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ON THE PREDUALS OF W^*-ALGEBRAS

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In the present paper ,we shall show some properties of weakly relatively compact subsets of predual of W^*-algebra, which were also discussed in [1], [10] and [12].

Let M be a W^*-algebra (namely, C^*-algebra with a dual structure as a Banach space [7]), M^* (resp. M_*) be the dual (resp. predual) of M, and let M_h, M_p, and M_{pi} be the set of all Hermitian elements, projections, and partial isometries in M, respectively.

The weak topology on M_* is $\sigma(M_*, M)$-topology in the sense of [3; p. 50].

For any linear functional φ in M, we define the functionals φa, $a\varphi$, φ^* and $|\varphi|$ on M as follows: $\varphi a(b) = \varphi(ab)$,$a\ \varphi(b) = \varphi(ba)$, $\varphi^*(b) = \overline{\varphi(b^*)}$ for all b $\in M$, where $\overline{\varphi(b^*)}$ is the complex conjugate of $\varphi(b^*)$. $|\varphi|$ is said the absolute value of φ [8]. If φ is in M_*, then φa, $a\varphi$, and φ^* are also in M_*. We denote the set $\{|\varphi|; \varphi \in K\}$ by $|K|$.

A functonal φ on M is positive if $\varphi(a^*a) \geqq 0$ for all $a \in M$. Denote the set of all positive functionals in M^* (resp. M_*) by M^{*+} (resp. M_*^+).

We may consider the following five typical topologies on M:

(1) The norm topology as a Banach space, (2) The Mackey topology τ on M, namely, the togology of uniform convergence on the weakly relatively compact sets of M_*, (3) The topology s^* defined by a family of semi-norms $\{\alpha_\varphi, \alpha_\varphi^*; \varphi \in M_*{}^+\}$, where $\alpha_\varphi(x) = \varphi(x^*x)^{1/2}$, and $\alpha_\varphi^*(x) = \varphi(xx^*)^{1/2}$ for $x \in M$, (4) The topology s defined by a family of semi-norms $\{\alpha_\varphi; \varphi \in M_*{}^+\}$, (5) The weak topology on M as point, which is merely called σ-topology. The topology s^*(resp. s and σ) coincides with strong $*$-operator topology, namely the operator topology defined by a family of semi-norms $\{\|x\xi\|\ \|x^*\xi\|\ ; \xi \in \mathfrak{H}\}$ (resp. the strong operator topology and the weak operator topology) on bounded spheres, when M is faithfully represented as a von Neumann algebra on a Hilbert apace $\mathfrak{H}$. The τ-topology is equivalent to the s^*-topology on bounded spheres. [1]

In the followings, theorem 1 shows a characterization of the finiteness of W^*-algebras. Theorem 2 and the following remark concern with a weak convergence property in the predual of an atomic W^*-algebra, which is a non-commutative generalization of a well known theorem in the Lebesgue L^1, and the last theorem 3 deals with weakly relatively compact subsets lying in the positive portion of the predual M_*.

Firstly, we state and prove the following

THEOREM 1. *Let M be a W^*-algebra, then M is finite if and only if, for any weakly relatively compact subset K of M_*, $|K|$ is also weakly relatively compact.*

PROOF. Necessity: By Eberlein-Šmulian theorem, it sufficies to prove only the case $K=\{\varphi_n\}_{n=1}^{\infty}$. By [1; Theorem 2], it is sufficient to prove that, for any orthogonal sequence of projections $\{e_k\}_{k=1}^{\infty}$ $\lim_{k\to\infty} |\varphi_n|(e_k)=0$ uniformly for n.

Let

$$|\varphi_n|(e_k) = \varphi_n(e_k u_n^*)=\varphi_n^*(u_n e_k) \quad (u_n \in M_{p.i.})$$

be the polar decomposition of φ_n in the sense of [7]. If the statement that $\lim_{k\to\infty} |\varphi_n|(e_k) = 0$ uniformly for n is false, then there exists an $\varepsilon>0$ such that for each n there is some φ_n' in K such that

$$(*) \qquad |\varphi_n'|(e_n) \geqq \varepsilon.$$

As the $*$-operation is continuous for the weak topology, K^* (the set $\{\varphi^*; \varphi\in K\}$) is also weakly relatively compact. Setting

$$a_n = u_n e_n,\ \|a_n\|\leqq 1,\ \text{and}\ a_n^* a_n = e_n e(|\varphi_n'|)e_n$$

where $e(|\varphi_n'|)$ is the carrier projection of $|\varphi_n'|$ [5], and a_n converges strongly to 0. Since M is a finite algebra, τ is equivalent to s on S, the unit sphere. Hence $\lim_{n\to\infty} a_n=0$ for τ- topology and then $\lim_{m\to\infty} \varphi_n^*(a_m) = 0$ uniformly for n, contradicting the inequality $(*)$.

Sufficiency: By [5], there exists a central projection e such that $M(1-e)$ is finite algebra, $e=0$ or Me is properly infinite algebra and $M=Me\oplus M(1-e)$. If $e\neq 0$, then Me is a properly infinite W^*-algebra and there is a family of orthogonal projections $\{e_i\}_{i=1}^{\infty}$ such that

$$e=\sum_{i=1}^{\infty} e_i,\ e_i\sim e_j (e_i\in M). \qquad [5].$$

Let ψ be a σ-continuous state on e_1Me_1 and putting

$$\varphi(a)=\psi(e_1ae_1),\ a\in M,$$

φ is a σ-continuous positive functional on M such that $\varphi(e_1)=1$. Setting $\varphi_n(a)=\varphi(v_n^* a)$, where v_n is a partial isometry in M such that $v_n^* v_n=e_1$, $v_n v_n^*=e_n$, φ_n is σ-continuous. Then we have

$$\varphi_n(a)=\varphi(v_n^* a)=\varphi(v_n^* a v_n^* v_n)=(v_n\varphi v_n^*)(av_n^*)=v_n^*(v_n\varphi v_n^*)(a).$$

Putting $\psi_n=v_n\varphi v_n^*$, ψ_n is positive and we have

$$\psi_n(e_n)=\varphi(v_n^* e_n v_n)=\varphi(e_1)=1,$$

$$\psi_n(1)=\|\psi_n\|=\varphi(v_n^* v_n)=\varphi(e_1)=1.$$

Hence we have

$$|\varphi_n(a)|^2\leqq\psi_n(aa^*)\cdot\psi_n(v_n v_n^*)=\psi_n(aa^*)\cdot\psi_n(e_n)$$
$$=\psi_n(aa^*)\cdot\|\psi_n\|.$$

By the unicity of polar decomposition [11], $\varphi_n=v_n^*\psi_n$ is the polar decomposition of φ_n and ψ_n is the absolute value of φ_n, that is, $|\varphi_n|=\psi_n$. By [9], we have

$$\varphi_n(a)=\varphi(v_n^* a)=(\pi_\varphi(a)\ \eta_\varphi(1),\ \eta_\varphi(v_n)),$$

where π_φ is a cyclic representation on $\mathfrak{H}_\varphi$ induced by φ and $\eta_p(a)$ is an element of $\mathfrak{H}_\varphi$ corresponding to a in M in the sense of I. E. Segal [9]. As $\{\eta_\varphi(v_n)\}$ is an orthogonal system in $\mathfrak{H}_\varphi$, we have

$$\lim_{n\to\infty}(\pi_\varphi(a)\eta_\varphi(1),\ \eta_p(v_n))=0,$$

that is, φ_n is weakly convergent to 0. Hence $\{\varphi_n\}_{n=1}^\infty$ is a weakly relatively compact subset of M_*.

If $\{|\varphi_n|\}_{n=1}^\infty$ is weakly relatively compact, then by [1], for the above family of orthogonal projections $\{e_i\}_{i=1}^\infty$, $\lim_{k\to\infty}|\varphi_n|(e_k)=0$ uniformly for n. On the other hand, we have

$$|\varphi_n|(e_n)=\psi_n(e_n)=\varphi(e_1)=1, \text{ for each } n.$$

This is a contradiction, that is, $\{|\varphi_n|\}_{n=1}^\infty$ is not weakly relatively compact. Therefore, if M is not finite, then there exists a weakly relatively compact subset $\{\varphi_n\}_{n=1}^\infty$ of M_* such that $\{|\varphi_n|\}_{n=1}^\infty$ is not weakly relatively compact, This completes the proof.

REMARK. If M is an abelian W^*-algebra, then $M = L^\infty(\Omega, \mu)$ where Ω is a locally compact Hausdorff space and μ is a positive Radon measure on Ω by [5]. Therefore, in the abelian case, the necessary condition of the above theorem is a well known result in the classical measure theory.

By an atomic W^*-algebra M we mean a W^*-algebra such that for every projection e in M, there exists a minimal subprojection f of e in M.

Then we obtain

THEOREM 2. *Let M be an atomic W^*-algebra and $\{\varphi_n\}_{n=1}^\infty$ be a sequence in M_* such that $\lim_{n\to\infty} \varphi_n(e)$ exists and is finite for each e in M_p and that $\{|\varphi_n|\}_{n=1}^\infty$, $\{|\varphi_n^*|\}_{n=1}^\infty$ are weakly relatively compact, then there exists φ in M_* such that $\lim_{n\to\infty} \|\varphi_n - \varphi\| = 0$.*

In the proof of our theorem we shall use the following lemma due to C.Akemann [1].

LEMMA. *Let M be a W^*-algebra and $\{e_\theta\}_{\theta\in\Theta}$ be an increasing net of projections in M such that $\sup_{\theta\in\Theta} e_\theta = 1$, then for bounded subset K of M_*, K is weakly relatively compact if and only if for every positive ε, there exists an e in $\{e_\theta\}_{\theta\in\Theta}$ such that $\|e^\perp\varphi e^\perp\| \leqq \varepsilon$ for each φ in K, where $e^\perp$ means the projection $1-e$.*

PFOOF OF THEOREM 2. From the result of Aarnes [2], there is a real number $r > 0$ such that $\|\varphi_n\| \leqq r$ for each n. Then by the spectral theory and Banach-Steinhaus theorem, there exists φ in M_* such that φ_n converges weakly to φ.

Denoting

$$K = \{|\varphi_n|, |\varphi_n^*|, |\varphi|, |\varphi^*|; n = 1, 2, \cdots\},$$

K is weakly relatively compact. By the hypothesis and Zorn's lemma, there exists a family of projections $\{e_\theta\}_{\theta\in\Theta}$ in M as follows;

(1) The algebra $e_\theta M e_\theta$ is finite dimensional for each θ.

(2) The $\{e_\theta\}_{\theta\in\Theta}$ are increasing net.

(3) $\sup_{\theta\in\Theta} e_\theta = 1$.

By scalar multiplication, we may assume that $\sup_k \|\varphi_k\| = 1$ without loss of generality.

By the above lemma, for $\varepsilon > 0$, there is a projection e in $\{e_\theta\}_{\theta\in\Theta}$ such that, $\|e^\perp \varphi e^\perp\| \leqq \varepsilon$ for all φ in K. Since eMe is finite dimensional, the weak and the norm topologies coincide on eMe, so that there exists an integer k_0 such that we have, for each a in S,

$$|(\varphi_k - \varphi)(eae)| < \varepsilon,$$

for $k > k_0$.

Thus, for any a in S and $k > k_0$, we have the inequalities:

$$\begin{aligned}|(\varphi_k - \varphi)(a)| &< |(\varphi_k - \varphi)(eae)| + |(\varphi_k - \varphi)(eae^\perp)| \\ &+ |(\varphi_k - \varphi)(e^\perp ae)| + |(\varphi_k - \varphi)(e^\perp ae^\perp)| \\ &< \varepsilon + |\varphi_k^*(e^\perp a^* e)| + |\varphi^*(e^\perp a^* e)| \\ &+ |\varphi_k(e^\perp ae)| + |\varphi(e^\perp ae)| + |\varphi_k(e^\perp ae^\perp)| \\ &+ |\varphi(e^\perp ae^\perp)|.\end{aligned}$$

Now let $\varphi_k = u_k|\varphi_k|$, (resp. $\varphi_k^* = v_k|\varphi_k^*|$) be the polar decomposition of φ_k (resp. φ_k^*); then, by the Schwarz inequality, we have

$$\begin{aligned}|\varphi_k(e^\perp ae)| &= ||\varphi_k|(e^\perp aeu_k)| \\ &< \{|\varphi_k|(e^\perp)\}^{1/2} \cdot \{|\varphi_k|((aeu_k)^*(aeu_k))\}^{1/2} \\ &< \{|\varphi_k|(e^\perp)\}^{1/2} < \varepsilon^{1/2}.\end{aligned}$$

Similarly we have

$$|\varphi_k^*(e^\perp a^* e)| < \varepsilon^{1/2}.$$

Combining the above estimations, we get

$$|(\varphi_k - \varphi)(a)| < \varepsilon + 6\varepsilon^{1/2}$$

for $k > k_0$ and $a \in S$. Since ε is arbitrary and a is an arbitrary element of S, we have that $\lim_{k\to\infty} \|\varphi_k - \varphi\| = 0$. This completes the proof.

REMARK. This theorem can be considered as a non-commutative version of [6; p. 295] and includes the result of C.Akemann [1;Theorem IV 1.]. In finite case, by Theorem 1, we can drop the condition that $\{|\varphi_n|\}_{n=1}^\infty, \{|\varphi_n^*|\}_{n=1}^\infty$ are weakly relatively compact, but in general case, we cannot drop it, as the following example shows. Let $\mathfrak{H}$ be an infinite dimensional separable Hilbert space, $\{\xi_i\}_{i=1}^\infty$ an orthonormal basis for it, and define functionals $\{\omega_n\}$ in $\boldsymbol{B}(\mathfrak{H})$ by;

$$\omega_n(a) = (a\xi_1, \xi_n), \text{ for } a \in \boldsymbol{B}(\mathfrak{H}).$$

(Note that $\boldsymbol{B}(\mathfrak{H})$ is an atomic W^*-algebra.) and we have

$$\omega_n^*(a) = (a\xi_n, \xi_1).$$

Then by the definition of ω_n, both ω_n and ω_n^* converge weakly to 0. Let v_n be a partial isometry defined by $v_n\xi = (\xi, \xi_1)\xi_n$ for $\xi \in \mathfrak{H}$.

Putting $\varphi_n(a) = (a\xi_n, \xi_n)$ and $\omega_n(a) = \varphi_n(av_n^*)$, we have

$$|\omega_n(a)|^2 \leqq \varphi_n(aa^*)\|\varphi_n\|, \ \|\varphi_n\| = 1 = \|\omega_n\|.$$

By the unicity of polar decomposition of functionals, we have

$$|\omega_n| = \varphi_n \text{ and } |\omega_n^*| = \varphi_1.$$

Hence $\{|\omega_n^*|\}_{n=1}^{\infty}$ is weakly relatively compact. On the other hand, $\{|\omega_n|\}_{n=1}^{\infty}$ is not weakly relatively compact. If otherwise, putting $e_n = p_{[\xi_n]}$, for the family of orthogonal projections $\{e_n\}_{n=1}^{\infty}$, we have

$$\lim_{k\to\infty} \varphi_n(e_k) = 0 \text{ uniformly for } n.$$

This is a contradiction. And ω_n cannot converge to 0 uniformly. Hence either of the above conditions can not be dropped.

THEOREM 3. *Let M be a W^*-algebra and K be a weakly relatively compact subset in M_*^+, then $\{a\varphi; \varphi \in K, a \in S\}$ is also weakly relatively compact.*

PROOF. By uniform boundedness theorem, $\Delta = \sup\{\|\varphi\|; \varphi \in K\} < \infty$. For any sequence of orthogonal projections $\{e_n\}_{n=1}^{\infty}$ in M, we have, by Schwarz inequality,

$$|\varphi(e_n a)| \leqq \varphi(a^*a)^{1/2}\varphi(e_n)^{1/2} \leqq \Delta^{1/2}\varphi(e_n)^{1/2}$$

for $a \in S$, $\varphi \in K$. Since K is weakly relatively compact,

$$\lim_{n\to\infty} \varphi(e_n) = 0 \text{ uniformly for } \varphi \in K.$$

Therefore $\{a\varphi; \varphi \in K, a \in S\}$ is weakly relatively compact. This completes the proof of Theorem 3.

REMARK. In the above theorem, we cannot drop the hypothesis that $K\subset M_*^+$. Considering $\boldsymbol{B}(\mathfrak{H})$ (where $\mathfrak{H}$ is a separable infinite dimensional Hilbert space), then by the above arguments, there is a weakly relatively compact subset K of $\boldsymbol{B}(\mathfrak{H})_*$ whose absolute value is not weakly relatively compact. $|\varphi| =v^*\varphi\in\{a\varphi;\ a\in S, \varphi\in K\}$ where v is in M_{pi}. Hence $|K|\subset\{a\varphi; a\in S, \varphi\in K\}$ and $\{a\varphi;\ a\in S, \varphi\in K\}$ is not weakly relatively compact.

Moreover, for a W^*-algebra M to be finite, it is necessary and sufficient that for every weakly relatively compact subset K of the predual M_* of M, $\{a\varphi; a\in S, \varphi\in K\}$ is also weakly relatively compact. Since $|K|\subset\{a\varphi; a\in S, \varphi\in K\}$, the proof is the same as that of Theorem 1, so we omit it.

COROLLARY. *Let M be a W^*-algebra and K a subset of M_* whose absolute value $|K|$ is weakly relatively compact, then K is also weakly relatively compact.*

PROOF. By the polar decomposition of functional, we have

$$K\subset\{a\varphi; a\in S, \varphi\in |K|\}.$$

Hence, by Theorem 2, K is weakly relatively compact. This completes the proof.

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