}^0$, and one consequently obtains a sequence

$$0 \rightarrow \boldsymbol{R} \xrightarrow{j} \mathfrak{F}^0 \xrightarrow{d} \mathfrak{F}^1 \xrightarrow{d} \cdots. \tag{3.1}$$

The first condition to be imposed on $\boldsymbol{X}$ should guarantee that (3.1) is an exact sequence, i.e., that it constitutes a cohomology resolution of $\boldsymbol{R}$. In virtue of Propositions 2.4 and 2.5, it will obviously suffice to assume that X is *locally differentiably contractible* (l.d.c.) in the following sense: Given $x \in X$ and a neighborhood U of x, there exists a neighborhood V of x such that $V \subset U$ and $\boldsymbol{X} | V$ is differentiably contractible.

At this point one must decide whether to consider arbitrary topological spaces X at the cost of using a paracompactifying family Φ, or instead restrict oneself to paracompact spaces X.[23] We will adopt the second course as a matter of convenience. In the case of manifolds this means that we are assuming seperability. With this stipulation there remains precisely one more condition to be imposed, i.e., one which will guarantee that the sheaves $\mathfrak{F}^p$ are *soft* (*mous*[24]) for every $p \geqq 0$. A sheaf over X is called soft if every section over a closed subset $S \subset X$ can be extended to a section over X. Since, however, for every $p > 0$, $\mathfrak{F}^p$ is clearly an $\mathfrak{F}^0$-module[25] and $\mathfrak{F}^0$ is a $\mathfrak{K}$-valued sheaf, it suffices to assume[26] that the sheaf $\mathfrak{F}^0$ is soft.

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21) See Godement [3], chapter 4.
22) Ibid., p. 113.
23) As previously remarked, Φ may then be taken to be the canonical paracompactifying family which contains X.
24) Ibid., p. 151.
25) Ibid., p. 127.
26) Ibid., Theorem 3. 7. 1, p. 156.

An object $\boldsymbol{X}$ in $\mathfrak{D}$ whose underlying space X is paracompact will be called a D-space provided (i) $\boldsymbol{X}$ is l.d.c.; (ii) $\mathfrak{F}^0$ is soft. It is perhaps not obvious that conditions (i) and (ii) are actually independent. The following simple examples may serve to demonstrate that this is the case: (1) Let M be the Euclidean plane and $\{X_i : i \geqq 1\}$ a family of circles in M whose radii r_i tend to 0 in the limit as $i \to \infty$, and let it be further assumed that all circles X_i are tangent at a given point $x_0 \in M$. We take $X = \cup_i X_i$, endow it with the relative topology, take $\phi : X \to M$ to be the inclusion map and let $\boldsymbol{X}$ denote the differentiable space obtained in accordance with general example (i) of Section 1. One may now verify that $\mathfrak{F}^0$ is soft. This is in fact quite apparent in the light of Theorem 3. 7. 2 (Godement [3], p. 156). (2) Take X to be R (considered as a topological space), and note that the totality of constant (R-valued) functions on X constitutes a differentiable structure. Let $\boldsymbol{X}$ denote the resultant object in $\mathfrak{D}$. Then every continuous map $h : I \times X \to X$ is a proper map from $I \times \boldsymbol{X}$ to $\boldsymbol{X}$, so that $\boldsymbol{X}$ is certainly l.d.c.. But now $\mathfrak{F}^0$ is obviously not soft. For example, the section over $\{0, 1\}$ which assigns the germ of the constant function $\boldsymbol{x}$ to x cannot be extended to all of X.

It is not difficult to see that there is no shortage of nonclassical D-spaces, i.e., D-spaces which are not differentiable manifolds. For instance, if the condition $r_i \to 0$ in the preceding example (1) be replaced by $r_i > r > 0$, the result will be such a D-space.

Given a paracompact topological space X, let $H^n(X, \boldsymbol{R})$ denote the n-dimensional cohomology group with coefficients in $\boldsymbol{R}$, as defined by means of the canonical resolution[27] of $\boldsymbol{R}$. For an arbitrary differentiable space $\boldsymbol{X}$ we will denote by $H^n(F(\boldsymbol{X}))$ the n-dimensional cohomology of $F(\boldsymbol{X})$.

THEOREM I. *Given that $\boldsymbol{X}$ is a D-space, there exists a canonical isomorphism*

$$H^n(F(\boldsymbol{X})) \to H^n(L(\boldsymbol{X}), \boldsymbol{R}), \quad n \geqq 0.$$

As previously noted, the fact that $\boldsymbol{X}$ is l.d.c. implies that (4.1) is a cohomology resolution of R. It was also seen that condition (ii) for D-spaces implies that $\mathfrak{F}$ is soft. The conclusion follows therefore from a known result[28].

4. Quotient spaces. We will now specialize example (ii) of Section 1 by taking X to be a quotient space of M and $\pi : M \to X$ the natural projection.

27) Ibid., p. 167.
28) Ibid., Theorem 4. 7. 1, p. 181.

Consequently π will be open and surjective. We recall that the resulting differentiable structure on X (which may be denoted by $F^0(\boldsymbol{X})$) is precisely the set of all continuous functions $f: X \to R$ such that $f\circ\pi \in C^\infty(M)$. Moreover, in accordance with the results of Section 2 we may identify $F(M)$ with the classical differential algebra of exterior differential forms on M. We now let A_π^0 denote the set of all $h \in C^\infty(M)$ such that $h=f\circ\pi$ with $f: X\to R$ continuous, and take A_π to be the differential subalgebra of $F(M)$ generated by A_π^0.

PROPOSITION 4.1. *The induced differential algebra homomorphism* $\pi^*: F(X)\to F(M)$ *constitutes an isomorphism of* $F(\boldsymbol{X})$ *onto* A_π.

One observes, in the first place, that π^* maps $F^0(\boldsymbol{X})$ onto A_π^0. Since π^* is known to be a differential algebra homomorphism, it follows that π^* maps $F(\boldsymbol{X})$ onto A_π. It remains to show that π^* is injective. On dimension zero this follows trivially from the fact that π is surjective. For $p>0$ let us consider $\omega \in G^p(\boldsymbol{X})$ and $\sigma: J^p \to X$ in $K_p(\boldsymbol{X})$ such that $\omega(\sigma)\neq 0$. We may choose a representation

$$\omega = \sum_j f_0^j\, df_1^j \wedge \cdots \wedge df_p^j\,, \quad f_i^j \in F^0(\boldsymbol{X}).$$

Let $g:\Omega \to X$ be an extension of σ (Ω being an open subset of R^p) and set $g_i^j = f_i^j \circ g$. Since

$$\omega(\sigma) = \sum_j \int\cdots\int_{J^p} g_0^j \frac{\partial(g_1^j,\cdots,g_p^j)}{\partial(t_1,\cdots,t_p)}\, dt_1\cdots dt_p\,,$$

there must exist a point $t^0 \in J^p$ such that

$$\sum_j g_0^j \frac{\partial(g_1^j,\cdots,g_p^j)}{\partial(t_1,\cdots,t_p)}\bigg|_{t_0} \neq 0\,, \tag{4.1}$$

the subscript t^0 being understood to indicate the function-value at t^0. There also exists a value k of the index j such that

$$g_0^k \frac{\partial(g_1^k,\cdots,g_p^k)}{\partial(t_1,\cdots,t_p)}\bigg|_{t_0} \neq 0\,. \tag{4.2}$$

Let $\phi: X\to R^p$ be defined by

$$\phi(x) = (g_1^k(x), \cdots, g_p^k(x)),$$

and set $g(t^0) = x^0$, $\phi(x^0) = s^0$. By (4.2), $\phi \circ g$ has maximal rank at t^0 and therefore admits a local inverse ψ, defined near s^0. Since π is surjective, there exists a point $y^0 \in M$ such that $\pi(y^0) = x^0$, and now the fact that π is open guarantees that the map $\psi \circ \phi \circ \pi$ (which is of class C^∞) has maximal rank at y^0. Let $\theta : \Omega \to M$ be a local inverse, defined near t^0. We have thus arrived at the following diagram:

$$\begin{tikzcd} & M \arrow[dr, "\pi"] & \\ \Omega \arrow[ur, "\theta"] \arrow[rr, "g"] & & X \arrow[dl, "\phi"] \\ & R^p \arrow[ul, "\psi"] & \end{tikzcd}$$

One observes that

$$\psi \circ \phi \circ g = \psi \circ \phi \circ \pi \circ \theta \quad \text{near} \quad t^0, \tag{4.3}$$

since both sides reduce to the identity map on some neighborhood U of t^0. Composing both sides of Equation (4.3) with g (on the left) yields

$$g = \pi \circ \theta \quad \text{on} \quad U. \tag{4.4}$$

Now consider a p-cube J^p in U with $t^0 \in J^p$, and let $\bar{\sigma} = g|U$, $\hat{\sigma} = \theta|U$. In virtue of (4.1) J^p may be chosen sufficiently small so that $\omega(\bar{\sigma}) \neq 0$. But by Equation (4.4), $\pi^*\omega(\hat{\sigma}) = \omega(\bar{\sigma})$. Consequently $\pi^*\omega \neq 0$, as was to be proved.

The differential forms on M belonging to A_π may be referred to as *baselike* (with respect to π), and we shall denote their cohomology by $H^n(A_\pi)$. Theorem I, together with Proposition 4.1, yields

THEOREM II. *Given that the differentiable space* $\boldsymbol{X}$ *induced by an open surjection* $\pi : M \to X$ *is a D-space, there exists a canonical isomorphism*

$$H^n(A_\pi) \to H^n(X, \boldsymbol{R}), \qquad n \geqq 0.$$

The hypothesis of Theorem II is verified in a considerable variety of situations. For instance, the question comes up in the context of transformation groups. Let us consider a Lie group G operating (to the left) differentiably on M, and take X to be the resulting orbit space M_G. It will now be shown[29] that when M is a *regular G-variety* in the sense of J. L. Koszul,[30] then $\boldsymbol{X}$ is a D-space. We briefly recall Koszul's definitions: Let H be a closed subgroup of G, and suppose that a linear representation of H on a vector space L has been given. One then defines an action of $G\times H$ on $G\times L$ by setting

$$(s,t)(r,a)=(srt^{-1},ta) \tag{4.5}$$

for $s,r\in G$, $t\in H$ and $a\in L$. For fixed s, Equation (4.5) defines an action of H on $G\times L$ (which is independent of s). The resulting quoteint $(G\times L)_H$ constitutes the total space of a fiber bundle, with fiber L and base space G/H. Moreover, taking O to be the origin of L, one obtains a preferred section $(G\times\{O\})_H$, referred to as the *principal section*. One also observes that the action of G on $G\times L$, as defined by Equation (4.5), induces a corresponding action on $(G\times L)_H$. Now consider a point $p\in M$ and let $H(p)$ denote the stability subgroup of p. The orbit through p is said to admit a *transversally fibered neighborhood* if there exists a linear representation of $H(p)$ on a real vector space L, together with a homeomorphism ϕ of a stable open neighborhood of the principal section of $(G\times L)_{H(p)}$ onto an open neighborhood of the orbit through p, subject to the condition that ϕ commutes with the action of G and maps O to p[31]. Finally, M is said to be a *regular G-variety* if $H(p)$ is compact for all $p\in M$, and every orbit admits a transversally fibered neighborhood. The second condition is always fulfilled when G is compact, a result established by Koszul[32].

Now let M be a regular G-variety, X its orbit space and $\boldsymbol{X}$ the differentiable space induced by the natural projection $\pi: M\to X$. Given $x\in X$, there exists a point $p\in M$ such that $\pi(p)=x$. Moreover, the orbit through p admits a transversally fibered neighborhood. In the previous notation, this may be identified with $(G\times V)_{H(p)}$, where V is a neighborhood of the origin $O\in L$. The corresponding neighborhood U of x may consequently be identified with the quotient of V under the action of $H(p)$. Since $H(p)$ is compact, U will be Hausdorff and consequently X itself is a Hausdorff space. It follows readily that X is paracompact.[28]

To show that $\boldsymbol{X}$ is a D-space, we must verify two conditions, both of which are purely local. For this reason it will suffice to consider the matter for the restriction space $\boldsymbol{X}|U$, where U is given as above. The functions

29) M is assumed to be separable, and therefore paracompact.

30) See J. L. Koszul [4].

31) A simple example illustrating the notion of transversally fibered neighborhoods will be given below.

32) Ibid., p. 139.

$f \in F^0(\boldsymbol{X}|U)$ may now be regarded as C^∞-functions $f: V \to R$, invariant under the action of $H(p)$. As an immediate consequence one sees that $\boldsymbol{X}$ is l.d.c.. For if $V_0 \subset V$ denotes a neighborhood of O which is starlike with respect to O, and U_0 denotes the corresponding neighborhood of x, then the map $h: I \times V_0 \to V_0$ given by $h(t, y) = ty$ (scalar multiplication in L) defines a contraction of $\boldsymbol{X}|U_0$.

It remains therefore to verify condition (ii) for a D-space. Let S be a closed subset of V and $f: S \to R$ a C^∞-function, invariant under $H(p)$. Since the sheaf of differentiable functions on a manifold is soft, f may be extended to a C^∞-function $\hat{f}: V \to R$. We will transform $\hat{f}$ into a function $\bar{f}$, invariant under $H(p)$, by means of an integration over $H(p)$ defined in terms of an invariant measure μ. We suppose $\mu(H(p))=1$ and set

$$\bar{f}(x) = \int_{H(p)} \hat{f}(t \cdot x)\, dt\,, \quad x \in V\,;$$

where $t \cdot x$ denotes the image of x under the action of t. It is clear that this averaging process yields a C^∞-function $\bar{f}$, invariant under $H(p)$. However, since $\hat{f}|S$ is already invariant, it follows that $\bar{f}$ extends f, as was to be shown.

As a corollary to Theorem II, one now obtains

PROPOSITION 4.2. *For a regular G-variety M, the cohomology $H^n(A_\pi)$ of baselike forms on M reduces to the real cohomology $H^n(X, \boldsymbol{R})$ of the orbit space.*

A corresponding result regarding the cohomology of *basic* forms on M was obtained by Koszul.[33] We recall that a differential form ω on a G-variety M is called *basic* if (i) ω is invariant under the action of G; (ii) the interior product of ω with every left-invariant vector field gives zero. It is clear for arbitrary G-varieties that every baselike form is also basic in this sense. The converse, on the other hand, does not hold even for regular G-varieties, as will now be shown. This means that Proposition 4.2 and the corresponding Theorem of Koszul represent independent results.

Let us take M to be the Euclidean plane and G the group of integers mod 2. Let (x, y) be Cartesian coordinates on M. The nonzero element of G shall operate on M by the rule $(x, y) \to (-x, -y)$, and this clearly defines a regular G-variety. Now let us consider the differential form

$$\omega = x dy - y dx\,,$$

33) Ibid., [4].

which is certainly basic. It will require a little calculation to show that $\omega \notin A_\pi$. Since $\omega = r^2 d\theta$ in terms of polar coordinates, one finds that the integral of ω around a (suitably oriented) circle γ_r of radius r with center at the origin is given by

$$\int_{\gamma_r} \omega = 2\pi r^2 .$$

On the other hand we shall see that

$$\int_{\gamma_r} \alpha = O(r^4) \tag{4.6}$$

for all $\alpha \in A_\pi$. To show this it will suffice to consider a monomial 1-form $\alpha = f_0 df_1$, where now f_i are C^∞-functions on M invariant under G. On account of this invariance one sees that the terms of odd order in a finite Taylor's series expansion of f_i must vanish, so that

$$f_i = a_i + b_i x^2 + c_i xy + d_i y^2 + O(r^4) .$$

But this implies the estimate (4.6).

It should be remarked that, besides regular G-varieties, the V-manifolds introduced by I. Satake[34] are also D-spaces, as may be verified by considerations analogous to the preceding ones. More precisely, if X is a V-manifold, the local uniformizing systems[34] which define the V-manifold structure give rise to a differentiable structure $\mathfrak{F}$ on X, and $(X, \mathfrak{F})$ will be a D-space. One finds that the differential forms on $(X, \mathfrak{F})$ corresponds in general to a proper subalgebra of the differential forms on the V-manifold[34] X. Consequently Satake's version of the de Rham theorem[34] is independent of Theorem II.

It should be noted that the content of Theorem II is by no means limited to situations which resemble the regular G-varieties or V-manifolds. In both these instances we have been dealing essentially with quotient spaces which arise from the action of a *compact* transformation group. On the other hand, quotients corresponding to the action of noncompact groups also turn out to be D-spaces in a large variety of situations, provided one agrees to identify nonseparated points, so that the resultant quotient will be a Hausdorff space. We will conclude this paper with a rather characteristic example of a nonregular G-variety for which the Hausdorff quotient, in the above sense, is a D-space.

Let G denote the group of real numbers, E a 3-dimensional Euclidean space

34) See Satake [6].

and $\boldsymbol{v}$ a nonsingular C^∞ vector field on E, subject to the condition that the function $p \to \|\boldsymbol{v}(p)\|$ (Euclidean norm) is bounded on E. The vector field $\boldsymbol{v}$ defines a differentiable action of G on E in an obvious way ($\boldsymbol{v}$ may be thought of as a velocity field, t as a time coordinate). In particular, let us introduce cylindrical coordinates[35] (r, θ, z) in E and consider

$$\boldsymbol{v} = \cos r \frac{\partial}{\partial z} + \sin r \frac{\partial}{\partial \theta}.$$

This is clearly well defined on the entire space E and defines an action of G. Apart from the z-axis (which is an orbit), the orbit through p is a circle when $r(p) = (n+1/2)\pi$ for some integer $n \geqq 0$, and a helix otherwise. We shall only need to concern ourselves with a cylindrical region $r < r_0$ for some $r_0 > \pi/2$, which is certainly stable under the action of G. To be specific, take M to be the region $r < \pi$. Let us ncw examine the orbit space corresponding to the given action of G on M. The orbits inside the cylinder $r < \pi/2$ may clearly be identified with points of an open disc D of radius $\pi/2$. Similarly those outside the cylinder $r = \pi/2$ correspond to points of an open annulus A, while the circular orbits in M correspond to points on a line L. This establishes a point set isomorphism $\mathcal{X}$ between the orbit space M_G and the set $D \cup L \cup A$. It is easy to see that with respect to the topology induced on $D \cup L \cup A$ by $\mathcal{X}$, the points of L are nonseparated. Identification of non-separated points yields a sphere with an attached disc[36], and this represents the Hausdorff quotient X of the G-variety M. Thus X has precisely the cohomology of a 2-sphere. The natural projection $\pi: M \to X$ induces a differentiable structure on X, and we let $\boldsymbol{X}$ denote the resulting differentiable space. We assert that $\boldsymbol{X}$ is a D-space.

In the first place one sees that stable neighborhoods in M involving only noncircular orbits actually admit a product structure, and it will therefore suffice to consider a stable neighborhood V of the cylinder $r = \pi/2$. To be specific, let V be given by $1 < r < 2$. As things stand, the circular orbits of M do *not* admit transversally fibered neighborhoods. On the other hand, consider the vector field

$$\boldsymbol{v}^* = 2\pi\sqrt{r^2 + \left(\frac{\pi}{2}\right)^2 \cot r} \quad \boldsymbol{v}$$

defined on V, which corresponds to a constant angular velocity $\dot{\theta} = 2\pi$. It gives rise to a new action of G on V. Since the two actions of G on V have

35) r denotes the radial, θ the angular and z the axial coordinate.
36) The point of attachment corresponds to L.

precisely the same orbits, we may now replace $\boldsymbol{v}$ by $\boldsymbol{v}^*$. With this modification, V becomes transversally fibered, as will now be shown.

Let (x, y, z) be Cartesian coordinates on E and let U denote the intersection of V with the plane $y=0$. Let p denote the point with Cartesian coordinates $(\pi/2, 0, 0)$. The stability group $H(p)$ is precisely the group of integers. Let L denote the vector space R^2. The preferred generator 1 of $H(p)$ shall operate on L by the formula

$$(\xi, \zeta) \rightarrow (\xi, \zeta-\xi),$$

and this defines a linear representation of $H(p)$. Now $(G \times L)_{H(p)}$ is simply a plane bundle over $G/H(p)$. Let W denote the region of L consisting of points (ξ, ζ) with

$$\pi^2 \cot 2 < \xi < \pi^2 \cot 1 .$$

This is stable under $H(p)$, and we propose to define a diffeomorphism ϕ: $(G \times W)_{H(p)} \rightarrow V$. To this end we observe that the map $\psi: U \rightarrow W$ given by

$$\psi[(x, 0, z)] = (\pi^2 \cot x, z)$$

maps U diffeomorphically onto W. We note further that a point $s \in (G \times W)_{H(p)}$ may be represented by (t, q), with $t \in G$, $q \in W$ and $0 \leqq t < 1$. The map ϕ is defined by

$$\phi(s) = t \cdot \psi^{-1}(q),$$

where the dot is understood to indicate the action of t as a transformation on V. One may verify without difficulty that ϕ is a diffeomorphism and commutes with the operation of G.

It will now suffice to show that $\boldsymbol{X}|\pi(V)$ is a D-space. By virtue of the preceding consideration, the elements of $F^0(\boldsymbol{X}|\pi(V))$ may be identified with C^∞-functions $f: W \rightarrow R$ invariant under the action of $H(p)$. By the simple argument previously employed in the case of regular G-varieties one sees that $\boldsymbol{X}|\pi(V)$ is l.d.c.. To verify the remaining condition, we will suppose that S and T are closed, disjoint subsets of W, stable under $H(p)$, and we let f denote the function defined on $S \cup T$ which assumes the constant value 1 on S and 0 on T. By a known result[37] it will suffice to show that f can be extended to a function in $F^0(\boldsymbol{X}|\pi(V))$. To this end we may assume without loss of generality that S contains the origin O of L. There must exist a closed

37) Theorem 3.7.2, Godement [3], p. 156.

neighborhood S^* of S, stable under $H(p)$, such that $S^* \cap T$ is empty. Thus for a sufficiently small $\varepsilon > 0$, the region W_* consisting of points (ξ, ζ) with $|\xi| < \varepsilon$ must be contained in S^*. On the other hand, the complementary set $W^* = W - W_*$ is stable under $H(p)$, and $(G \times W^*)_{H(p)}$ is clearly a differentiable manifold (i.e., a pair of cylinders). Consequently there exists a C^∞-function $g: W^* \to R$, invariant under $H(p)$, such that $g | S^* \cap W^* = 1$ and $g | T = 0$. We extend g to W by setting $g | W - W^* = 1$ and obtain thus a function in $F^0(\boldsymbol{X} | \pi(V))$ extending f.

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