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IMBEDDINGS OF HOMOGENEOUS SPACES WITH MINIMUM TOTAL CURVATURE

SHOSHICHI KOBAYASHI*)

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1. Introduction. Let M be an n-dimensional compact differentiable manifold immersed in the Euclidean space R^{n+k}. Let B be the set of unit normal vectors of M. Then B is a bundle of $(k-1)$-sphere over M and is a manifold of dimension $n+k-1$. Let S be the unit $(n+k-1)$-sphere about the origin of R^{n+k}. Let $d\sigma$ be the volume element of S and $c_{n+k-1} = \int_S d\sigma$ the volume of S. If we denote by ν the canonical map $B \to S$, then the total curvature of the immersed manifold M is defined by (cf. Chern and Lashof [3])

$$\frac{1}{c_{n+k-1}} \int_B \nu^*(d\sigma).$$

Since the total curvature defined above depends not only on M but on the immersion $\varphi: M \to R^{n+k}$, we shall denote it by $\tau(M, \varphi, R^{n+k})$ or simply by $\tau(\varphi)$.

Let F be the set of functions f on M whose critical points are all non-degenerate. The number of the critical points of index i of $f \in F$ will be denoted by $\beta_i(M, f)$. We set

$$\begin{cases} \beta_i(M) = \min_{f \in F} \beta_i(M, f), \\ \beta(M, f) = \sum_{i=0}^{n} \beta_i(M, f), \\ \beta(M) = \min_{f \in F} \beta(M, f). \end{cases}$$

Since $\beta_i(M, f) = \beta_{n-i}(M, -f)$, we have

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$$\beta_i(M) = \beta_{n-i}(M).$$

Evidently we have also

$$\beta(M) \geqq \sum_{i=0}^{n} \beta_i(M).$$

For an arbitrarily fixed coefficient field, let $b_i(M)$ be the i-th Betti number of M. Then the Morse inequalities state

$$\beta_i(M) \geqq b_i(M).$$

Chern-Lashof [4] proved the inequality

$$\tau(M, \varphi, R^{n+k}) \geqq \sum_{i=0}^{n} b_i(M).$$

Kuiper [8] obtained a stronger inequality:

$$\tau(M, \varphi, R^{n+k}) \geqq \beta(M).$$

Thus

$$\tau(M, \varphi, R^{n+k}) \geqq \beta(M) \geqq \sum_{i=0}^{n} \beta_i(M) \geqq \sum_{i=0}^{n} b_i(M).$$

He also proved that if M is fixed, then for variable immersion $\varphi : M \to R^{n+k}$ and variable k:

$$\inf \tau(M, \varphi, R^{n+k}) = \beta(M).$$

An immersion $\varphi : M \to R^{n+k}$ is said to be minimal if $\tau(\varphi) = \beta(M)$. Given a compact manifold M, in general there does not exist a minimal immersion of M. Kuiper pointed out that an exotic sphere cannot be minimally immersed. In fact, if M is an exotic sphere, it admits a function with two isolated singularities (one maximum and one minimum) and hence $\beta(M)=2$. On the other hand, by a theorem of Chern-Lashof [3] an immersed compact manifold M with $\tau(M, \varphi, R^{n+k}) = 2$ is a convex hypersurface in some $R^{n+1} \subset R^{n+k}$, which implies that M is diffeomorphic with a usual sphere. It is an interesting but difficult problem to decide which manifolds can be minimally immersed, since it involves not only topological but differentiable structures of manifolds. Kuiper [9] proved that every orientable closed surface and also every non-orientable closed surface with Eular number $\leqq -2$ can be minimally immersed in R^3 and that the real projective plane and the Klein bottle can not be minimally immersed in R^3. In a earlier paper (Kuiper [8]) he exhibited a minimal immersion of the real projective plane in R^4. [1)]

The purpose of this paper is to prove

THEOREM 1. *Every compact homogeneous Kaehler manifold can be minimally imbedded into a Euclidean space.*

We can also estimate the dimension of the receiving Euclidean space. Every compact homogeneous Kaehler manifold M can be written as a coset space G/H of a compact Lie group G, (see references given in § 2). Let S be the semi-simple part of G and C the center of G. Then M can be minimally imbedded into a Euclidean space of dimension $\dim S + \frac{3}{2} \dim C$.

For further comments on the imbedding we construct, see the last section.

2. Reduction to the simply connected case. Let M and M' be compact manifolds and $\varphi : M \to R^N$ and $\varphi' : M' \to R^{N'}$ be immersions with total curvature $\tau(M, \varphi, R^N)$ and $\tau(M', \varphi', R^{N'})$. Then the total curvature $\tau(M \times M', \varphi \times \varphi', R^{N+N'})$ is given by

$$\tau(M \times M', \varphi \times \varphi , R^{N+N'}) = \tau(M, \varphi, R^N) \cdot \tau(M', \varphi', R^{N'}) ;$$

see Kuiper [8] for the proof.

From the definition of $\beta(M)$ and $\beta(M')$, it is clear that

$$\beta(M)\beta(M') \geqq \beta(M \times M') \geqq \sum_k b_k(M \times M')$$
$$= \Big(\sum_i b_i(M)\Big)\Big(\sum_j b_j(M')\Big).$$

Hence if $\beta(M) = \sum_i b_i(M)$ and $\beta(M') = \sum_j b_j(M')$, then

$$\beta(M)\beta(M') = \beta(M \times M').$$

We may now conclude

1) For the total curvature of immersed manifolds, see also an excellent exposition by D. Ferus, Die absolute Totalkrümmung Riemannscher Immersionen, Diplomarbeiten, Bonn 1966.

LEMMA 1. *Let M and M' be compact manifolds minimally immersed in R^N and $R^{N'}$ respectively. If $\beta(M)=\sum_i b_i(M)$ and $\beta(M')=\sum_j b_j(M')$, then $M\times M'$ is minimally immersed in $R^{N+N'}$ in a natural manner.*

Since a torus T is a product of circles, we have

LEMMA 2. *For a torus T, we have the equality $\beta(T)=\sum_i b_i(T)$.*

The following is due to Frankel [5].

LEMMA 3. *Let M be a compact Kaehler manifold with $b_1(M)=0$. If it admits a Killing vector field X with isolated zeros, then $\beta(M)=\sum_i b_i(M)$.*

Since we need the proof in the next section, we shall describe it briefly. Let ξ be the 1-form corresponding to X under the duality defined by the metric. Let J be the complex structure of M. Then $J\xi$ is a closed 1-form and hence $J\xi=df$ for some function f. Clearly the critical points of f coincide with the zeros of X. Frankel shows that the isolated critical points of f are all non-degenerate and of even index, i.e.,

$$\beta_i(M,f)=0 \quad \text{for all odd } i\,.$$

From the Morse relations it follows that

$$b_i(M)=\beta_i(M,f) \quad \text{for all } i\,,$$

which implies Lemma 3.

LEMMA 4. *A torus T of dimension r can be minimally imbedded into R^N where $N=\frac{3}{2}r$ if r is even and $N=\frac{3}{2}(r-1)+2$ if r is odd.*

If r is even, we write T as a product of 2-dimensional tori. If r is odd, we write T as a product of 2-dimensional tori and a circle. A 2-dimensional torus can be minimally imbedded into R^3 (in an ordinary doughnut shaped form) and a circle can be minimally imbedded into R^2 in a usual manner. Lemma 4 follows from Lemmas 1 and 2.

The following main lemma will be proved in the next section.

LEMMA 5. *Let $M=G/H$ be a simply connected homogeneous Kaehler* *manifold with G compact. Then* $\beta(M)=\sum_i b_i(M)$ *and M can be minimally imbedded into* R^N *where* $N=\dim G$.

The following lemma is due to Borel-Remmert [2].

LEMMA 6. *A compact homogeneous Kaehler manifold is a direct product of a complex torus and a compact simply connected homogeneous Kaehler manifold and can be written as a coset space* G/H *of a compact Lie group* G.

The following lemma is due to Matsushima [12].

LEMMA 7. *Let* $M=G/H$ *be a homogeneous Kaehler manifold with G compact. Then* $G=S\times C$ *where S is semi-simple and C is the center of G. Moreover S contains H so that* $G/H=(S/H)\times C$ *is the decomposition described in Lemma 6, i.e.,* S/H *is a simply connected homogeneous Kaehler manifold and C is a complex torus.*

It is now clear that the proof of Theorem is reduced to that of the main lemma (lemma 5).

3. Proof of Lemma 5. Let $M=G/H$ be a simply connected homogeneous Kaehler manifold on which a compact Lie group G acts effectively. Let $\mathfrak{g}$ denote the Lie algebra of G whose elements are considered as Killing vector fields on M. Let $\mathfrak{g}^*$ denote the space of 1-forms corresponding to the Killing vector fields $X\in\mathfrak{g}$ under the duality defined by the metric. Let $\Delta=d\delta+\delta d$ denote the Laplacian. We define a space E of functions on M by

$$E=\{\delta(J\xi)\,;\,\xi\in\mathfrak{g}^*\}\,.$$

Let E^* be the dual space of E and $\varphi: M\rightarrow E^*$ the evaluation map, i.e.,

$$<\varphi(x),f>=f(x)\quad\text{for } x\in M \text{ and } f\in E\,.$$

We shall show that φ gives a minimal imbedding of M into E^*.

A compact simply connected homogeneous Kaehler manifold M carries an Einstein-Kaehler metric (cf. Borel [1] and Koszul [7]). If ξ is a 1-form corresponding to a Killing vector field of M, then (cf. Yano and Bochner [14; p. 33])

$$\Delta(J\xi)=J\Delta(\xi)=2J\xi\,.$$

Since $J\xi$ is closed (cf. Matsushima [13; lemme 4]), we have

$$d\delta J\xi = 2J\xi\,.$$

This shows that the mapping $\xi \in \mathfrak{g}^* \to \delta J\xi \in E$ is a linear isomorphism and that the critical points of the function $f = \delta J\xi$ coincide with the zeros of ξ. From the proof of Lemma 3 described above, we have

LEMMA 8. *Let $f = \delta J\xi \in E$ be a function having only isolated points. Then the critical points of f are all non-degenerate and of even index. Hence*

$$\beta(M, f) = \sum_{\text{even } i} \beta_i(M, f) = \sum_{\text{even } i} b_i(M) = \beta(M)\,.$$

Note that the last equality follows from the first two and from

$$\beta(M, f) \geqq \beta(M) \geqq \sum_i b_i(M)\,.$$

Since G is transitive on M, it follows that the set

$$\{(df)_x\,; f \in E\} = \{(d\delta J\xi_x\,; \xi \in \mathfrak{g}^*\} = \{(J\xi)_x\,; \xi \in \mathfrak{g}^*\}$$

coincides with the cotangent space of M at x. Hence the evaluation map $\varphi : M \to E^*$ is an immersion.

The linear isomorphism $\xi \in \mathfrak{g}^* \to \delta J\xi \in E$ induces the dual linear isomorphism $E^* \to \mathfrak{g}$. Let ρ be the representation of G on E^* corresponding to the adjoint representation of G on $\mathfrak{g}$. It is easy to verify that the immersion φ is equivariant with ρ in the sense that

$$\rho(s)(\varphi(x)) = \varphi(s(x)) \text{ for } s \in G \text{ and } x \in M\,.$$

The group G acts on $\varphi(M)$ in a natural manner; $s \in G$ sends $\varphi(x)$ into $\rho(s)\varphi(x) = \varphi(s(x))$. Let $o \in M$ be the origin corresponding to the coset H of G/H and let H^* be the isotropy subgroup of G acting on $\varphi(M)$ at $\varphi(o)$, i.e.,

$$H^* = \{s \in G\,; \rho(s)(\varphi(o)) = \varphi(o)\} = \{s \in G\,; \varphi(s(o)) = \varphi(o)\}\,.$$

Since $\varphi : M \to E^*$ is an equivariant immersion, it follows that $\varphi : M \to \varphi(M)$ is a covering projection. In other words, the natural map $G/H \to G/H^*$ is a covering projection.

Following Lichnerowicz [11; p. 166] we shall show that $H^* = H$ thereby proving that $\varphi: M \to \varphi(M)$ is one-to-one. The group H^* is the isotropy subgroup of G at $\varphi(o)$ in the representation ρ. Under the identification $E^* \approx \mathfrak{g}$, H^* is the isotropy subgroup of G at $\varphi(o)$ in the adjoint representation, i.e., the centralizer of the 1-parameter subgroup generated by $\varphi(o) \in \mathfrak{g}$. Since G is compact, the closure of this 1-parameter subgroup in G is a torus and H^* is the centralizer of this torus in G. It follows that H^* is connected and hence $H^* = H$.

Now the fact that the imbedding $\varphi: M \to E^*$ is minimal follows from Lemma 8 and from the following lemma of Kuiper [10].

LEMMA 9. *In general let φ be an immersion of M into a vector space E^*. Consider each element f of the dual space E as a function $f \circ \varphi$. Then the immersion φ is minimal if and only if*

$$\beta(M, f) = \beta(M)$$

for every function $f \in E$ having only isolated non-degenerate critical points on M.

4. Some comments on the minimal imbedding φ. Throughout this section $M = G/H$ will be a compact simply connected homogeneous Kaehler manifold and G will be a compact semi-simple Lie group.

The minimal imbedding φ of M constructed in § 3 may be considered as an imbedding of M into the Lie algebra $\mathfrak{g}$ of G. The imbedding φ is then equivariant with the adjoint representation of G.

There is no proper affine subspace of E^* which contains $\varphi(M)$. Otherwise there would be a nonzero function $f \in E$ which is constant on M. Since $f = \delta J \xi$ for some $\xi \in \mathfrak{g}^*$ and $0 = df = 2J\xi$, f must vanish identically on M and hence f is the zero element of E. This is absurd.

If $2m$ denotes the real dimension of M, then $\dim E^* = \dim G = \dim M + \dim H \leqq \dim M + \dim U(m) = 2m + m^2$, and the equality is attained when M is the complex projective space.

For the complex projective space $P_m(C)$ the minimal imbedding φ may be described as follows. Let $(z^0, z^1, \cdots, z^m)$ be a homogeneous coordinate system of $P_m(C)$ with the condition $z^0\bar{z}^0 + z^1\bar{z}^1 + \cdots + z^m\bar{z}^m = 1$. Let $R^{(m+1)^2}$ be a Euclidean space with coordinate system (X^h, X^{hk}, Y^{hk}) where $h, k = 0, \cdots, m$ and $h \neq k$. Consider the imbedding $P_m(C) \to R^{(m+1)^2}$ defined by

$$X^h = \sqrt{2}\, z^h \bar{z}^h, \quad X^{hk} = z^h \bar{z}^k + \bar{z}^h z^k, \quad Y^{hk} = i(z^h \bar{z}^k - \bar{z}^h z^k).$$

Then $P_m(C)$ lies in the hyperplane of $R^{(m+1)^2}$ defined by

$$X^0 + \cdots + X^m = \sqrt{2}\,.$$

It can be shown that the imbedding of $P_m(C)$ into this hyperplane is the same as φ. I found this imbedding in Hodge [6; p. 151]. [2)]

Since $P_{2m-1}(C)$ can be written also as $Sp(m)/Sp(m-1)\times U(1)$, $P_{2m-1}(C)$ can be minimally imbedded into R^N where $N = \dim Sp(m) = m(2m+1)$.

ADDED IN PROOF. Generalizing the minimal imbedding of a complex projective space described in §4, S. S. Tai has recently discovered minimal imbeddings of real and quaternionic projective spaces.

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UNIVERSITY OF CALIFORNIA,
BERKELEY, U. S. A.

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2) Hodge attributes this imbedding to G. Mannoury, Niew Archief voor Wiskunde 4(1898), 112.