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NOTES ON COVARIANT ALMOST ANALYTIC VECTOR FIELDS

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1. Introduction. In a previous paper [4] we defined almost analytic vector fields in an almost complex space and generalized some of well known results for analytic vector fields in a Kähler space to those for almost analytic vector fields in the most general almost Hermitian space.

To define a contravariant almost analytic vector field we proceeded as follows:

In a complex manifold M covered by a system of neighborhoods U with complex coordinates $(z^\kappa, z^{\bar\kappa})$[1], a self conjugate contravariant vector field $(v^\kappa, v^{\bar\kappa})$, that is, a contravariant vector field $(v^\kappa, v^{\bar\kappa})$ satisfying $\bar{v}^\kappa = v^{\bar\kappa}$, is said to be analytic when the components v^κ and $v^{\bar\kappa}$ are analytic functions of z and $\bar{z}$ respectively:

$$(1.1) \qquad v^\kappa = v^\kappa(z), \quad v^{\bar\kappa} = v^{\bar\kappa}(\bar{z}).$$

The condition (1.1) is equivalent to

$$(1.2) \qquad \partial_{\bar\lambda} v^\kappa = 0, \quad \partial_\lambda v^{\bar\kappa} = 0,$$

where $\partial_{\bar\lambda}$ means $\partial/\partial z^{\bar\lambda}$ and ∂_λ means $\partial/\partial z^\lambda$.

On the other hand, we have, in a complex manifold, a numerical tensor F of type (1, 1) given by

$$(1.3) \qquad F_i{}^h = \begin{pmatrix} \sqrt{-1}\,\delta_\lambda^\kappa & 0 \\ 0 & -\sqrt{-1}\,\delta_{\bar\lambda}^{\bar\kappa} \end{pmatrix}^{2)}$$

and consequently, putting

1) Here and in the sequel the Greek indices $\kappa, \lambda, \mu, \cdots$ run over the range $\{1, 2, \cdots, n\}$ and $\bar\kappa, \bar\lambda, \bar\mu, \cdots$ the range $\{\bar{1}, \bar{2}, , \cdots, \bar{n}\}$.

2) Here and in the sequel, the Roman indices $h, i, j, \cdots$ run over the range $1, 2, \cdots, n$; $\bar{1}, \bar{2}, \cdots, \bar{n}$.

$$
(1.4)\qquad \begin{cases} O_{ir}^{sh} = \dfrac{1}{2}(A_i^s A_r^h - F_i{}^s F_r{}^h)\,, \\[2ex] {}^*O_{ir}^{sh} = \dfrac{1}{2}(A_i^s A_r^h + F_i{}^s F_r{}^h)\,, \end{cases}
$$

where A_i^s is the unit tensor, we can write (1.2) in the form

$$
(1.5)\qquad {}^*O_{ir}^{sh}\,\partial_s v^r = 0
$$

or

$$
(1.6)\qquad \underset{v}{\pounds} F_i{}^h = v^t \partial_t F_i{}^h - F_i{}^t \partial_t v^h + F_t{}^h \partial_i v^t = 0\,,
$$

which is easily verified to be a tensor equation, where $\underset{v}{\pounds}$ denotes the Lie derivation with respect to v.

Thus, we define a *contravariant almost analytic vector field* v^h in an almost complex space with structure tensor $F_i{}^h$ to be a contravariant vector field which satisfies (1.6).

Similary, a self-adjoint covariant vector field $(w_\lambda, w_{\bar\lambda})$ in a complex space is said to be analytic when the components w_λ and $w_{\bar\lambda}$ are analytic functions of z and $\bar z$ respectively:

$$
(1.7)\qquad w_\lambda = w_\lambda(z)\,, \qquad w_{\bar\lambda} = w_{\bar\lambda}(\bar z)\,.
$$

The condition (1.7) is equivalent to

$$
(1.8)\qquad \partial_{\bar\mu} w_\lambda = 0\,, \qquad \partial_\mu w_{\bar\lambda} = 0
$$

or

$$
(1.9)\qquad {}^*O_{ji}^{ts}\,\partial_t w_s = 0
$$

or

$$
(1.10)\qquad (\partial_j F_i{}^s - \partial_i F_j{}^s)\, w_s - F_j{}^s \partial_s w_i + F_i{}^s \partial_j w_s = 0\,,
$$

which is also easily verified to be a tensor equation.

Thus we define a covariant almost analytic vector field w_i in an almost complex space to be a covariant vector field which satisfies (1.10).

On the other hand I. Sato [1] and one of the present authors [3] found another way of defining a covariant almost analytic vector field.

We suppose that a manifold M is covered by a system of coordinate neighborhoods $\{U; x^h\}$ where x^h is a system of local coordinates in the neighborhood U. Let (p_i) be the system of Cartesian coordinates in each cotangent space ${}^cT_P(M)$ of M at a point P in U with respect to the natural base dx^i. Then we can introduce, in the open set $\pi^{-1}(U)$ of ${}^cT(M)$, local coordinates (x^h, p_i) for a point in ${}^cT(M)$, π being the projection ${}^cT(M) \to M$. We recall (x^h, p_i) the *induced coordinates* in $\pi^{-1}(U)$.

Suppose that the manifold M has an almost complex structure F, then we can prove that the cotangent bundle ${}^cT(M)$ has an almost complex structure $\widetilde{F}$ whose components in the induced coordinate system (x^h, p_i) are given by

$$\begin{pmatrix} F_i{}^h & 0 \\ p_r(\partial_i F_h{}^r - \partial_h F_i{}^r + \frac{1}{2} N_{it}{}^r F_h{}^t) & F_h{}^i \end{pmatrix}, \tag{1.11}$$

where $N_{it}{}^r$ is the Nijenhuis tensor of F:

$$N_{ji}{}^h = F_j{}^t \partial_t F_i{}^h - F_i{}^t \partial_t F_j{}^h - (\partial_j F_i{}^t - \partial_i F_j{}^t) F_t{}^h . \tag{1.12}$$

We also can prove that the cross-section in ${}^cT(M)$ determined by a covariant vector field w_i in M is almost analytic, that is, the tangent plane to the cross-section is invariant with respect to the almost complex structure defined above, if and only if w_i satisfy

$$(\partial_i F_h{}^r - \partial_h F_i{}^r) w_r - F_i{}^t \partial_t w_h + F_h{}^t \partial_i w_t + \frac{1}{2} N_{it}{}^r F_h{}^t w_r = 0. \tag{1.13}$$

Thus, we define a *covariant almost analytic vector field* w_i to be a vector field which satisfies (1.13).

The main purpose of the present paper is to study the properties of covariant almost analytic vector fields in this sense.

2. Covariant almost analytic vector fields. We consider an almost Hermitian space with almost complex structure $F_i{}^h$ and almost Hermitian metric g_{ji}:

$$F_j{}^h F_i{}^j = -A_h^i , \qquad F_j{}^t F_i{}^s g_{ts} = g_{ji} , \tag{2.1}$$

$$F_{ji} = -F_{ij} , \qquad F_{ji} = F_j{}^t g_{ti} , \tag{2.2}$$

and we denote by ∇_j the covariant differentiation with respect to g_{ji}.

In an almost Hermitian space, the equation (1.13) may be written as

$$*O_{ji}^{ts}(\nabla_t F_s{}^a - \nabla_s F_t{}^a) w_a - F_j{}^a \nabla_a w_i + F_i{}^a \nabla_j w_a = 0 \tag{2.3}$$

or

$$*O_{ji}^{ts}(\nabla_t F_s{}^a - \nabla_s F_t{}^a) w_a - 2F_j{}^a *O_{ai}^{ts} \nabla_t w_s = 0 \tag{2.4}$$

or

$$*O_{ji}^{ts}\{(\nabla_t F_s{}^a - \nabla_s F_t{}^a) w_a - F_t{}^a \nabla_a w_s + F_s{}^a \nabla_t w_a\} = 0 . \tag{2.5}$$

Taking the symmetric part of (2.5) with respect to j and i, we find

$$*O_{ji}^{ts}(\nabla_t w_s - \nabla_s w_t) = 0 . \tag{2.6}$$

The equation (2.6) shows that $\nabla_j w_i - \nabla_i w_j$ is pure[3] for a covariant almost analytic vector w_i in an almost Hermitian space. Transvecting g^{ji} to (2.5), we find

$$F^{ji} \nabla_j w_i = 0 \tag{2.7}$$

for a covariant almost analytic vector field w_i.

Now we define tensors P_{ji} and Q_{ji} by

$$P_{ji} = *O_{ji}^{ts}(\nabla_t F_s{}^a - \nabla_s F_t{}^a) w_a , \tag{2.8}$$

and

$$Q_{ji} = (F_j{}^a \nabla_a w_i - F_i{}^a \nabla_j w_a) \tag{2.9}$$

respectively. Then for a covariant almost analytic vector field w_i, we have

$$P_{ji} = Q_{ji} . \tag{2.10}$$

In an almost Kähler space, we have

$$\nabla_t F_{sa} + \nabla_s F_{at} + \nabla_a F_{ts} = 0 ,$$

and consequently, from (2.8),

$$P_{ji} = *O_{ji}^{ts}(\nabla_a F_{ts}) w^a , \tag{2.11}$$

which is zero because of the pureness of $\nabla_a F_{ts}$ with respect to t and s. Thus, for a covariant almost analytic vector field in an almost Kähler space, we have

3) See, e.g. [2].

$$Q_{ji} = 0,$$

which is equiva ent to

$$*O_{ji}^{ts} \nabla_t w_s = 0. \tag{2.12}$$

On the other hand, $N_{jih} = 2F_j{}^t(\nabla_h F_{it})$ is valid in an almost Kähler space. Therefore the equation (1. 13) reduces to

$$2w^t \nabla_t F_{ji} - Q_{ji} = 0,$$

from which we have

$$w^t \nabla_t F_{ji} = 0,$$

for a covariant almost analytic vector in an almost Kähler space.

Conversely, if we have

$$Q_{ji} = 0 \quad \text{and} \quad w^a \nabla_a F_{ji} = 0$$

for a covariant vector field w_i in an almost Kähler space, then w_i is a covariant almost analytic vector.

In an almost Tachibana space[4], we have

$$\nabla_j F_{ia} + \nabla_i F_{ja} = 0,$$

and the similar argument shows that

$$P_{ji} = 0 \quad \text{and} \quad w^a \nabla_a F_{ji} = 0,$$

if we take account of the equations

$$N_{ji}{}^h = -4(\nabla_j F_i{}^a) F_a{}^h$$

in an almost Tachibana space. Thus we have

THEOREM 1. *A necessary and sufficient condition for a covariant vector field w_i in an almost Kähler or in an almost Tachibana space to be covariant almost analytic is that*

$$w^a \nabla_a F_{ji} = 0, \tag{2.13}$$

4) See, e. g. [2].

(2.14) $$F_j{}^a \nabla_a w_i - F_i{}^a \nabla_j w_a = 0. \quad ({}^*O_{ji}^{ts} \nabla_t w_s = 0\ )$$

If we suppose that $w^h = g^{hi} w_i$ is a contravariant and w_i is a covariant almost analytic vector field in an almost Hermitian space, then adding

$$w^a \nabla_a F_j{}^h - F_j{}^a \nabla_a w^h + F_a{}^h \nabla_j w^a = 0$$

or

$$w^a \nabla_a F_{ji} - F_j{}^a \nabla_a w_i - F_i{}^a \nabla_j w_a = 0$$

and

$${}^*O_{ji}^{ts}(\nabla_t F_{sa} - \nabla_s F_{ta})\, w^a - F_j{}^a \nabla_a w_i + F_i{}^a \nabla_j w_a = 0,$$

we find

(2.15) $${}^*O_{ji}^{ts} F_{tsa} w^a + w^a \nabla_a F_{ji} - 2F_j{}^a \nabla_a w_i = 0.$$

In an almost Kähler space, equation (2.15) reduces to

$$F_j{}^a \nabla_a w_i = 0$$

by virtue of (2.13).

In an almost Tachibana space, (2.15) is written as

$$3w^a\, {}^*O_{ji}^{ts} \nabla_a F_{ts} + w^a \nabla_a F_{ji} - 2F_j{}^a \nabla_a w_i = 0$$

or

$$F_j{}^a \nabla_a w_i = 0$$

because of ${}^*O_{ji}^{ts} \nabla_a F_{ts} = 0$ and (2.14). Thus we have

THEOREM 2. *If, in an almost Kähler or almost Tachibana space, w_i is a contravariant and at the same time covariant almost analytic vector field, then it is covariantly constant.*

The equation (2.3) is written as

(2.16) $${}^*O_{ji}^{ts}(\nabla_t \widetilde{w}_s - \nabla_s \widetilde{w}_t) = F_j{}^a\, {}^*O_{ai}^{ts}(\nabla_t w_s - \nabla_s w_t),$$

where

(2.17) $$\widetilde{w}_i = F_i{}^a w_a.$$

The equation (2.16) may also be written as

$$-{}^*O_{ji}^{ts}(\nabla_t w_s - \nabla_s w_t) = F_j{}^a\, {}^*O_{ai}^{ts}(\nabla_t \tilde{w}_s - \nabla_s \tilde{w}_t)\,. \tag{2.18}$$

The equations (2.17) and (2.18) give

THEOREM 3. *If a vector field w_i in an almost Hermitian space is covariant almost analytic, then the vector field $\tilde{w}_i = F_i{}^a w_a$ is also covariant almost analytic.*

If vectors w_i and $\tilde{w}_i$ are both closed, or more weakly, $\nabla_j w_i - \nabla_i w_j$ and $\nabla_j \tilde{w}_i - \nabla_i \tilde{w}_j$ are both pure, then the equation (2.16) is satisfied. Thus we have

THEOREM 4. *If vectors w_i and $\tilde{w}_i = F_i{}^a w_a$ in an almost Hermitian space are both closed, or more weakly $\nabla_j w_i - \nabla_i w_j$ and $\nabla_j \tilde{w}_i - \nabla_i \tilde{w}_j$ are both pure, then they are both covariant almost analytic vectors.*

The equation (1.13) reduces to

$$\nabla_j w_i - \nabla_i w_j - \frac{1}{2} N_{ji}{}^t w_t = F_i{}^t(\nabla_t \tilde{w}_j - \nabla_j \tilde{w}_t)$$

in an almost Hermitian space. Thus we have

THEOREM 5. *If, in an almost Hermitian space, a covariant almost analytic vector w_i and $\tilde{w}_i$ are both closed, then w_i satisfies*

$$N_{ji}{}^a w_a = 0\,.$$

Applying $g^{ji}\nabla_i$ *to* $F_j{}^a \tilde{w}_a = -w_j$, we find

$$-g^{ji}\nabla_i w_j = F^a \tilde{w}_a - F^{ja}\nabla_j \tilde{w}_a\,, \quad (F^a = g^{ji}\nabla_j F_i{}^a)$$

from which, together with Theorem 3 and (2.7), we have

THEOREM 6. *If, in an almost Hermitian space with $F^i = 0$, a covariant almost analytic vector field w_i is closed, then it is harmonic.*

Now transvecting ${}^*O_{ml}^{ji}(\nabla^m F^{lc} + \nabla^l F^{mc})$ to the equation (2.3), we have

$$-F_j{}^a(\nabla_a w_i)\, {}^*O_{ml}^{ji}(\nabla^m F^{lc} + \nabla^l F^{mc}) + F_i{}^a(\nabla_j w_a)\, {}^*O_{ml}^{ji}(\nabla^m F^{lc} + \nabla^l F^{mc}) = 0\,.$$

A straightforward computation shows that the first term of the equation above is zero.

Consequently we have

$$F_a{}^i(\nabla^j w^a)\, {}^*O_{ji}^{ml} G_{mlb} w^b = 0 \tag{2.19}$$

for a covariant almost analytic vector in an almost Hermitian space.

Applying $F_a{}^i \nabla^j$ to (2. 5) and changing indices, we have

$$\begin{aligned}\nabla^a\nabla_a w_i - K^*{}_{ji} w^j &+ (\nabla^b w^a)(F_{la}\nabla_b F_i{}^l + F_{ia}F_b) \\ &+ \frac{1}{2} w^a(F^b F_{bia} + F_b{}^t F_c{}^s F_{tsa}\nabla^b F_i{}^c) \\ &- F_i{}^j\, {}^*O_{kj}^{ts}\nabla^k(F_{tsa}w^a) = 0 .\end{aligned} \tag{2.20}$$

For T_{ji} defined by

$$T_{ji} = {}^*O_{ji}^{ts}\{(\nabla_t F_s{}^a - \nabla_s F_t{}^a)\, w_a - F_t{}^a\nabla_a w_s + F_s{}^a\nabla_t w_a\} , \tag{2.21}$$

we have the identity

$$\begin{aligned}\nabla^j(T_{ji}F_a{}^i w^a) &+ [\nabla^a\nabla_a w_i - K_{ji}{}^* w^j + F_i{}^c\, {}^*O_{cb}^{ts}\nabla^b(F_{sta}w^a) \\ &+ (\nabla^b w^a)(F_{ca}\nabla_b F_i{}^c + F_{ia}F_b) + \frac{1}{2} w^a(F^b F_{bia} + F_c{}^t F_b{}^s F_{sta}\nabla^b F_i{}^c)]\, w^i \\ &- F_a{}^b(\nabla^j w^a)\, {}^*O_{jb}^{ts} G_{tsi} w^i + \frac{1}{2} T_{ji}T^{ji} = 0 .\end{aligned} \tag{2.22}$$

Thus, in a compact almost Hermitian space, we have

$$\begin{aligned}\int_M [\{\nabla^a\nabla_a w_i - K^*{}_{ji}w^j + F_i{}^c\, {}^*O_{cb}^{ts}\nabla^b(F_{sta}w^a) &+ (\nabla^b w^a)(F_{ca}\nabla_b F_i{}^c + F_{ia}F_b) \\ &+ \frac{1}{2} w^a(F^b F_{bia} + F_c{}^t F_b{}^s F_{sta}\nabla^b F_i{}^c) \\ &- F_a{}^b(\nabla^j w^a)\, {}^*O_{jb}^{ts} G_{tsi}\}\, w^i + \frac{1}{2} T_{ji}T^{ji}]\, d\sigma = 0 ,\end{aligned} \tag{2.23}$$

and consequently

THEOREM 7. *A necessary condition for a vector field w_i in an almost Hermitian space to be covariant almost analytic is that* (2. 19) *and* (2. 20) *are satisfied and a sufficient condition for w_i in a compact almost Hermitian* *space to be covariant almost analytic is*

$$\nabla^a \nabla_a w_i - K^*{}_{ji} w^j + F_i{}^c {}^*O_{cb}^{ts} \nabla^b (F_{sta} w^a) + (\nabla^b w^a)(F_{ca} \nabla_b F_i{}^c + F_{ia} F_b) + \frac{1}{2} w^a (F^b F_{bia} + F_c{}^t F_b{}^s F_{sta} \nabla^b F_i{}^c) - F_a{}^b (\nabla^j w^a) {}^*O_{jb}^{ts} G_{tsi} = 0 . \tag{2.24}$$

COROLLARY 1. *A necessary condition for a covariant vector field* w_i *in an almost Kähler space to be covariant almost analytic is that*

$$F_a{}^i (\nabla^j w^a) w_b {}^*O_{ji}^{ts} G_{ts}{}^b = 0 \tag{2.25}$$

and

$$\nabla^a \nabla_a w_i - K^*{}_{ji} w^j + (\nabla^b w^a) F_{ca} \nabla_b F_i{}^c = 0 \tag{2.26}$$

are satisfied and a sufficient condition for w_i *in a compact almost Kähler space to be covariant almost analytic is*

$$\nabla^a \nabla_a w_i - K^*{}_{ji} w^j + (\nabla^b w^a) F_{ca} \nabla_b F_i{}^c - F_a{}^b (\nabla^c w^a) {}^*O_{cb}^{ts} G_{tsi} = 0 . \tag{2.27}$$

The equation (2.26) can be written as

$$\nabla^a \nabla_a w_i - w^j F_i{}^k \nabla_a \nabla_j F_k{}^a - K_{ji} w^j + (\nabla^b w^a) F_{ca} \nabla_b F_i{}^c = 0 .$$

On the other hand, we have

$$\begin{aligned} & w^j F_i{}^k \nabla_a \nabla_j F_k{}^a + (\nabla^b w^a) F_{ca} \nabla_b F_i{}^c \\ & = F_i{}^k (\nabla_a w^j)(\nabla_j F_k{}^a) + (\nabla^b w^a) F_{ca} \nabla_b F_i{}^c \\ & = (\nabla^b w^a)(F_{ca} \nabla_b F_i{}^c + F_i{}^c \nabla_a F_{cb}) \\ & = -(\nabla^b w^a) F_i{}^c \nabla_c F_{ba} \end{aligned}$$

and consequently, taking account of Theorem 3 and (2.13),

$$(\nabla^a \nabla_a w_i - K_{ji} w^j) w^i = 0 .$$

Thus the integral formula (K. Yano [2])

$$\int_M [(\nabla^a \nabla_a w_i - K_{ji} w^j) w^i + \frac{1}{2} (\nabla^j w^i - \nabla^i w^j)(\nabla_j w_i - \nabla_i w_j) + (\nabla_j w^j)(\nabla_i w^i)] \, d\sigma = 0 \tag{2.28}$$

shows that

$$\nabla_j w_i - \nabla_i w_j = 0, \qquad \nabla_i w^i = 0,$$

that is, w_i is harmonic. Thus we have

COROLLARY 2. *A covariant almost analytic vector in a compact almost Kähler space is harmonic.*

For a covariant almost analytic vector field w_i in an almost Tachibana space, we have, taking account of (2.13),

$$(\nabla^b w^a) F_i^{\ c} \nabla_b F_{ca} = K^*{}_{ji} w^j - K_{ji} w^j, \tag{2.29}$$

from which we find

COROLLARY 3. *A necessary condition for a vector field w_i in an almost Tachibana space to be covariant almost analytic is that*

$$\nabla^a \nabla_a w_i - 2K^*{}_{ji} w^j + K_{ji} w^j = 0 \tag{2.30}$$

are satisfied and a sufficient condition for w_i in a compact almost Tachibana space to be covariant almost analytic is

$$\nabla^a \nabla_a w^i - K^*{}_{ji} w^j + (\nabla^b w^a) F_{ca} \nabla_b F_i^{\ c} + \frac{3}{2} w^a \nabla_a F_{cb} \nabla^b F_i^{\ c} - 3F_i^{\ b} \, {}^*O_{cb}^{ts} \nabla^c (\nabla_t F_{sa} w^a) = 0.$$

From (2.29) we have

$$(K^*{}_{ji} - K_{ji}) w^j w^i = 0$$

and consequently, taking account of (2.28) and (2.29), we have

COROLLARY 4. *A covariant almost analytic vector in a compact almost Tachibana space is harmonic.*

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