Tôhoku Math. Journ.
Vol. 19, No. 2, 1967

ON THE EXISTENCE OF O-CURVES II *)

JUNJI KATO

(Received November 19, 1966)

Cooke [1] has discussed asymptotic behaviors of solutions of a functional differential equation

$$(1) \qquad \dot{u}(t) + au(t - r(t)) = 0$$

under the assumption that $r(t)$ is a non-negative continuous function which satisfies the conditions

$$r(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad \int_0^\infty r(t)\,dt < \infty .$$

In the previous paper [3], we have obtained some results concerning the existence of O-curves and some kind of the asymptotic equivalence, which we shall call the asymptotic semi-equivalence (for the definition, see the below). By applying the similar arguments to those used in [3], we shall discuss the same problems as discussed by Cooke, for more general equations.

Here, we shall give the following definitions:

DEFINITION 1. A solution of a system will be called to be *an O-curve* of the system, if it tends to zero as $t \to \infty$.

DEFINITION 2. Two systems (E_1) and (E_2) are said to be *asymptotically semi-equivalent*, provided that for any bounded solution of (E_1) (or (E_2)) we can find a solution of (E_2) (or (E_1)) which approaches the bounded solution of (E_1) (or (E_2), respectively) for infinitely increasing t. In the case where we can remove the boundedness for the given solution, two systems (E_1) and (E_2) are *asymptotically equivalent* (cf. [2]).

Let $r \geqq 0$ be a given constant. C^n denotes the space of continuous functions mapping the interval $[-r, 0]$ into the Euclidean n-space E^n with a norm $\|\varphi\|_r$ defined by

*) This work was partially supported by the Sakkokai Foundations.

$$\|\varphi\|_r = \sup\{\|\varphi(\theta)\|\ ;\ \theta \in [-r, 0]\}\ ,$$

where $\|\varphi(\theta)\|$ is a Euclidean norm. For an E^n-valued continuous function $x(t)$, $\dot{x}(t)$ denotes the right-hand derivative and we represent by x_t the function in C^n such that

$$x_t(\theta) = x(t+\theta),\quad \theta \in [-r, 0]\,.$$

Let M be an (m, n)-matrix. Then, $\|M\|$ denotes the supremum of $\|Mx\|$ for all $x \in E^n$ such that $\|x\| = 1$.

Our purpose is to discuss the existence of O-curves of a system

$$(2)\qquad \dot{x}(t) = Ax(t) + B(t)\{x(t) - x(t-r(t))\} + f(t, x_t)$$

and the asymptotic semi-equivalence between the system (2) and the system

$$(3)\qquad \dot{x}(t) = Ax(t)\,,$$

where x is an n-vector and A is a real constant (n, n)-matrix.

Throughout this paper, the following assumptions will be made:

(i) *An (n, n)-matrix $B(t)$ is continuous and bounded on* $[0, \infty)$.

(ii) *$r(t)$ is a continuous function which satisfies the condition*

$$0 \leqq r(t) \leqq r \qquad \textit{for all}\quad t \geqq 0$$

and

$$\int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{p-1}}^\infty r(t_p)\, dt_p\, dt_{p-1} \cdots dt_1 < \infty\,,$$

where r is the constant given in the definition of C^n and p is a positive integer which will be determinded below.

(iii) *$f(t, \varphi)$ is defined and continuous on $[0, \infty) \times C^n$, and for any $\alpha \geqq 0$ there exists a continuous function $\lambda(t, \alpha)$ which is bounded uniformly in $t \in [0, \infty)$ and satisfies the conditions*

$$\int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{p-1}}^\infty \lambda(t_p, \alpha)\, dt_p\, dt_{p-1} \cdots dt_1 < \infty$$

and

$$\|f(t, \varphi)\| \leqq \lambda(t, \alpha), \qquad \textit{if}\quad \|\varphi\|_r \leqq \alpha\,,$$

where p is the same one as in (ii).

Here, p is determinded in the following way; if A has characteristic roots with zero real part, then p is the maximum degree of the elementary divisors corresponding to such roots, and otherwise $p=1$.

THEOREM 1. *Under the assumptions* (i) *through* (iii), *there exists an O-curve of the system* (2).

PROOF. As was shown in [3], we can find a non-singular (n, n)-matrix $P(t)$ such that both of $P(t)$ and $P(t)^{-1}$ are continuous and bounded on $[0, \infty)$ and that by the transformation

$$x = P(t)y, \quad y = (u, v, w_1, \cdots, w_{p-1}), \tag{4}$$

the system (3) is transformed into a system

$$\dot{u}(t) = A_1 u(t), \ \dot{v}(t) = A_2 v(t), \ \dot{w}_j(t) = C_j w_{j-1}(t), \tag{5}$$

where and in the followings j stands for 1 through $p-1$, u, v, w_j are k, m, n_j-vectors, $n_1+n_2+\cdots+n_{p-1}=n-k-m$, and $w_0=v$. Here, A_1, A_2 and C_j are constant matrices, and all characteristic roots of A_1 have negative real parts, those of A_2 have non-negative real parts and the elementary divisor corresponding to each characteristic root of A_2 with zero real part is linear. In the above, if $p=1$, the system (5) becomes

$$\dot{u}(t) = A_1 u(t), \quad \dot{v}(t) = A_2 v(t)$$

and $y=(u, v)$. Moreover, we can find two continuous Liapunov functions $V(t, u)$ and $W(t, v)$ as follows; $V(t, u)$ is defined on $[0, \infty) \times E^k$ and satisfies the conditions

$$\|u\| \leqq V(t, u) \leqq K\|u\|, \ |V(t, u) - V(t, u')| \leqq K\|u-u'\|,$$

$$\overline{\lim_{\delta \to +0}} \frac{1}{\delta} \{V(t+\delta, u+\delta A_1 u) - V(t, u)\} \leqq -cV(t, u),$$

and $W(t, v)$ is defined on $[0, \infty) \times E^m$ and satisfies the conditions

$$\|v\| \leqq W(t, v) \leqq K\|v\|, \ |W(t, v) - W(t, v')| \leqq K\|v-v'\|,$$

$$\lim_{\delta \to +0} \frac{1}{\delta} \{W(t+\delta, v+\delta A_2 v) - W(t, v)\} \geqq 0,$$

where K and c are positive constants.

By the transformation (4), the system (2) is transformed into a system

$$(6) \qquad \begin{cases} \dot{u}(t) = A_1 u(t) + B_1(t)\{y(t) - y(t-r(t))\} + f_1(t, y_t) \\ \dot{v}(t) = A_2 v(t) + B_2(t)\{y(t) - y(t-r(t))\} + f_2(t, y_t) \\ \dot{w}_j(t) = C_j w_{j-1}(t) + D_j(t)\{y(t) - y(t-r(t))\} + g_j(t, y_t) . \end{cases}$$

Clearly, $B_1(t)$, $B_2(t)$, $D_j(t)$ are continuous and bounded on $[0, \infty)$, and for any $\alpha > 0$ we can find a continuous function $\lambda^*(t, \alpha)$ such that

$$\int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{p-1}}^\infty \lambda^*(t_p, \alpha)\, dt_p dt_{p-1} \cdots dt_1 < \infty$$

and that

$$\| f_1(t, \varphi)\| , \ \| f_2(t, \varphi)\| , \ \|g_j(t, \varphi)\| \leqq \lambda^*(t, \alpha) \quad \text{if} \quad \|\varphi\|_r \leqq \alpha ,$$

and moreover, there exists a continuous function $\lambda_0(\alpha)$ such that $\lambda^*(t, \alpha) \leqq \lambda_0(\alpha)$, because we assume that $\lambda(t, \alpha)$ is bounded uniformly in $t \in [0, \infty)$. Let C_0 and B_0 be chosen so that $\|C_i\| \leqq C_0$ and

$$\|B_1(t)\| , \ \|B_2(t)\| , \ \|D_j(t)\| \leqq B_0$$

for all $t \geqq 0$. For a given $\alpha > 0$, set

$$\beta(\alpha) = \{\lambda_0(4K\alpha) + 8B_0 K\alpha\}(p+1) + \{2\|A_1\| + \|A_2\| + (p-1)C_0\}K\alpha .$$

By our assumptions, we can choose a $T(\alpha)$ so that

$$R_1(T(\alpha), \alpha) < \alpha$$

and

$$\sum_{j=1}^{p-1} \left\{ C_0{}^j K R_{j+1}(T(\alpha), \alpha) + \sum_{l=1}^{j} C_0{}^{l-1} R_l(T(\alpha), \alpha) \right\} < K\alpha ,$$

where

$$R_j(t, \alpha) = \int_t^\infty \int_{t_1}^\infty \cdots \int_{t_{j-1}}^\infty \{B_0 \beta(\alpha)\, r(t_j) + \lambda^*(t_j, 4K\alpha)\}\, dt_j dt_{j-1} \cdots dt_1 .$$

Let $f_1{}^*(t, \varphi)$, $f_2{}^*(t, \varphi)$ and $g_j{}^*(t, \varphi)$ be defined by replacing (t, φ) in $f_1(t, \varphi)$, $f_2(t, \varphi)$ and $g_j(t, \varphi)$, respectively, by $(t, \min\{1, 4K\alpha/\|\varphi\|_r\}\varphi)$, and let $F_1(t, \varphi)$, $F_2(t, \varphi)$ and $G_j(t, \varphi)$ be defined by

$$F_1(t, \varphi) = \min\left\{1, \frac{B_0\beta(\alpha)\, r(t)}{\|B_1(t)\{\varphi(0)-\varphi(-r(t))\}\|}\right\} B_1(t)\{\varphi(0) - \varphi(-r(t))\}\ ,$$

$$F_2(t, \varphi) = \min\left\{1, \frac{B_0\beta(\alpha)\, r(t)}{\|B_2(t)\{\varphi(0)-\varphi(-r(t))\}\|}\right\} B_2(t)\{\varphi(0) - \varphi(-r(t))\}\ ,$$

$$G_j(t, \varphi) = \min\left\{1, \frac{B_0\beta(\alpha)\, r(t)}{\|D_j(t)\{\varphi(0)-\varphi(-r(t))\}\|}\right\} D_j(t)\{\varphi(0) - \varphi(-r(t))\}\ ,$$

where if the denominator in the minimum sign is zero, then we understand that the minimum is 1. Obviously, f_1^*, f_2^*, g_j^*, F_1, F_2 and G_j are continuous and satisfy

$$\|f_1^*(t, \varphi)\|,\ \|f_2^*(t, \varphi)\|,\ \|g_j^*(t, \varphi)\| \leqq \lambda^*(t, 4K\alpha)\,,$$
$$\|F_1(t, \varphi)\|,\ \|F_2(t, \varphi)\|,\ \|G_j(t, \varphi)\| \leqq B_0\beta(\alpha)\, r(t)$$

for all $(t, \varphi) \in [0, \infty) \times C^n$.

Consider the system

$$(7)\qquad \begin{cases} \dot{u}(t) = A_1^*(u(t)) + F_1(t, y_t) + f_1^*(t, y_t) \\ \dot{v}(t) = A_2^*(v(t)) + F_2(t, y_t) + f_2^*(t, y_t) \\ \dot{w}_j(t) = C_j^*(w_{j-1}(t)) + G_j(t, y_t) + g_j^*(t, y_t)\,, \end{cases}$$

where

$$A_1^*(u) = \min\left\{1, \frac{2K\alpha}{\|u\|}\right\} A_1 u,$$

$$A_2^*(v) = \min\left\{1, \frac{K\alpha}{\|v\|}\right\} A_2 v,$$

$$C_j^*(w_{j-1}) = \min\left\{1, \frac{K\alpha}{\|w_{j-1}\|}\right\} C_j w_{j-1}\,.$$

Since the right-hand sides of the systm (7) are continuous and bounded on $[0, \infty) \times C^n$, for any given τ, $s \in [0, \infty)$, $\tau < s$, and for any $\xi \in E^k$ we can find a solution of the system (7) such that

$$u(\tau) = \xi\,, \quad v(s) = 0\,, \quad w_j(s) = 0$$

and that $y(\tau+\theta) = (u(\tau+\theta), v(\tau+\theta), w_1(\tau+\theta), \cdots, w_{p-1}(\tau+\theta))$ is a constant for $\theta \in [-r, 0]$ (refer Theorem 1 in [2]).

By the similar arguments to those used in [3] and by using the Liapunov functions $V(t, u)$ and $W(t, v)$, we have

$$(8)\qquad \begin{gathered} \|u(t)\| \leqq K\alpha e^{-c(t-\tau)} + KR_1(t, \alpha),\ \|v(t)\| \leqq KR_1(t, \alpha) \\ \|w_j(t)\| \leqq C_0{}^j KR_{j+1}(t, \alpha) + \sum_{l=1}^{j} C_0{}^{l-1} R_l(t, \alpha) \end{gathered}$$

and

$$(9)\qquad \|u(t)\| < 2K\alpha,\ \|v(t)\| < K\alpha,\ \|w(t)\| < K\alpha$$

for all t, $\tau \leqq t \leqq s$, if $\|\xi\| \leqq \alpha$ and $\tau \geqq T(\alpha)$, where $w=(w_1, \cdots, w_{p-1})$. From (9), we obtain

$$(10)\qquad \|y(t)\| \leqq \|u(t)\| + \|v(t)\| + \|w(t)\| < 4K\alpha \quad \text{for all } t,\ \tau \leqq t \leqq s.$$

Now, we shall show that $y(t) = (u(t), v(t), w(t))$ is a solution of the system (6). Since $y(\tau+\theta) = y(\tau)$ for $\theta \in [-r, 0]$, by (10) we have $\|y(t)\| < 4K\alpha$ on $[\tau-r, s]$ which implies that $\|y_t\|_r < 4K\alpha$ for all t, $\tau \leqq t \leqq s$. Hence,

$$f_1^*(t, y_t) = f_1(t, y_t),\ f_2^*(t, y_t) = f_2^*(t, y_t),\ g_j^*(t, y_t) = g_j(t, y_t)$$

for all t, $\tau \leqq t \leqq s$, and

$$\|F_1(t, y_t)\|,\ \|F_2(t, y_t)\|,\ \|G_j(t, y_t)\| \leqq B_0\{\|y(t)\| + \|y(t-r(t))\|\} < 8B_0K\alpha,$$

which implies that $\|\dot{y}(t)\| < \beta(\alpha)$ for all t, $\tau - r \leqq t \leqq s$, because y_τ is a constant, $\|A_1^*(u)\| \leqq 2\|A_1\|K\alpha$, $\|A_2^*(v)\| \leqq \|A_2\|K\alpha$, $\|C_j^*(w_{j-1})\| \leqq C_0K\alpha$ and $\|f_1^*(t, \varphi)\|$, $\|f_2^*(t, \varphi)\|$, $\|g_j^*(t, \varphi)\| \leqq \lambda^*(t, 4K\alpha) \leqq \lambda_0(4K\alpha)$. On the other hand, since $F_1(t, y_t)$, $F_2(t, y_t)$ and $G_j(t, y_t)$ are bounded by $B_0\|y(t) - y(t-r(t))\|$ and $\|\dot{y}(t)\| < \beta(\alpha)$, we have

$$\|F_1(t, y_t)\|,\ \|F_2(t, y_t)\|,\ \|G_j(t, y_t)\| \leqq B_0\beta(\alpha)r(t),$$

which shows that

$$\begin{gathered} F_1(t, y_t) = B_1(t)\{y(t) - y(t-r(t))\},\ F_2(t, y_t) = B_2(t)\{y(t) - y(t-r(t))\}, \\ G_j(t, y_t) = D_j(t)\{y(t) - y(t-r(t))\}. \end{gathered}$$

Clearly,

$$A_1^*(u(t)) = A_1u(t),\ A_2^*(v(t)) = A_2v(t),$$

$$C_j^*(w_{j-1}(t)) = C_j w_{j-1}(t)$$

and hence $y(t) = (u(t), v(t), w(t))$ is a solution of the system (6). Thus, we can show the existence of an O-curve of the system (6) by the inequalities (8) and by the same arguments as used in [3], which implies the existence of an O-curve of the system (2). The proof is completed.

Now, we shall discuss the asymptotic semi-equivalence of the system (2) and (3) under the assumptions (i) through (iii).

Let $x^*(t)$ be a bounded solution of the system (3) starting at $t=t_0$. Then, there exists a constant $B^* > 0$ such that

$$\|x^*(t)\| \leqq B^* \quad \text{for all } t \geqq t_0 . \tag{11}$$

By the transformation

$$x(t) = y(t) + x^*(t), \tag{12}$$

the system (2) is transformed into a system

$$\dot{y}(t) = Ay(t) + B(t)\{y(t) - y(t-r(t))\} + g(t, y_t),$$

where

$$g(t, \varphi) = B(t)\{x^*(t) - x^*(t-r(t))\} + f(t, x^*_t+\varphi).$$

Since $\|\dot{x}^*(t)\| \leqq \|A\|B^*$ for all $t \geqq t_0$, we have

$$\|g(t, \varphi)\| \leqq \|B(t)\|\,\|A\|B^*r(t) + \lambda(t, \alpha+B^*) \qquad \text{for all } t \geqq t_0 + r$$

if $\|\varphi\|_r \leqq \alpha$, and hence, $g(t, \varphi)$ satisfies the similar condition to the condition (iii) for $f(t, \varphi)$. Conversely, if $x^*(t)$ is a solution of the system (2) satisfying the condition (11) for a constant B^*, then by the transformation (12), the system (3) is transformed into the system

$$\dot{y}(t) = Ay(t) + g(t),$$

where

$$g(t) = -B(t)\{x^*(t) - x^*(t-r(t))\} - f(t, x^*_t).$$

Bacause $\|\dot{x}^*(t)\| \leqq \beta^*$ for all $t \geqq t_0 + r$, where

$$\beta^* = \sup_{t \geqq t_0} \{\|A\| B^* + 2\|B(t)\| B^* + \lambda(t, B^*)\} ,$$

we have

$$\|g(t)\| \leqq \|B(t)\| \beta^* r(t) + \lambda(t, B^*) \qquad \text{for all } t \geqq t_0 + 2r ,$$

which implies that

$$\int_{t_0+2r}^{\infty} \int_{t_1}^{\infty} \cdots \int_{t_{p-1}}^{\infty} \|g(t_p)\| \, dt_p \, dt_{p-1} \cdots dt_1 < \infty .$$

Thus, by Theorem 1 we can prove the following theorem.

THEOREM 2. *Under the assumptions* (i) *through* (iii), *the systems* (2) *and* (3) *are asymptotically semi-equivalent.*

By the definition, it is obvious that if the systems (2) and (3) are asymptotically semi-equivalent and if all solutions of these systems are bounded in the future, then the systems (2) and (3) are asymptotically equivalent.

Here, we shall state lemmas concerning the boundedness of solutions of the system (2).

LEMMA 1. *In the system*

$$\dot{x}(t) = Ax(t) + B(t)\{x(t) - x(t-r(t))\} , \tag{13}$$

we assume the conditions (i) *and* (ii) *with* $p=1$. *Furthermore, suppose that*

(iv) *all solutions of the system* (3) *are bounded in the future.*

Then, there exists a continuous Liapunov functional $U(t, \varphi)$ *defined on* $[0, \infty) \times C^n$ *which satisfies the following conditions*;

$$\|\varphi\|_r \leqq U(t, \varphi) \leqq K \|\varphi\|_r , \tag{1°}$$

$$|U(t, \varphi) - U(t, \varphi')| \leqq K \|\varphi - \varphi'\|_r , \tag{2°}$$

$$\overline{\lim_{\delta \to +0}} \frac{1}{\delta} \{U(t+\delta, x_{t+\delta}) - U(t, \varphi)\} \leqq 0 , \tag{3°}$$

where K *is a constant and* $x(s)$ *is a solution of the system* (13) *through* (t,φ).

PROOF. Since the system (13) is linear, we can find a desired Liapunov functional, if there exists a constant K such that for any $(t_0, \varphi_0) \in [0, \infty) \times C^n$

the solution $x(t;\varphi_0, t_0)$ of the system (13) through (t_0, φ_0) satisfies the inequality

$$(14) \qquad \|x_t(\varphi_0, t_0)\|_r \leqq K\|\varphi_0\|_r \qquad \text{for all } t \geqq t_0$$

(cf. Theorem 33.4 in [4]). Hence, it is sufficient to show the existence of a constant K for which we have the inequality (14).

Now, we shall prove this. Let $X(t)$ be the fundamental matrix of the system (3) such that $X(0)$ is the unit (n, n)-matrix. By the assumption (iv), $X(t)$ is bounded in the future, that is, there exists a constant $M_1 > 1$ such that $\|X(t)\| \leqq M_1$ for all $t \geqq 0$. Let $M_2 > 0$ be the supremum of $\|B(t)\|$ on $[0, \infty)$. Since

$$(15) \quad x(t;\varphi_0, t_0)$$
$$= X(t-t_0)\varphi_0(0) + \int_{t_0}^{t} X(t-\tau)B(\tau)\{x(\tau;\varphi_0, t_0) - x(\tau-r(\tau);\ \varphi_0, t_0)\}\, d\tau$$

for any (t_0, φ_0) and for all $t \geqq t_0$, we have

$$(16) \qquad \|x(t;\varphi_0, t_0)\| < M_1\|\varphi_0\|_r \exp[2M_1M_2(t-t_0)]$$

for all $t \geqq t_0$ if $\|\varphi_0\|_r \neq 0$. In fact, if not, then there exists a $t_1 > t_0$ such that

$$\|x(t_1;\varphi_0,t_0)\| = M_1\|\varphi_0\|_r \exp[2M_1M_2(t_1-t_0)]$$

and that we have the inequality (16) for all t, $t_0 \leqq t < t_1$.

Here, easily we can see

$$\|x(\tau-r(\tau);\varphi_0, t_0)\| \leqq M_1\|\varphi_0\|_r e^{2M_1M_2(\tau-t_0)}$$

for all $\tau \in [t_0, t_1]$. Hence, by (15), we have

$$\|x(t;\varphi_0, t_0)\| < M_1\|\varphi_0\|_r \left\{1 + \int_{t_0}^{t} 2M_1M_2 \exp[2M_1M_2(\tau-t_0)]d\tau\right\}$$
$$< M_1\|\varphi_0\|_r \exp[2M_1M_2(t-t_0)]$$

for all t, $t_0 \leqq t \leqq t_1$ from which there arises a contradiction. The inequality (16) implies that all solutions of the system (13) are continuable in the future. Let us choose a $T \geqq 0$ so that

$$\int_{T}^{\infty} r(t)\, dt < \frac{1}{2M_1M_2(\|A\| + 2M_2)},$$

and set

$$\alpha = M_1\|\varphi_0\|_r \exp[2M_1M_2 \max(2r, T)]$$

for $\|\varphi_0\|_r \neq 0$. Clearly, for given $(t, \varphi_0) \in [0, \infty)\times C^n$ we have

$$\|x(t;\varphi_0, t_0)\| < \alpha \qquad \text{for all } t, t_0 - r \leqq t \leqq t^* \tag{17}$$

by the inequality (16), where

$$t^* = \max(t_0+2r, T).$$

Suppose that $x(t)=x(t;\varphi_0, t_0)$ is bounded by $\beta(\alpha) = 2M_1\alpha$ on $[t_0, s]$ for an $s > t^*$. Then, we have

$$\|\dot{x}(\tau)\| \leqq \{\|A\|+2M_2\}\,\beta(\alpha) \qquad \text{for all } \tau \in [t_0+r, s)$$

by the equations (13). Since

$$x(t) = X(t-t^*)\,x(t^*) + \int_{t^*}^{t} X(t-\tau)B(\tau)\{x(\tau)-x(\tau-r(\tau))\}\,d\tau,$$

we have

$$\|x(t)\| \leqq M_1\|x(t^*)\| + M_1M_2(\|A\|+2M_2)\,\beta(\alpha)\int_{t^*}^{t} r(\tau)\,d\tau < \beta(\alpha)$$

for all $t, t^* \leqq t \leqq s$, which implies that

$$\|x(t)\| \leqq 2M_1\alpha \qquad \text{for all } t \geqq t^*.$$

From this and the inequality (17), it follows that

$$\|x_t\|_r \leqq 2M_1\alpha \qquad \text{for all } t \geqq t_0,$$

that is, we have the inequality (14), where

$$K = 2M_1^2 \exp[2M_1M_2 \max(2r, T)].$$

LEMMA 2. *In addition to the assumptions* (i), (ii) *with* $p=1$ *and* (iv), *we assume that*

(v) *there exist continuous functions* $\lambda(t) \geqq 0$ *and* $\eta(\alpha) > 0$ *such that* $\eta(\alpha)$ *is non-decreasing,* $\lambda(t)$ *is bounded,*

$$\int_0^\infty \lambda(t)\,dt < \infty\,, \quad \int_0^\infty \frac{d\alpha}{\eta(\alpha)} = \infty$$

and that

$$\|f(t, \varphi)\| \leqq \lambda(t)\,\eta(\|\varphi\|_r)$$

for all $(t, \varphi) \in [0, \infty) \times C^n$.

Then, all solutions of the system (2) *are bounded in the future.*

PROOF, By Lemma 1, we can find a continuous Liapunov functional $U(t, \varphi)$ defined on $[0, \infty) \times C^n$ which satisfies the conditions (1°) through (3°) given in Lemma 1. Let $x(t)$ be a solution of the system (2) through (t_0, φ_0). Then, by calculating the upper right derivative of $U(t, x_t)$, we have

$$\dot{U}(t, x_t) \leqq K\lambda(t)\,\eta(\|x_t\|_r) \leqq K\lambda(t)\,\eta(U(t, x_t))$$

for all $t \geqq t_0$, as long as $x(t)$ exists. Hence, comparing $U(t, x_t)$ with the maximum solution of the equation

$$\frac{dv}{dt} = K\lambda(t)\,\eta(v), \quad v(t_0) = U(t_0, \varphi_0)\,,$$

we can see the boundedness of $x(t)$. Thus, we complete the proof of this lemma.

Combining Theorem 2 with Lemma 2, immediately we have the following theorem. Here, it should be noted that under the assumption (iv), the elementary divisor corresponding to each characteristic root of A with zero real part is linear, even if such a root exists.

THEOREM 3. *Under the assumptions in Lemma* 2, *the systems* (2) *and* (3) *are asymptotically equivalent.*

Now, consider a system

$$\dot{u}(t) + Bu(t - r(t)) = 0\,, \tag{18}$$

where u is an n-vector and B is a constant (n, n)-matrix, and assume the condition (ii) with $p=1$. Then, obviously the equation (1) is a special case of the system (18), where $n=1$ and $B=a$. Put

$$x(t) = e^{Bt} u(t) .$$

Then, the system (18) is transformed into the system

$$\dot{x}(t) = B\{x(t)-x(t-r(t))\} + B(E-e^{Br(t)})\, x(t-r(t)) , \tag{19}$$

because the matrices B and e^{Bt} are commutative and also so are Bt and $Br(t)$, where E is the unit (n, n)-matrix. Since

$$\|E-e^{Br(t)}\| \leqq \|B\| e^{\|B\|r} r(t) ,$$

the equation (19) is a special form of the system (2) and the conditions (i), (ii) with $p=1$, (iv) and (v) are satisfied. Hence, by [illegible] the system (19) and the system

$$\dot{x}(t) = 0$$

are asymptotically equivalent. Namely, we have the following corollary of Theorem 3.

COROLLARY 1. *Suppose that*

(vi) $r(t)$ *is a non-negative, continuous and bounded function on* $[0, \infty)$ *such that*

$$\int_0^\infty r(t)\, dt < \infty .$$

Then, for any solution $u(t)$ *of the system* (18), *there exists a constant* n*-vector* c *such that*

$$e^{Bt} u(t) \to c \quad as \quad t \to \infty , \tag{20}$$

and conversely for any given constant n*-vector* c *we can show the existence of a solution* $u(t)$ *of the system* (18) *which satisfies the condition* (20).

Similarly, we have the following corollary of Theorem 3.

any (n, n)-matrix function $G(t)$ satisfying the condition

(vii) *$G(t)$ is continuous and bounded on $[-r, \infty)$, and any two of the matrices B, $G(t)$, $\int_0^t G(\tau)\, r(\tau)\, d\tau$, $\int_0^s G(\tau)\, r(\tau)\, d\tau$ are commutative for any $t \geqq -r$ and $s \geqq -r$*

and for any solution $u(t)$ of the system (18), *there exists a constant n-vector c such that*

$$\exp\left[\int_0^t \{B + G(\tau)\, r(\tau)\}\, d\tau\right] u(t) \to c \quad as \quad t \to \infty\,. \tag{21}$$

Furthermore, for any constant n-vector c and for any (n, n)-matrix function $G(t)$ satisfying the condition (vii), *we can find a solution $u(t)$ of the system* (18) *which has the property* (21).

PROOF. Let $r(t) = r(0)$ for $t \in [-r, 0]$, and let $S(t)$ be the (n, n)-matrix function defined by

$$S(t) = \int_0^t \{B + G(\tau)\, r(\tau)\}\, d\tau$$

for a given function $G(t)$ which satisfies the condition (vii). Then, clearly we have

$$\left\{\begin{aligned} &\frac{d}{dt}\exp[S(t)] = (B + G(t)\, r(t))\exp[S(t)]\,, \\ &\exp[S(t)] \cdot B = B\exp[S(t) - S(t - r(t))] \cdot \exp[S(t - r(t))] \end{aligned}\right. \tag{22}$$

by the condition (vii). Let

$$x(t) = \exp[S(t)]\, u(t)\,.$$

Then, noting the relations (22), the system (18) is transformed into the system

$$\begin{aligned} \dot{x}(t) = {} & B\{x(t) - x(t - r(t))\} + G(t) r(t) x(t) \\ & + B\{E - \exp[S(t) - S(t - r(t))]\}\, x(t - r(t))\,. \end{aligned} \tag{23}$$

Since

$$S(t)-S(t-r(t))=\int_{t-r(t)}^{t}\{B+G(\tau)\,r(\tau)\}\,d\tau=Br(t)+\int_{t-r(t)}^{t}G(\tau)\,r(\tau)\,d\tau\,,$$

we have

$$\|E-\exp\,[S(t)-S(t-r(t))]\|\leqq\exp[\|B\|r+G_0r^2]\cdot(\|B\|r(t)+G_0\,rr(t))\,,$$

where G_0 and r are non-negative constants such that

$$\|G(t)\|\leqq G_0 \quad\text{and}\quad r(t)\leqq r \quad\text{for all}\quad t\geqq 0\,,$$

which shows that the function

$$f(t,\varphi)=G(t)r(t)\varphi(0)+B\{E-\exp\,[S(t)-S(t-r(t))]\}\,\varphi(-r(t))$$

satisfies the inequality

$$\|f(t,\varphi)\|\leqq\lambda(t)\|\varphi\|_r,$$

$$\lambda(t)=G_0r(t)+\|B\|\exp\,[\|B\|r+G_0r^2]\{\|B\|r(t)+G_0\,rr(t)\}\,,$$

that is, the condition (v) holds good by the assumption (vi). Hence, the system (23) is a special form of the system (2), and the assumptions (i), (ii) with $p=1$, (iv) and (v) hold good. Thus, the proof of this corollary follows from Theorem 3.

REMARK 1. If $n=1$, then the second part of the assumption (vii) is obviously satisfied.

REMARK 2. Here, we should note that our results are not exact generalizations of Cook's. For he discussed the existence of a solution $u(t)$ of the equation (1), which satisfies the condition (20) for a given constant c, on the whole interval, while we can only show the existence of such a solution on $[T,\infty)$ for a suitably large T.

REFERENCES

[1] K. L. COOKE, Functional differential equations close to differential equations, Bull. Amer. Math. Soc., 72 (1966), 285–288.

[2] J. KATO, Asymptotic behaviors in functional differential equations, Tôhoku Math. Journ., 18(1966), 174–215.
[3] J. KATO, On the existence of *O*-curves, Tôhoku Math. Journ., 19(1967), 49–62.
[4] T. YOSHIZAWA, Stability theory by Liapunov's second method, Math. Soc. Japan, Tokyo, 1966.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY,
SENDAI, JAPAN.