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REMARKS ON GROTHENDIECK RINGS

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R.G.Swan has obtained several important results on Grothendieck rings of a finite group. In this note we derive generalizations of some of his results. Throughout this note, R denotes a noetherian integral domain and K denotes its quotient field. All modules we consider are finitely generated unitary left modules. If A is a finite R-algebra (or K-algebra), $G(A)$ denotes the Grothendieck group of A-modules, $P(A)$ denotes the Grothendieck group of projective A-modules, and $C_0(A)$ its reduced class group, i.e, the subgroup of $P(A)$ generated by the elements of the form $[P]-[Q]$, where P, Q are projective and $K\otimes_R P \cong K\otimes_R Q$.

1. R is called regular if its localization $R_\mathfrak{p}$ is a regular local ring for each prime ideal $\mathfrak{p}$. A regular domain is integrally closed [1. Proposition 4. 2]. In this section we calculate $G(R\pi)$ for a regular domain R of prime characteristic p and for any finite group π.

PROPOSITION 1. *Any finitely generated module over a regular domain R has a finite projective dimension.*

PROOF. Let M be such a module and let

$$\to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \to X_0 \to M \to 0$$

be its projective resolution, where we assume every X_k is finitely generated. Let Y_n be the kernel of d_n. Then Y_n is a finitely generated torsion-free R-module. To show that some Y_n is projective, we first prove the following lemma.

LEMMA. *Let R be an integral domain (not necessarily noetherian), and Y be a finitely generated torsion-free R-module. Let $\mathfrak{p}$ be a prime ideal of R. If $Y_\mathfrak{p}=R_\mathfrak{p}\otimes_R Y$ is $R_\mathfrak{p}$-projective, then $Y_\mathfrak{q}$ is $R_\mathfrak{q}$-projective for each $\mathfrak{q}$ which does not contain a certain element $r\notin\mathfrak{p}$.*

PROOF. Let $F\xrightarrow{f} Y\to 0$ be exact where F is a finitely generated free R-module. Then the sequence $F_\mathfrak{p}\xrightarrow{f_\mathfrak{p}} Y_\mathfrak{p}\to 0$ splits by assumption, and we have a homomorphism $g_{\mathfrak{p}}: Y_{\mathfrak{p}} \to F_{\mathfrak{p}}$ such that $f_{\mathfrak{p}} \cdot g_{\mathfrak{p}} = \text{identity}$. Let $y_1, \cdots, y_n$ be a set of generators for Y, and let $g_{\mathfrak{p}}(y_i) = v_i / r_i$, where $v_i \in F$, $r_i \in R - \mathfrak{p}$. If a prime $\mathfrak{q}$ does not contain $r = r_1 r_2 \cdots r_n$, it is clear that $g_{\mathfrak{q}}: y_i \to v_i / r_i$ induces a homomorphism of $Y_{\mathfrak{q}}$ into $F_{\mathfrak{q}}$ which splits $f_{\mathfrak{q}}$. Then $Y_{\mathfrak{q}}$ is $R_{\mathfrak{q}}$-projective.

We now continue the proof of Proposition 1. As $R_{\mathfrak{p}}$ is regular for each $\mathfrak{p}$, it has the finite global dimension [11. Theorem 1]. Therefore $Y_{n,\mathfrak{p}}$ is $R_{\mathfrak{p}}$-projective for some $n = n(\mathfrak{p})$. By the lemma there exists an element $r = r(\mathfrak{p})$ not contained in $\mathfrak{p}$ such that $Y_{n,\mathfrak{q}}$ is $R_{\mathfrak{q}}$ projective for every $\mathfrak{q}$ which does not contain $r(\mathfrak{p})$. As $\{r(\mathfrak{p}), \mathfrak{p} \text{ prime}\}$ generates a unit ideal, there exist a finite number of $r(\mathfrak{p})$ which generate a unit ideal. Let n be the maximal value of corresponding $n(\mathfrak{p})$. Then it is clear that $Y_{n,\mathfrak{p}}$ is $R_{\mathfrak{p}}$-projective for every prime $\mathfrak{p}$. Then Y_n is R-projective by [2. Proposition 2.6].

Let R be a regular domain of prime characteristic p, and let π be a finite group. Then R contains a prime field F_0 of characteristic p. Let F be the set of the elements of K which are algebraic over F_0. Then F is a field contained in R because R is integrally closed. Let N_0 denote the radical of $F\pi$. Then $N = R \otimes_F N_0$ is the nil-radical of $R\pi \cong R \otimes_F F\pi$, and

$$R\pi/N \cong R \otimes_F (F\pi/N_0) \cong R \otimes_F M_1 \oplus \cdots \oplus R \otimes_F M_r$$

holds. Where $M_i = M(n_i, F_i)$ is the total matric algebra of degree n_i over a finite extension field F_i of F. $R_i = R \otimes_F F_i$ is an integral domain with the quotient field $K_i = K \otimes_F F_i$. Then we have

$$R\pi/N \cong M(n_1, R_1) \oplus \cdots \oplus M(n_r, R_r),$$

and so

$$G(R\pi/N) \cong G(M(n_1, R_1)) \oplus \cdots \oplus G(M(n_r, R_r)).$$

As F_i is separable over F, any finitely generated R_i-module has a finite projective dimension by Proposition 1,[4. IX. Theorem 7.10] and [6. Proposition 2]. By Morita theorem [9. Theorem 3.4], it is also true for any finitely generated $M(n_i, R_i)$-module. So $G(M(n_i, R_i)) \cong P(M(n_i, R_i))$ holds by [12. Proposition 11]. Then by [15. Proposition 4.1] and [15. Proposition 1.1],

$$0 \to C_0(M(n_i, R_i)) \to G(M(n_i, R_i)) \to G(M(n_i, K_i)) \to 0$$

holds. So also holds

$$0 \to C_0(R\pi/N) \to G(R\pi/N) \to G(K\pi/N) \to 0.$$

Now as N is nilpotent, $P(R\pi) \cong P(R\pi/N)$ by [3. Lemma 18.1]. This isomorphism is obtained by corresponding P/NP to any finitely generated projective module P. If $K \otimes_R (P/NP) \cong K \otimes_R P / K \otimes_R NP$ is $K\pi/KN$-free, $K \otimes_R P$ is $K\pi$-free because

KN is the radical of $K\pi$. So we have an isomorphism $C_0(R\pi)\cong C_0(R\pi/N)$. There exist natural homomorphisms

$$G(R\pi/N)\to G(R\pi),\ G(K\pi/KN)\to G(K\pi).$$

They are isomorphisms by [7. Proposition 9.4].
Hence

THEOREM 1. *Let R be a regular domain of prime characteristic p, and let π be a finite group. Then we have an exact sequence*

$$0\to C_0(R\pi)\xrightarrow{\phi} G(R\pi)\to G(K\pi)\to 0,$$

where $$\phi([P_1]-[P_2])=[P_1/NP_1]-[P_2/NP_2].$$

This theorem generalizes Theorem 1 of [15]. $G(R\pi)$ has a ring structure similarly to [13. §1] by Proposition 1. If R is a Dedekind ring, ϕ is a ring homomorphism. In fact $C_0(R\pi)^2=(\mathrm{Im}\phi)^2=0$ holds. In order to prove this analogously to [15. §12], we need only to note that P/NP is R-projective if P is $R\pi$-projective and

$$0\to F/NF\to P/NP\to A/NA\to 0$$

is exact if

$$0\to F\to P\to A\to 0$$

is exact, F is $R\pi$-projective and A is of R-torsion. We do not know if it is true for any regular ring, but we have

THEOREM 2. *The ring extension*

$$0\to Im\ \phi\to G(R\pi)\to G(K\pi)\to 0$$

splits.

PROOF. Every simple $K\pi$-module is of the form $K_i^{n_i}$ which is a minimal left ideal of $M(n_i, K_i)$. Let $R_i^{n_i}$ be a corresponding ideal of $M(n_i, R_i)$. Let

$$K_i^{n_i}\otimes_K K_j^{n_j} \sim \sum_l m_l\cdot K_l^{n_l}$$

be the decomposition into simple factors as $K\pi$-modules. As $K_i^{n_i}\cong K\otimes_F F_i^{n_i}$, we can take a F-basis of $F_i^{n_i}\otimes_F F_j^{n_j}$ as a K-basis of $K_i^{n_i}\otimes_K K_j^{n_j}$. As every simple $F\pi$-module $F_i^{n_i}$ induces simple $K\pi$-module $K_i^{n_i}$, the above decomposition comes from a transformation of F-basis. As F is contained in R, and a F-basis of $F_i^{n_i}$ becomes an R-basis of $R_i^{n_i}$, this transformation induces a transformation of R-basis of $R_i^{n_i}\otimes_R R_j^{n_j}$. Therefore $R_i^{n_i}\otimes_R R_j^{n_j}\sim \sum_l m_l\cdot R_l^{n_l}$ holds, so the correspondence $K_i^{n_i}\to R_i^{n_i}$ induces a ring homomorphism which splits the ring extension.

2. Let A be a finite R-algebra. We assume that A is torsion-free as an R-module and $K\otimes_R A$ is a separable algebra. Let $\mathfrak{o}$ denote a maximal order containing A. Then by [8], there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
W(K\otimes A) & \longrightarrow & G_t(\mathfrak{o}) & \longrightarrow & G(\mathfrak{o}) & \longrightarrow & G(K\otimes A) & \longrightarrow & 0\\
\| & & \downarrow\varphi & & \downarrow\psi & & \| & & \\
W(K\otimes A) & \longrightarrow & G_t(A) & \longrightarrow & G(A) & \longrightarrow & G(K\otimes A) & \longrightarrow & 0.
\end{array}$$

Where $W(K\otimes A)$ is the Whitehead group of $K\otimes A$-modules, and $G_t(A)$, $G_t(\mathfrak{o})$ are Grothendieck groups of R-torsion A-modules and $\mathfrak{o}$-modules respectively. φ, ψ are natural homomorphisms. From this diagram we have

$$G_t(A)/\varphi G_t(\mathfrak{o}) \cong G(A)/\psi G(\mathfrak{o}).$$

If R is of Krull dimension one, $G_t(A)\cong \sum_{\mathfrak{p}} G(A/\mathfrak{p}A)$, $G_t(\mathfrak{o})\cong\sum_{\mathfrak{p}} G(\mathfrak{o}/\mathfrak{p}\mathfrak{o})$ where sums are direct sums over all non-zero prime ideals $\mathfrak{p}$. Then φ is also a direct product of $\varphi_{\mathfrak{p}}: G(\mathfrak{o}/\mathfrak{p}\mathfrak{o})\to G(A/\mathfrak{p}A)$. So we have

$$G(A)/\psi G(\mathfrak{o})\cong \sum_{\mathfrak{p}} G(A/\mathfrak{p}A)/\varphi_{\mathfrak{p}} G(\mathfrak{o}/\mathfrak{p}\mathfrak{o}) \quad \text{(direct)}.$$

Z denotes the rational integers and Q denotes the rationals. Let $A=\{a+b\sqrt{m},\ a,b\in Z\}$ be a subring of $Q(\sqrt{m})$ where $m\equiv 1 \pmod 4$. Then $\mathfrak{o}=\left\{a+b\frac{1+\sqrt{m}}{2}\right\}$ is the ring of integers of $Q(\sqrt{m})$. If $\mathfrak{q}\neq\mathfrak{p}=(2, 1+m)$ is a prime ideal of

A, $A/\mathfrak{q} \cong \mathfrak{o}/\mathfrak{q}\mathfrak{o}$ so that $G(A/\mathfrak{q}) = \varphi_{\mathfrak{q}} G(\mathfrak{o}/\mathfrak{q}\mathfrak{o})$. It is well known that

$$\mathfrak{p}\mathfrak{o} = 2\mathfrak{o} = \begin{cases} \mathfrak{P}_1\mathfrak{P}_2 \text{ if m} \equiv 1 \bmod 8 \\ \text{prime in } \mathfrak{o} \text{ if m} \equiv 5 \bmod 8. \end{cases}$$

Hence

$$\mathfrak{o}/\mathfrak{p}\mathfrak{o} = \begin{cases} \mathfrak{o}/\mathfrak{P}_1 \oplus \mathfrak{o}/\mathfrak{P}_2 & \text{if } m \equiv 1 \bmod 8 \\ \text{simple } \mathfrak{o}\text{-module} & \text{if } m \equiv 5 \bmod 8. \end{cases}$$

As $\mathfrak{o}/\mathfrak{p}\mathfrak{o} \cong A/\mathfrak{p} \oplus A/\mathfrak{p}$ as A-modules,

$$G(A/\mathfrak{p})/\varphi_{\mathfrak{p}} G(\mathfrak{o}/\mathfrak{p}\mathfrak{o}) \begin{cases} = 0 \text{ if } m \equiv 1 \bmod 8 \\ \cong Z/2Z \text{ if } m \equiv 5 \bmod 8. \end{cases}$$

So we have $G(A) \cong \psi G(\mathfrak{o})$ if $m \equiv 1 \bmod 8$ and $G(A)/\psi G(\mathfrak{o}) \cong Z/2Z$ if $m \equiv 5 \bmod 8$. In the latter case, the sequence

$$C_0(\mathfrak{o}) \longrightarrow G(A) \longrightarrow G(Q(\sqrt{m})) \longrightarrow 0$$

is not exact. As $C_0(A) \to G(A)$ factors nto $C_0(A) \to C_0(\mathfrak{o}) \to G(A)$, the sequence

$$C_0(A) \longrightarrow G(A) \longrightarrow G(Q(\sqrt{m})) \longrightarrow 0$$

is not exact. This shows the analogy of Theorem 1 of [15] does not hold in general even if A is commutative (We consider A as a Z-algebra).

3. In this section we consider special cases of Corollary 2 of [15]. T.Obayashi [10] has determined the ring structure of $G(Z\pi)$ more explicitly in the case of a finite abelian p-group.

THEOREM 3. *Let π be a finite p-group and $\mathfrak{o}$ be a maximal order of $Q\pi$ containing $Z\pi$. Then*

$$0 \longrightarrow C_0(\mathfrak{o}) \longrightarrow G(Z\pi) \longrightarrow G(Q\pi) \longrightarrow 0$$

is exact.

PROOF. It suffices to prove that $0 \to C_0(\mathfrak{o}) \to G(Z\pi)$ is exact. Let $Z_{(p)}$ denote the ring of the rationals whose denominators are powers of p. Then the sequence

$$G((Z/pZ)\pi) \longrightarrow G(Z\pi) \longrightarrow G(Z_{(p)}\pi) \longrightarrow 0$$

is exact by [15. Proposition 1.1]. But the unique simple $(Z/pZ)\pi$-module is Z/pZ, and

$$0 \longrightarrow Z \xrightarrow{p} Z \longrightarrow Z/pZ \longrightarrow 0$$

is exact. So the class of Z/pZ in $G(Z\pi)$ is zero. Therefore $G(Z\pi) \cong G(Z_{(p)}\pi)$ holds. In the commutative diagrams

$$\begin{array}{ccccccccc} & & C_0(Z\pi) & \longrightarrow & G(Z\pi) & \longrightarrow & G(Q\pi) & \longrightarrow & 0 \\ & & \downarrow \text{onto} & & \uparrow \text{onto} & & \| & & \\ 0 & \longrightarrow & C_0(\mathfrak{o}) & \longrightarrow & G(\mathfrak{o}) & \longrightarrow & G(Q\pi) & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} & & C_0(Z\pi) & \longrightarrow & G(Z\pi) & \longrightarrow & G(Q\pi) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & C_0(Z_{(p)}\pi) & \longrightarrow & G(Z_{(p)}\pi) & \longrightarrow & G(Q\pi) & \longrightarrow & 0 \end{array}$$

all the rows are exact by [15. Theorem 1. Proposition 5.1]. The last row is exact because $Z_{(p)}\pi$ is a maximal order [15. Lemma 5.1]. As $Z_{(p)}\pi$ contains $\mathfrak{o}$, the kernel of $C_0(Z\pi) \to C_0(\mathfrak{o})$ is contained in the kernel of $C_0(Z\pi) \to C_0(Z_{(p)}\pi)$. If we show they are equal, $\text{Ker}(C_0(Z\pi) \to G(Z\pi))$ is equal to Ker $(C_0(Z\pi) \to C_0(\mathfrak{o}))$. So $G(\mathfrak{o}) \to G(Z\pi)$ becomes the isomorphism, and we have the assertion.

Let $[P]-[F]$ be an element of the kernel of $C_0(Z\pi) \to C_0(Z_{(p)}\pi)$. Here P is a projective $Z\pi$-module and F is a free $Z\pi$-module. By assumption

$$Z_{(p)} \otimes_Z P \oplus Z_{(p)} \otimes_Z F' \cong Z_{(p)} \otimes_Z F \oplus Z_{(p)} \otimes_Z F'$$

for some free $Z\pi$-module F'. This isomorphism, by multiplying some power of p if necessary, can be assumed to be induced from an injection

$$\varphi_0 : P \oplus F' \longrightarrow F \oplus F'$$

whose cokernel has a finite order of some power of p. So we may assume P is contained in F, and $(F : P)$ is a power of p. Tensoring with $\mathfrak{o}$ over $Z\pi$ we have

$$\varphi_0 : \mathfrak{o} \otimes p \longrightarrow \mathfrak{o}^r$$

for some r. The order of the cokernel is also a power of p, and φ_0 is an injection because $\mathfrak{o} \otimes P$ is Z-torsion-free. In general, let A be a semi-simple algebra over Q, and $\mathfrak{o}$ its maximal order. Let M be a sub-module of $\mathfrak{o}^r$ of a finite index. Put $\mathfrak{o}^r = \mathfrak{o}_1 \oplus \cdots \oplus \mathfrak{o}_r$ for convenience. Then $M \cap \mathfrak{o}_1$ is a submodule of $\mathfrak{o}_1$ of a finite index and $M/M \cap \mathfrak{o}_1$ is torsion-free. It is projective because

$\mathfrak{o}$ is hereditary, so $M \cong M \cap \mathfrak{o}_1 \oplus M'$. M' is isomorphic to the projection of M into $\mathfrak{o}_2 \oplus \cdots \oplus \mathfrak{o}_r$. Similarly we have $M \cong M_1 \oplus \cdots \oplus M_r$, where M_j is isomorphic to a submodule of $\mathfrak{o}_j$ of a finite index. If the index $(\mathfrak{o}^r : M)$ is a power of p, so is every $(\mathfrak{o}_j : M_j)$. If $A \cong A_1 \oplus \cdots \oplus A_r$ where each A_i is a simple algebra, $\mathfrak{o}$ has corresponding decomposition

$$\mathfrak{o} \cong \mathfrak{A}_1 \oplus \cdots \oplus \mathfrak{A}_r .$$

If $(\mathfrak{o}^r : M)$ is a power of p, every $(\mathfrak{A}_i : \mathfrak{A}_i M_j)$ is also a power of p. Applying the above argument to $\mathfrak{o} \otimes P \subset \mathfrak{o}^r$, we have

$$[\mathfrak{o} \otimes P] - [\mathfrak{o}^r] = \sum_i ([L_i] - [\mathfrak{A}_i]) ,$$

where $\mathfrak{A}_i$ is a component of $\mathfrak{o}$ and L_i is a left $\mathfrak{A}_i$-ideal of index of a power of p. The center K_i of every simple component A_i of $Q\pi$ is contained in $Q(\zeta_n)$ because $Q\pi$ splits over $Q(\zeta_n)$. Where p^n is the order of π, and ζ_n is a primitive p^n-th root of unity. Therefore p has a unique prime factor $\mathfrak{p}_i$ in K_i. $\mathfrak{p}_i$ is a principal ideal generated by $N_{Q(\zeta_n)/K_i}(1-\zeta_n)$. If K_i is real, it is therefore generated by a total positive element. Hence if the reduced norm of an ideal L_i is a power of $\mathfrak{p}_i$, then holds either $L_i \cong \mathfrak{A}_i$ or $L_i \oplus \mathfrak{A}_i \cong \mathfrak{A}_i \oplus \mathfrak{A}_i$ [5.Satz 1.See also 14]. Therefore $[L_i] = [\mathfrak{A}_i]$ in $C_0(\mathfrak{A}_i)$ and $[\mathfrak{o} \otimes P] - [\mathfrak{o}^r] = 0$ in $C_0(\mathfrak{o})$ holds. We have $\mathrm{Ker}\ (C_0(Z\pi) \to C_0(Z_{(p)}\pi)) = \mathrm{Ker}(C_0(Z\pi) \to C_0(\mathfrak{o}))$, and this concludes the proof.

It is known that the homomorphism ϕ in the exact sequence

$$C_0(\mathfrak{o}) \xrightarrow{\phi} G(Z\pi) \longrightarrow G(Q\pi) \longrightarrow 0$$

is not injective even if π is a cyclic group. But we can show

THEOREM 4. *The exact sequence*

$$0 \longrightarrow Im\phi \longrightarrow G(Z\pi) \longrightarrow G(Q\pi) \longrightarrow 0$$

splits as a ring extension when π is a finite abelian group.

PROOF. Put $Q\pi \cong Q_1 \oplus \cdots \oplus Q_s$, where every Q_i is a field. Let $\rho_i : \pi \to Q_i$ be a corresponding representation. The image of ρ_i consists of roots of unity in Q_i.

If H_i is the kernel of ρ_i, G/H_i is a cyclic group. This correspondence is bijective, and $Q_i = Q(\zeta_i)$ where ζ_i is a primitive $(G : H_i)$-th root of unity.

Let $\mathfrak{o}_i = Z[\zeta_i]$ and $\mathfrak{o}_j = Z[\zeta_j]$ be the rings of integers in Q_i and in Q_j

respectively. Let $f(X)$ be an irreducible polynomial over Q such that $f(\zeta_j)=0$. Let $f(X)=g(X)g(X)^\sigma\cdots g(X)^\tau$ be a factorization into irreducible polynomials over Q_i. Let $\zeta_j, \zeta_j^\sigma, \cdots, \zeta_j^\tau$ be representatives of their roots. Then

$$\mathfrak{o}_i\otimes_Z\mathfrak{o}_j = Z[\zeta_i]\otimes_Z Z[\zeta_j]\cong \sum_\sigma Z[\zeta_i, \zeta_j{}^\sigma]\,.$$

Let x be an element of π. Then x acts on $Z[\zeta_i, \zeta_j^\sigma]$ as multiplication by $\rho_i(x)\rho_j(x)^\sigma$. If we put H_k the kernel of this action, $Z\pi$-module structure of $Z[\zeta_i, \zeta_j^\sigma]$ is the same as $\mathfrak{o}_k$-module structure. As $Z[\zeta_i, \zeta_j^\sigma]$ is $\mathfrak{o}_k$-free, $\mathfrak{o}_i\otimes\mathfrak{o}_j$ is a direct sum of $\mathfrak{o}_k$'s. Hence we know that $Q_i\to\mathfrak{o}_i$ is a ring homomorphism which splits the extension.

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