Tôhoku Math. Journ.
Vol. 19, No. 3, 1967

STRONGLY CURVATURE-PRESERVING TRANSFORMATIONS OF PSEUDO-RIEMANNIAN MANIFOLDS

SHÛKICHI TANNO

(Received January 26, 1967)

Introduction. Let (M, g) and (M', g') be two (pseudo-) Riemannian manifolds with metric tensors g and g' respectively. A diffeomorphism φ of M to M' is called a strongly curvature-preserving transformation if φ maps $\nabla^k R$ into $\nabla'^k R'$ for $k=0, 1, \cdots$, where $\nabla^k R$ $(k \geqq 1)$ denotes the k-th covariant derivative of the Riemannian curvature tensor R of g and $\nabla^0 R = R$.

As a different version of the equivalence problem in Riemannian geometry, K. Nomizu and K. Yano [4, 5] have obtained the following result:

THEOREM A. *If M and M' are both irreducible and analytic Riemannian manifolds, and if φ is a strongly curvature-preserving transformation of M to M', then φ is a homothety.*

However this kind of problem is important also in pseudo-Riemannian geometry. When g is an indefinite Riemannian metric, at any point it is reducible to

$$(dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^m)^2$$

with respect to some local coordinates $(x^1, \cdots, x^m)$, where $m = \dim M$ and the integer $2p-m$ is called the signature of g. We may assume that the signature is not smaller than 0. Of course the signature m implies that the metric is positive definite. Then the purpose of this note is to show the following

THEOREM 1. *Let M and M' be both irreducible and analytic pseudo-Riemannian manifolds, and assume that the signature of g is not zero in the case where $\dim M$ is even $\geqq 4$, then any strongly curvature-preserving transformation of M to M' is a homothety.*

The proof of Theorem 1 gives the proof also to the following Theorem which is a generalization of a result in [3] to pseudo-Riemannian manifolds:

THEOREM 2. *Let M and M' be irreducible pseudo-Riemannian manifolds, if the signature of g is not zero in the case where* $\dim M$ *is even* $\geqq 4$, *then any affine transformation of M to M' is a homothety.*

1. The linear transformation A. Let $g^* = \varphi^* g'$ be the metric in M induced by φ from g'. We take an arbitrary point x of M and fix it. Then by the quite similar argument on infinitesimal holonomy group as in [5] we have

LEMMA 1.1. *The restricted holonomy group $\psi(x)$ of g at x is contained in that $\psi^*(x)$ of g^*.*

Now we define a linear transformation A of the tangent space M_x at x by

$$g(AX, Y) = g^*(X, Y) \tag{1.1}$$

for X, Y in M_x. We show that A commutes with every element σ of $\psi(x)$. As g, and g^* by Lemma 1.1, are invariant by $\psi(x)$, we have

$$g(\sigma X, \sigma Y) = g(X, Y)\,, \tag{1.2}$$

$$g^*(\sigma X, \sigma Y) = g^*(X, Y)\,. \tag{1.3}$$

Then by (1.1), (1.2) and (1.3), we have

$$g(A\sigma X, \sigma Y) = g(\sigma A X, \sigma Y)\,. \tag{1.4}$$

As g and σ are non-singular we have $A\sigma=\sigma A$ for any σ of $\psi(x)$. Since $\psi(x)$ acts on M_x irreducibly, A must be of the form either $A=aI$, or $A=aI+bJ$, where I is the identity transformation of M_x and J is a linear transformation such that $J^2=-I$, a and b are real numbers. (p. 277, [2]) If $A=aI$ we have $g^*=ag$ at x, namely φ is conformal. Thus the essential point is to obtain the conditions for bJ to vanish. Suppose that $A=aI+bJ$, then by the symmetry of g^*, A and J are symmetric. This implies:

$$g(JX, JY) = -g(X, Y) \tag{1.5}$$

for any X, Y in M_x. Thus J satisfies:

(i) If X and Y are orthogonal, so are JX and JY.
(ii) If X is null, so is JX.
(iii) If X is positive (negative), then JX is negative (positive resp.);

Thus we have

LEMMA 1.2. *The only possible case of existence of J is that* $\dim M$ *is even and the signature is zero.*

Next we find out the system of equations which J must satisfy. By this we obtain the next:

LEMMA 1.3. *When* $\dim M$ *is* 2 *and the signature is zero, J does not exist.*

Assume that $\dim M = m = 2n$ and take a basis of M_x such that

$$J = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \quad E\colon \text{unit } n\times n \text{ matrix.} \tag{1.6}$$

As any $\sigma \in \psi(x)$ commutes with A and hence J, $\psi(x)$ is considered as the real representation of the unitary group $U(n)$:

$$\sigma = \begin{pmatrix} Q & -R \\ R & Q \end{pmatrix} \quad Q, R\colon n\times n \text{ matrices.} \tag{1.7}$$

By (1. 5) namely ${}^tJgJ = -g$, g is of the form

$$g = \begin{pmatrix} B & C \\ C & -B \end{pmatrix} \quad B, C\colon n\times n \text{ matrices.} \tag{1.8}$$

By (1. 2) namely ${}^t\sigma g\sigma = g$, Q and R satisfy the following:

$${}^tQBQ + {}^tQCR + {}^tRCQ - {}^tRBR = B\,, \tag{1.9}$$

$${}^tQCQ - {}^tQBR - {}^tRBQ - {}^tRCR = C\,. \tag{1.10}$$

So if $m = 2$, we put $Q = q$, $R = r$, $B = \beta$ and $C = \gamma$, then (1. 9) and (1. 10) reduce to

$$\beta q^2 + 2\gamma rq - \beta r^2 = \beta\,, \tag{1.11}$$

$$\gamma q^2 - 2\beta rq - \gamma r^2 = \gamma\,. \tag{1.12}$$

The solution (q, r) are at most two pairs. This implies $\psi(x) = \{\text{identity}\}$. If

$m \geqq 4$, the number of variables in Q, R exceeds the number of the equations (1.9), (1.10). For example, if $m = 4$, σ has 8 variables and 6 equations.

2. Conformal transformations preserving R and ∇R. In this section we use tensor calculus in a coordinate neighborhood. First we recall the followings.

LEMMA 2.1. *For a conformal transformation of pseudo-Riemannian manifolds $\varphi: M \to M'$ such that $\varphi^* g' = e^{2\alpha} g$, if φ maps R into R' and if $m \geqq 3$, then*

$$\nabla_j \alpha_k = \alpha_j \alpha_k - (1/2)\alpha_r \alpha^r g_{jk} \tag{2.1}$$

holds, where $\alpha_k = \partial_k \alpha$. (cf. [1, 5])

LEMMA 2.2. *Suppose that* (2.1) *holds for a conformal transformation, and that $d\alpha$ vanishes at a point, then α is constant on M.* [5]

LEMMA 2.3. *If $m=3$, we have*

$$R^i{}_{jkl} = \delta^i_l R_{jk} - \delta^i_k R_{jl} + R^i_l g_{jk} - R^i_k g_{jl} - (1/2)S(\delta^i_l g_{jk} - \delta^i_k g_{jl}),$$

where $R_{jk} = R^i{}_{jki}$ is the Ricci curvature tensor and S is the scalar curvature.

Now we prove the following Proposition which was obtained when the metrics are positive definite in [4, 5]:

PROPOSITION 2.4. *Let (M, g) and (M', g') be pseudo-Riemannian manifolds such that $\dim M \geqq 3$. If a conformal transformation φ of M to M' maps R into R' and ∇R into $\nabla' R'$, then either $d\alpha = 0$ or $R = 0$ holds on M.*

PROOF. Generally we have

$$\begin{aligned} \nabla_h({}^{\varphi}R')^i{}_{jkl} - {}^{\varphi}(\nabla' R')_h{}^i{}_{jkl} \\ = -W^i_{rh}({}^{\varphi}R')^r{}_{jkl} + W^r{}_{jh}({}^{\varphi}R')^i{}_{rkl} + W^r{}_{kh}({}^{\varphi}R')^i{}_{jrl} + W^r{}_{lh}({}^{\varphi}R')^i{}_{jkr} \end{aligned} \tag{2.2}$$

where ${}^{\varphi}(\ \ )$ means the tensor field transformed by φ, and

$$W^i_{jk} = \alpha_j \delta^i_k + \alpha_k \delta^i_j - \alpha^i g_{jk}, \tag{2.3}$$

if φ is a conformal transformation. As ${}^{\varphi}R' = R$ and ${}^{\varphi}(\nabla' R') = \nabla R$, the left hand side of (2.2) vanishes. We substitute the relation (2.3) into (2.2) and get

$$(2.4)\qquad \begin{aligned} &2\alpha_h R^i{}_{jkl} + \alpha^i R_{hjkl} + \alpha_j R^i{}_{hkl} + \alpha_k R^i{}_{jhl} + \alpha_l R^i{}_{jkh} \\ &\quad - \alpha_r R^r{}_{jkl}\delta^i_h - \alpha^r R^i{}_{rkl} g_{jh} - \alpha^r R^i{}_{jrl} g_{kh} - \alpha^r R^i{}_{jkr} g_{lh} = 0 . \end{aligned}$$

Transvecting (2.4) with g^{jk} and δ^h_i, we have

$$(2.5)\qquad 2\alpha_h R^i{}_l + \alpha^i R_{hl} + \alpha_l R^i{}_h - \alpha_r R^r{}_l \delta^i_h - \alpha^r R^i{}_r g_{lh} = 0 ,$$

$$(2.6)\qquad (m-3)\,\alpha_r R^r{}_{jkl} = R_{jk}\alpha_l - \alpha_k R_{jl} .$$

If we contract (2.5) with respect to i and l, we get $\alpha_h S = 0$. We assume that $d\alpha \neq 0$ everywhere on M otherwise by Lemma 2.2 we have $d\alpha = 0$ on M. Then $S = 0$ holds. Transvecting (2.6) with g^{jk}, we have $(m-2)\alpha_r R^r{}_l = 0$. Then (2.5) implies

$$2\alpha_k R_{jl} + \alpha_j R_{kl} + \alpha_l R_{jk} = 0 .$$

By taking cyclic sum of this equation, we have $\alpha_j R_{kl} = 0$, and hence $R_{kl} = 0$. If $m = 3$ we have $R = 0$ by Lemma 2.3. So we assume that $m > 3$, then (2.6) means $\alpha_r R^r{}_{jkl} = 0$. Then lowering the index i in (2.4) we have

$$(2.7)\qquad 2\alpha_h R_{ijkl} + \alpha_i R_{hjkl} + \alpha_j R_{ihkl} + \alpha_k R_{ijhl} + \alpha_l R_{ijkh} = 0 .$$

Take any point x and a coordinate neighborhood about x such that the vector α_h has the components $(\alpha_1, 0, \cdots, 0)$ at x.

(i) If we put $h=1$, $i, j, k, l \neq 1$, then $R_{ijkl} = 0$.

(ii) If we put $h=1$, $i=1$, $j, k, l \neq 1$, then $R_{1jkl} = 0$.

(iii) If $h=1$, $i=1$, $k=1$, $j, l \neq 1$, then $R_{1j1l} = 0$.

Thus $R=0$, this completes the proof.

3. Proof of Theorem 1. By Lemma 1.2 and 1.3, we have $A = aI$ namely φ is a conformal transformation of M to M' : $\varphi^* g' = e^{2\alpha} g$. Then by Proposition 2.4, if $m \geqq 3$, we have either $d\alpha = 0$ or $R = 0$. Since M is irreducible we have $d\alpha = 0$. For the case of $\dim M = 2$, the proof in [5] is valid.

4. Proof of Theorem 2. Contrary to Theorem 1, in Theorem 2 analyticity is not assumed. As for this we refer Theorem 9.1 in [2], p. 151.

Applications of Theorem 2 will be seen in another paper.

REFERENCES

[1] S. ISHIHARA, Groups of projective transformations and groups of conformal transformations, Journ. Math. Soc. Japan, 9(1957), 195-227.
[2] S. KOBAYASHI AND K. NOMIZU, Foundations of Differential Geometry, Vol. I, Interscience Tracts, No. 15, New York, 1963.
[3] K. NOMIZU, Sur les transformations affines d'une variété riemannienne, Comptes Rendus, Paris, 237(1953), 1308-1310.
[4] K. NOMIZU AND K. YANO, Some results related to the equivalence problem in Riemannian geometry, Proc. United States-Japan Sem. Diff. Geom., Kyoto, 1965.
[5] K. NOMIZU AND K. YANO, Some results related to the equivalence problem in Riemannian geometry, Math. Zeitshr., 97(1967), 29-37.

MATHEMATICAL INSTITUTE,
TÔHOKU UNIVERSITY,
SENDAI, JAPAN.