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SOME CRITERIA FOR THE ABSOLUTE SUMMABILITY OF A FOURIER SERIES

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Let

$$f(x) \sim a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x) \tag{1}$$

and $\sigma_n^k(x) (k > -1)$ denote the n-th (C, k) mean of Fourier series (1). If the series

$$\sum_{n=0}^{\infty} |\sigma_n^k(x) - \sigma_{n-1}^k(x)|$$

is convergent, we say that the series (1) is absolutely summable (C, k) or summable $|C, k|$. We denote the integral modulus of continuity of f by

$$\omega_p(t, f) = \sup_{0<h<t} \left\{ \int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^p dx \right\}^{1/p} \quad (1 \leqq p < \infty).$$

H. C. Chow [1] proved the following theorem.

THEOREM A. *Let* $1 \leqq p \leqq 2$. *If*

$$\omega_p(t, f) = O\left\{ \left(\log \frac{1}{t} \right)^{-1-\delta} \right\} \qquad (\delta > 0),$$

then the series (1) *is summable* $|C, \alpha|$ *almost everywhere for* $\alpha > 1/p$.

On the other hand, P. L. Ul'yanov [5] proved the following theorem.

THEOREM B. *If*

$$\omega_2(t, f) = O\left\{ \left(\log \frac{1}{t} \right)^{-1/2-\delta} \right\} \qquad (\delta > 0),$$

then the series (1) *is summable* $|C,\alpha|$ *almost everywhere for* $\alpha > 1/2$.

We will shsrpen Theorem A into the form of Theorem B.

THEOREM 1. *Let* $1 < p \leqq 2$. *If*

$$\omega_p(t, f) = O\left\{\left(\log \frac{1}{t}\right)^{-1/2-\delta}\right\} \qquad (\delta > 0), \tag{2}$$

then the series (1) *is summable* $|C, \alpha|$ *almost everywhere for* $\alpha > 1/p$.

For proof of Theorem, we need three lemmas.

LEMMA 1. *If* (2) *is true, then the n-th partial sum of series* (1), $s_n(f)$ *has the approximation such as*

$$\left\{\int_{-\pi}^{\pi} |f(x) - s_n(x)|^p dx\right\}^{1/p} = O\{(\log n)^{-1/2-\delta}\}$$

PROOF. If $T_n(x)$ is an arbitrary trigonometric polynomial of degree n, then

$$\|f - s_n(f)\|_p \leqq \|f - T_n\|_p + \|s_n(f - T_n)\|_p \leqq A_p\|f - T_n\|_p$$

by the M. Riesz theorem and the order of best approximation is $\omega_p(1/n) = (\log n)^{-1/2-\delta}$ in this case.

LEMMA 2. *Under the condition of Theorem,*

$$\sum_{n=0}^{\infty} \lambda(n) A_n(x) \tag{3}$$

is a Fourier series of a function of class L^p, *where*

$$\lambda(0) = \lambda(1) = 1,$$
$$\lambda(n) = (\log n)^{1/2+\varepsilon} \qquad (\delta > \varepsilon > 0) \ \ (n = 2, 3 \cdots).$$

PROOF. Let denote by $t_n(x)$ the partial sum of (3), then

$$t_n(x) - f(x)$$
$$= \lambda(0)\{A_0(x) - f(x)\} + \sum_{k=1}^{n} \lambda(k) A_k(x)$$

$$=\sum_{k=0}^{n-1}\{s_k(x)-f(x)\}\Delta\lambda(k)+\{s_n(x)-f(x)\}\lambda(n).$$

Applying Minkowski's inequality,

$$\|t_n-\lambda(0)f\|_p$$
$$\leqq\sum_{k=0}^{n-1}\|s_k-f\|_p\Delta\lambda(k)+\|s_n-f\|_p\lambda(n)$$
$$\leqq C_1+C_2\sum_{k=2}^{n-1}(\log k)^{-1/2-\delta}\frac{1}{k}(\log k)^{-1/2+\varepsilon}+C_3(\log n)^{-1/2-\delta}(\log n)^{1/2+\varepsilon}$$
$$\leqq C_1+C_2\sum_{k=2}^{n-1}\frac{1}{k(\log k)^{1+\delta-\varepsilon}}+C_3\frac{1}{(\log n)^{\delta-\varepsilon}}\leqq C,$$

which is an absolute constant by Lemma 1. Hence

$$\|t_n(x)\|_p=O(1)$$

and (1) is a Fourier series of a function of the class L^p.

LEMMA 3. *If $f(x)$ belongs to the class $L^p(1<p\leqq 2)$, then the series $\sum\mu(n)A_n(x)$ is summable $|C,\alpha|(\alpha>1/p)$ almost everywhere, provided that*

$$\mu(0)=\mu(1)=1,\qquad \mu(n)=(\log n)^{-1/2-\varepsilon}(\varepsilon>0),\ (n=2,3,\cdots).$$

This is known, [Chow, 2].

The proof of Theorem is almost completed. That is to say, from Lemma 2 and Lemma 3

$$\sum_{n=0}^{\infty}A_n(x)$$

is $|C,\alpha|$ summable $(\alpha>1/p)$ almost everywhere.

We can prove the following theorem with the same method also.

THEOREM 2. *If $1<p\leqq 2$ and*

$$\omega_p(t,f)=O\left\{\left(\log\frac{1}{t}\right)^{-(1-\frac{1}{p}+\frac{1}{2}+\delta)}\right\}$$

then (1) *is* $|C, 1/p|$ *summable almost everywhere.*

PROOF. We apply a result of Kojima [3] instead of Lemma 3.

THEOREM 3. *If* $f(e^{i\theta})$ *belongs to the class H, and the integral modulus of continuity of complex* $f(e^{i\theta})$ *be*

$$\omega_1(t, f) = O\left\{\left(\log \frac{1}{t}\right)\right\}^{-(1/2+\delta)}$$

then the complex Fourier series of $f(e^{i\theta})$ *is* $|C, 1|$ *summable almost everywhere.*

PROOF. We use a result of the author [4] and the fact that the power series of bounded variation is absolutely continuous.

Leteratures

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