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ON C-KILLING FORMS IN A COMPACT SASAKIAN SPACE

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Introduction. On a normal contact metric space (=Sasakian space), we studied in a former paper [4] C-harmonic forms which are in certain sense an extension of harmonic forms. There we defined two operators Γ and D which correspond to the exterior differential operator d and the co-differential operator δ in a Riemann space. In this paper we get in the first place an integral formula for Γ and D.

On the other hand it is well known that for any harmonic form in a compact Riemann space, the Lie derivative with respect to a Killing form always vanishes. In order to obtain its analogy for C-harmonic forms in a compact Sasakian space, we introduce the notion of C-Killing forms. Then we shall show as an application of the integral formula for Γ and D that for any C-harmonic form in a compact Sasakian space its Lie derivative with respect to a C-Killing form must be zero. Lastly we treat with the case of a compact regular Sasakian space.

We suppose that manifolds are connected and the differentiable structures are of class C^{∞}.

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1. Preliminaries. An n-dimensional Riemannian space M^n is called a Sasakian space if it admits a unit Killing vector field η^{λ} [1] satisfying

$$(1.\ 1) \qquad \nabla_{\lambda}\nabla_{\mu}\eta_{\nu} = \eta_{\mu}g_{\lambda\nu} - \eta\, g_{\lambda\mu}$$

where $g_{\lambda\mu}$ is the metric tensor of M^n. It is well known that M^n is orientable and n is odd. We put $\varphi_{\lambda\mu}=\nabla_{\lambda}\eta_{\mu}$, $\varphi_{\lambda}{}^{\mu}=\varphi_{\lambda\rho}g^{\rho\mu}$. Then there exist some well-known identities, as follows (see Tachibana [2])

$$(1.\ 2) \qquad \nabla_{\lambda}\varphi_{\mu\nu}=\eta_{\mu}g_{\lambda\nu}-\eta_{\nu}g_{\lambda\mu}\,,$$

$$(1.\ 3) \qquad R_{\lambda\mu\nu\omega}\eta^{\omega}=\eta_{\lambda}g_{\mu\nu}-\eta_{\mu}g_{\lambda\nu}\,,$$

$$(1.\ 4) \qquad \nabla^{\lambda}\varphi_{\lambda\mu}=-(n-1)\eta_{\mu}, \quad \eta^{\lambda}R_{\lambda\mu}=(n-1)\eta_{\mu},$$

1) The Greek indices $\lambda, \mu, \nu, \ldots$ run from 1 to n.

(1. 5) $$R_{\mu\rho}\varphi_\lambda^{\ \rho}=-R_{\lambda\rho}\varphi_\mu^{\ \rho},\quad R_\mu^{\ \rho}\varphi_\rho^{\ \lambda}=R_\rho^{\ \lambda}\varphi_\mu^{\ \rho}.$$

We denote by $e(\eta)$ and L (resp. $i(\eta)$ and Λ) the exterior product (resp. inner product) of 1-form η and 2-form $d\eta$, then we have [4]

(1. 6) $$L=e(\eta)d+de(\eta),$$

(1. 7) $$\Lambda=i(\eta)\delta+\delta i(\eta).$$

We take an arbitrary 1-form ξ. Then the inner product $\Lambda_\xi=i(d\xi)$ for any p-form $u=(u_{\lambda_1\cdots\lambda_p})$ can be written by

$$(\Lambda_\xi u)_{\lambda_3\cdots\lambda_p}=\nabla^\rho\xi^\sigma u_{\rho\sigma\lambda_3\cdots\lambda_p}\qquad (p\geqq 2)$$

and $\Lambda_\xi u=0$ if p is 0 or 1. We can easily obtain

(1. 8) $$\Lambda_\xi=\delta i(\xi)+i(\xi)\delta.$$

The operators Φ, ∇_η, Γ and D for any p-form $u=(u_{\lambda_1\cdots\lambda_p})$ are defined by

(1. 9) $$(\Phi u)_{\lambda_1\cdots\lambda_p}=\sum_{i=1}^p\varphi_{\lambda_i}^{\ \sigma}u_{\lambda_1\cdots\overset{i}{\sigma}\cdots\lambda_p},\qquad (p\geqq 1)$$

(1.10) $$(\nabla_\eta u)_{\lambda_1\cdots\lambda_p}=\eta^\sigma\nabla_\sigma u_{\lambda_1\cdots\lambda_p},\qquad (p\geqq 0)$$

(1.11) $$(\Gamma u)_{\lambda_0\cdots\lambda_p}=\sum_{\alpha=0}^p(-1)^\alpha\varphi_{\lambda_\alpha}^{\ \ \sigma}\nabla_\sigma u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p},\qquad (p\geqq 0)$$

(1.12) $$(Du)_{\lambda_2\cdots\lambda_p}=\varphi^{\sigma\rho}\nabla_\sigma u_{\rho\lambda_2\cdots\lambda_p},\qquad (p\geqq 1)$$

where $u_{\lambda_1\cdots\overset{i}{\sigma}\cdots\lambda_p}$ means the subscript σ appears at the i-th position and $u_{\lambda_0\cdots\hat{\alpha}\cdots\lambda_p}$ means the α-th subscript λ_α is omitted.

We denote by $\theta(\xi)$ the Lie derivative with respect to a vector field ξ^λ. For a 1-form $\xi=(\xi_\lambda)$, identifying the covariant vector field with a contravariant vector field by the metric tensor, we also denote by $\theta(\xi)$ the Lie derivative of the vector field $\xi^\lambda=g^{\lambda\mu}\xi_\mu$.

Let the space M^n be compact. Then the global inner product of any p-forms u and v is given by

$$(u,v)=\int_{M^n}u\wedge *v$$

where the notations $*$ and $\wedge$ represent the dual operator and exterior product respectively. The dual operator $*$ satisfies

$$(1.13) \qquad ** = \text{identity}.$$

A p-form u on a Sasakian space is called to be C-harmonic if it satisfies

$$(1.14) \qquad du=0, \quad \delta u = e(\eta)\Lambda u.$$

Then clearly C-harmonic 1-forms are harmonic, and the converse is true. On C-harmonic forms, the following results are known [3], [4].

PROPOSITION 1.1. *In a compact Sasakian space, any C-harmonic p-form* u $(p \leqq (n-1)/2)$ *satisfies* $i(\eta)u=0$.

PROPOSITION 1.2. *In a compact Sasakian space, for any C-harmonic p-form* u $(p \leqq (n-1)/2)$ Λu *is also C-harmonic.*

PROPOSITION 1.3. *In a compact Sasakian space, a p-form* u $(p \leqq (n-1)/2)$ *is C-harmonic if and only if it satisfies* $i(\eta)u=0$ *and* $\triangle u = L\Lambda u$, *where* $\triangle$ *is the Laplacian.*

2. Integral formulas. In the following we consider a compact Sasakian space M^n. As for the operators Γ and D, we know the following relations.

LEMMA 2.1. [4] *In a Sasakian space, we have for any p-form u*

$$(2.\ 1) \qquad Du = \delta\nabla_\eta u - \nabla_\eta \delta u + (n-p)i(\eta)u,$$

$$(2.\ 2) \qquad \Gamma u = d\nabla_\eta u - \nabla_\eta du - pe(\eta)u.$$

LEMMA 2.2. *In a Sasakian space, we have for any p-form u and q-form v*

$$(2.\ 3) \qquad \Gamma(u\wedge v) = \Gamma u \wedge v + (-1)^p u \wedge \Gamma v,$$

$$(2.\ 4) \qquad *\Gamma * v = (-1)^p Du.$$

PROOF. Since it holds good for forms $u=(u_{\lambda_1\cdots\lambda_p})$ and $v=(v_{\lambda_1\cdots\lambda_q})$

$$(u\wedge v)_{\lambda_1\cdots\lambda_{p+q}} = \frac{1}{p!q!}\varepsilon^{\sigma_1\cdots\sigma_p\sigma_{1'}\cdots\sigma_{q'}}_{\lambda_1\cdots\cdots\cdots\lambda_{p+q}} u_{\sigma_1\cdots\sigma_p} v_{\sigma_{1'}\cdots\sigma_{q'}} \quad (k' = p+k,\ k \neq 1, \cdots q),$$

we have

$$(\Gamma(u\wedge v))_{\lambda_0\cdots\lambda_{p+q}} = \varphi_{\lambda_0}^{\rho}\nabla_\rho(u\wedge v)_{\lambda_1\cdots\lambda_{p+q}} - \sum_{i=1}^{p}\varphi_{\lambda_i}^{\rho}\nabla_\rho(u\wedge v)_{\lambda_1\cdots\overset{i}{\hat{\lambda}_0}\cdots\lambda_p\lambda_{1'}\cdots\lambda_{q'}}$$

$$-\sum_{j=1}^{q}\varphi_{\lambda_{j'}}^{\rho}\nabla_{\rho}(u\wedge v)_{\lambda_1\cdots\lambda_p\lambda_{1'}\cdots\overset{j'}{\hat{\lambda}_0}\cdots\lambda_{q'}}$$

$$=\frac{1}{p!q!}\{\varphi_{\lambda_0}^{\rho}\varepsilon_{\lambda_1\cdots\lambda_{p+q}}^{\sigma_1\cdots\sigma_{q'}}\nabla_{\rho}(u_{\sigma_1\cdots\sigma_p}v_{\sigma_{1'}\cdots\sigma_{q'}})-\sum_{i=1}^{\rho}\varphi_{\lambda_i}^{\rho}\varepsilon_{\lambda_1\cdots\lambda_0\cdots\lambda_p\lambda_{1'}\cdots\lambda_{q'}}^{\sigma_1\cdots\cdots\cdots\sigma_p\sigma_{1'}\cdots\sigma_{q'}}$$

$$\times\nabla_{\rho}(u_{\sigma_1\cdots\sigma_p}v_{\sigma_{1'}\cdots\sigma_{q'}})-\sum_{j=1}^{q}\varphi_{\lambda_{j'}}^{\rho}\varepsilon_{\lambda_1\cdots\lambda_p\lambda_{1'}\cdots\lambda_0\cdots\lambda_{q'}}^{\sigma_1\cdots\sigma_p\sigma_{1'}\cdots\cdots\cdots\sigma_{q'}}\nabla_{\rho}(u_{\sigma_1\cdots\sigma_p}v_{\sigma_{1'}\cdots\sigma_{q'}})\}$$

$$=\frac{1}{p!q!}\varphi_{\sigma_0}^{\rho}\varepsilon_{\lambda_0\lambda_1\cdots\lambda_{p+q}}^{\sigma_0\sigma_1\cdots\sigma_{p+q}}\nabla_{\rho}(u_{\sigma_1\cdots\sigma_p}v_{\sigma_1\cdots\sigma_{q'}}).$$

On the other hand we have

$$(\Gamma u\wedge v)_{\lambda_0\cdots\lambda_{p+q}}=\frac{1}{(p+1)!q!}\varepsilon_{\lambda_0\lambda_1\cdots\lambda_p\lambda_{1'}\cdots\lambda_{q'}}^{\sigma_0\sigma_1\cdots\sigma_p\sigma_{1'}\cdots\sigma_{q'}}\{\varphi_{\sigma_0}^{\rho}\nabla_{\rho}u_{\sigma_1\cdots\sigma_p}v_{\sigma_{1'}\cdots\sigma_{q'}}$$

$$-\sum_{i=1}^{\rho}\varphi_{\sigma_i}^{\rho}\nabla_{\rho}u_{\sigma_1\cdots\overset{i}{\hat{\sigma}_0}\cdots\sigma_q}v_{\sigma_{1'}\cdots\sigma_{p'}}\}$$

$$=\frac{1}{p!q!}\varepsilon_{\lambda_0\cdots\lambda_{q'}}^{\sigma_0\cdots\sigma_{q'}}\varphi_{\sigma_0}^{\rho}\nabla_{\rho}u_{\sigma_1\cdots\sigma_p}v_{\sigma_{1'}\cdots\sigma_{q'}}.$$

In the same way, we get

$$(\Gamma v\wedge u)_{\lambda_0\cdots\lambda_{p+q}}=\frac{1}{p!q!}\varepsilon_{\lambda_0\cdots\lambda_{p+q}}^{\sigma_0\cdots\sigma_{p+q}}\varphi_{\sigma_0}^{\rho}\nabla_{\rho}v_{\sigma_1\cdots\sigma_q}u_{\sigma_{q+1}\cdots\sigma_{q+p}}.$$

Therefore we have

$$(u\wedge\Gamma v)_{\lambda_0\cdots\lambda_{p+q}}=(-1)^{p(q+1)}\frac{1}{p!q!}\varepsilon_{\lambda_0\cdots\lambda_q\lambda_{q+1}\cdots\lambda_{q+p}}^{\sigma_0\cdots\sigma_q\sigma_{q+1}\cdots\sigma_{q+p}}\varphi_{\rho_0}^{\rho}u_{\sigma_{q+1}\cdots\sigma_{p+q}}\nabla_{\rho}v_{\sigma_1\cdots\sigma_p}$$

$$=(-1)^{p(q+1)+qp}\frac{1}{p!q!}\varepsilon_{\lambda_0\lambda_1\cdots\lambda_p\lambda_{1'}\cdots\lambda_{q'}}^{\sigma_0\sigma_1\cdots\sigma_p\sigma_{1'}\cdots\sigma_{q'}}\varphi_{\sigma_0}^{\rho}u_{\sigma_1\cdots\sigma_p}\nabla_{\rho}v_{\sigma_{1'}\cdots\sigma_{q'}}$$

$$=(-1)^{p}\frac{1}{p!q!}\varepsilon_{\lambda_0\lambda_1\cdots\lambda_{p+q}}^{\sigma_0\sigma_1\cdots\sigma_{p+q}}\varphi_{\sigma_0}^{\rho}u_{\sigma_1\cdots\sigma_p}\nabla_{\rho}v_{\sigma_{1'}\cdots\sigma_{q'}}.$$

Thus we can obtain

$$(\Gamma u\wedge v)_{\lambda_0\cdots\lambda_{p+q}}+(-1)^{p}(u\wedge\Gamma v)_{\lambda_0\cdots\lambda_{p+q}}=\frac{1}{p!q!}\varepsilon_{\lambda_0\lambda_1\cdots\lambda_{p+q}}^{\sigma_0\sigma_1\cdots\sigma_{p+q}}\varphi_{\sigma_0}^{\rho}$$

$$\times(\nabla_\rho u_{\sigma_1\cdots\sigma_q} v_{\sigma_1'\cdots\sigma_q'} + u_{\sigma_1\cdots\sigma_p} \nabla_\rho v_{\sigma_1'\cdots\sigma_q'})$$
$$= \Gamma(u\wedge v)_{\lambda_0\cdots\lambda_{p+q}}.$$

Next we calculate (2.4). We write $g=\det(g_{\lambda\mu})$ and $\varepsilon_{\lambda_1\cdots\lambda_n}=\varepsilon^{1\cdots n}_{\lambda_1\cdots\lambda_n}$. Then we have for any p-form u

$$(*\Gamma*u)_{\lambda_2\cdots\lambda_p} = \frac{1}{(n-p+1)!}\sqrt{g}\, g^{\sigma_0\mu_0}\cdots g^{\sigma_{n-p}\mu_{n-p}}\Big(\sum_{\alpha=0}^{n-p}(-1)^\alpha \varphi^{\rho}_{\sigma_\alpha}\nabla_\rho(*u)_{\sigma_0\cdots\hat{\sigma}_\alpha\cdots\sigma_{n-p}}\Big)\varepsilon_{\mu_0\cdots\mu_{n-p}\lambda_2\cdots\lambda_p}$$

$$= \frac{1}{(n-p)!}(-1)^{\alpha+\alpha-1}\sqrt{g}\, g^{\sigma_0\mu_0}\cdots g^{\sigma_{n-p}\mu_{n-p}}\varphi^{\rho}_{\sigma_0}\nabla_\rho(*u)_{\sigma_1\cdots\sigma_{n-p}}\varepsilon_{\mu_\alpha\mu_1\cdots\overset{\alpha}{\hat{\mu}_0}\cdots\mu_{n-p}\lambda_2\cdots\lambda_p}$$

$$= \frac{g}{(n-p)!\,p!}g^{\sigma_1\mu_1}\cdots g^{\sigma_{n-p}\mu_{n-p}}g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}\varphi^{\mu_0\rho}\nabla_\rho u_{\beta_1\cdots\beta_p}\varepsilon_{\alpha_1\cdots\alpha_p\sigma_1\cdots\sigma_{n-p}}\varepsilon_{\mu_0\cdots\mu_{n-p}\lambda_2\cdots\lambda_p}$$

$$= \frac{1}{(n-p)!\,p!}(-1)^{(n-p)(p-1)}\varepsilon^{\beta_1\cdots\beta_p\mu_1\cdots\mu_{n-p}}\varepsilon_{\mu_0\lambda_2\cdots\lambda_p\mu_1\cdots\mu_{n-p}}\varphi^{\mu_0\rho}\nabla_\rho u_{\beta_1\cdots\beta_p}$$

$$= \frac{1}{p!}(-1)^{n(p+1)}\varepsilon^{\beta_1\cdots\beta_p}_{\mu_0\lambda_2\cdots\lambda_p}\varphi^{\mu_0\rho}\nabla_\rho u_{\beta_1\cdots\beta_p}$$

$$=(-1)^{np+n+1}(Du)_{\lambda_2\cdots\lambda_p}.$$

Since n is odd, we have $(-1)^{np+n+1}=(-1)^p$, hence (2.4) is obtained.

LEMMA 2.3. *In a Sasakian space, we have for any forms u and v*

$$\nabla_\eta(u\wedge v)=\nabla_\eta u\wedge v+u\wedge\nabla_\eta v, \tag{2.5}$$

$$*\nabla_\eta*u=\nabla_\eta u. \tag{2.6}$$

PROOF. We take $u=(u_{\lambda_1\cdots\lambda_p})$, $v=(v_{\lambda_1\cdots\lambda_q})$. Then

$$(\nabla_\eta(u\wedge v))_{\lambda_1\cdots\lambda_{p+q}}=\eta^\rho\nabla_\rho(\varepsilon^{\sigma_1\cdots\sigma_{p+q}}_{\lambda_1\cdots\lambda_{p+q}}u_{\sigma_1\cdots\sigma_p}v_{\sigma_1'\cdots\sigma_q'})$$
$$=\varepsilon^{\sigma_1\cdots\sigma_{p+q}}_{\lambda_1\cdots\lambda_{p+q}}(\eta^\rho\nabla_\rho\, u_{\sigma_1\cdots\sigma_p}v_{\sigma_1'\cdots\sigma_p'}+u_{\sigma_1\cdots\sigma_p}\eta^\rho\nabla_\rho v_{\sigma_1'\cdots\sigma_q'})$$
$$=(\nabla_\eta u\wedge v)_{\lambda_1\cdots_{p+q}}+(u\wedge\nabla_\eta v)_{\lambda_1\cdots\lambda_{p+q}},$$

and

$$(*\nabla_\eta*u)_{\lambda_1\cdots\lambda_p}=\frac{1}{(n-p)!}\sqrt{g}g^{\sigma_1\mu_1}\cdots g^{\sigma_{n-p}\mu_{n-p}}\eta^\rho\nabla_\rho(*u)_{\sigma_1\cdots\sigma_{n-p}}\varepsilon_{\mu_1\cdots\mu_{n-p}\lambda_1\cdots\lambda_p}$$

$$= \frac{g}{(n-p)!\,p!} g^{\sigma_1\mu_1}\cdots g^{\sigma_{n-p}\mu_{n-p}} g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}\eta^\rho \nabla_\rho u_{\beta_1\cdots\beta_p}\varepsilon_{\alpha_1\cdots\alpha_p\sigma_1\cdots\sigma_{n-p}}\varepsilon_{\mu_1\cdots\mu_{n-p}\lambda_1\cdots\lambda_p}$$

$$= \frac{1}{(n-p)!\,p!} \varepsilon^{\beta_1\cdots\beta_p\mu_1\cdots\mu_{n-p}}\varepsilon_{\mu_1\cdots\mu_{n-p}\lambda_1\cdots\lambda_p}\eta^\rho \nabla_\rho u_{\beta_1\cdots\beta_p}$$

$$=(-1)^{p(n-p)}(\nabla_\eta u)_{\lambda_1\cdots\lambda_p}.$$

As n is odd, $(-1)^{p(n-p)}=1$ for any p, thus we get (2.6).

Since η is a Killing form, the Lie derivative $\theta(\eta)$ with respect to η satisfies the following relations [4].

$$\theta(\eta) = -\delta e(\eta) - e(\eta)\delta, \qquad (2.7)$$
$$*\theta(\eta)* = \theta(\eta).$$

The operator Φ satisfies for any u

$$\Phi u = \theta(\eta)u - \nabla_\eta u. \qquad (2.8)$$

Therefore by virtue of Lemma 2.3 and the fact that $\theta(\eta)$ is a derivation, we have

LEMMA 2.4. [6] *In a Sasakian space, we have for any forms u and v*

$$\Phi(u\wedge v) = \Phi u \wedge v + u\wedge \Phi v, \qquad (2.9)$$
$$*\Phi * u = \Phi u.$$

Using these Lemmas, we study some integral formulas in compact case.

THEOREM 2.5. *In a compact Sasakian space, we have for any p-forms u and v*

$$(\nabla_\eta u, v) = -(u, \nabla_\eta v), \qquad (2.10)$$

$$(\theta(\eta)u, v) = -(u, \theta(\eta)v), \qquad (2.11)$$

$$(\Phi u, v) = -(u, \Phi v). \qquad (2.12)$$

PROOF. From (2.5) and (2.6) we have

$$\nabla_\eta u \wedge * v = \nabla_\eta(u\wedge * v) - u\wedge * \nabla_\eta v.$$

As $u\wedge * v$ is an n- form, there exists a function f on M^n such that $u\wedge * v$ is equal to $f\omega$, where ω is the volume element of M^n. It is known that ω is written as $c\eta\wedge(d\eta)^m (m=(n-1)/2)$, c is a constant [5]. Since the relations

$$\nabla_\eta \eta = 0, \quad \nabla_\eta d\eta = 0$$

hold good, we have

$$\nabla_\eta (u\wedge *v) = \nabla_\eta f\omega = -\delta(f\eta)\omega.$$

Hence we get

$$\int_{M^n} \nabla_\eta (u\wedge *v) = 0,$$

which means (2.10) is true. (2.11) is the result of (2.7) and

$$\theta(\eta) = di(\eta) + i(\eta)d.$$

(2.12) follows from (2.10), (2.11) and (2.8).

THEOREM 2.6. *In a compact n-dimensional Sasakian space, we have for any p-form u and (p+1)-form v*

$$(\Gamma u, v) = (u, Dv) - (n-1)(e(\eta)u, v). \tag{2.13}$$

PROOF. By virtue of Lemma 2.1 and Theorem 2.5, we see

$$\begin{aligned}(\Gamma u, v) &= (\nabla_\eta u, \delta v) - (\nabla_\eta du, v) - p(u, i(\eta)v) \\ &= (u, -\nabla_\eta \delta v + \delta \nabla_\eta v - pi(\eta)v) \\ &= (u, Dv - (n-p-1)i(\eta)v - pi(\eta)v) \\ &= (u, Dv) - (n-1)(u, i(\eta)v).\end{aligned}$$

This is the required result.

3. *C*-Killing forms. Let M^n be an n-dimensional Sasakian space. We call a 1-form ξ of M^n to be C-Killing if it satisfies

$$\delta\xi = 0, \tag{3. 1}$$

$$\theta(\xi)(g_{\lambda\mu} - \eta_\lambda \eta_\mu) = 0. \tag{3. 2}$$

Clearly the 1-form η is C-Killing. The vector space of all vector fields identified with C-Killing forms is a Lie algebra. Especially we call a C-Killing form ξ such that

$$\xi' \equiv i(\eta)\xi = \text{const.} \tag{3. 3}$$

to be special C-Killing. η is a special C-Killing form. A Killing form which is at the same time a special C-Killing form is of the type $\xi'\eta$, $\xi'=$const.

LEMMA 3.1. *In a Sasakian space, we have for a C-Killing form*

$$\nabla_\eta \xi' = 0, \tag{3. 4}$$

$$\theta(\eta)\xi_\lambda = 0. \tag{3. 5}$$

PROOF. The equation (3.2) can be expressed as

$$\nabla_\lambda \xi_\mu + \nabla_\mu \xi_\lambda = 2\xi^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda) + \nabla_\lambda \xi' \eta_\mu + \nabla_\mu \xi' \eta_\lambda. \tag{3. 6}$$

Transvecting (3.6) with $g^{\lambda\mu}$, we obtain

$$\delta\xi = -\nabla_\eta \xi',$$

hence (3.4) follows by virtue of (3.1). Next transvecting (3.6) with η^λ, we have

$$\eta^\lambda \nabla_\lambda \xi_\rho + \varphi_\rho{}^\lambda \xi_\lambda = (\nabla_\eta \xi')\eta_\rho = 0$$

which means $\theta(\eta)\xi_\lambda = 0$.

Let ξ be a special C-Killing form. Then by virtue of (3.3) and (3.6) it satisfies

$$\nabla_\lambda \xi_\mu + \nabla_\mu \xi_\lambda = 2\xi^\rho(\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda). \tag{3. 7}$$

Conversely, we show that (3.7) is a sufficient condition for a 1-form ξ to be special C-Killing, in a compact case. Evidently (3.1) follows from (3.7). Differentiating (3.7) by ∇^λ, we have

$$\nabla^\lambda \nabla_\lambda \xi_\mu + R^\rho_\mu \xi_\rho = -2D\xi\eta_\mu + 2n\xi'\eta_\mu - 2\xi_\mu + 2\eta^\lambda \nabla_\lambda \xi^\rho \varphi_{\rho\mu}. \tag{3. 8}$$

LEMMA 3.2. *In a compact Sasakian space, for a* 1-*form* ξ *satisfying* (3.7) *the scalar* $\xi' = i(\eta)\xi$ *is a constant function.*

PROOF. Calculating the Laplacian of ξ', we have by virtue of (3.8)

$$\begin{aligned}\nabla^\lambda\nabla_\lambda\xi' &= \eta^\mu\nabla^\lambda\nabla_\lambda\xi_\mu+2D\xi-(n-1)\xi' \\ &= -(n-1)\xi'-2D\xi+2n\xi'-2\xi'+2D\xi-(n-1)\xi' \\ &= 0.\end{aligned}$$

Therefore if M^n is compact, ξ' must be constant.

From this Lemma 3.2, we see that the form ξ having the property (3.7) also satisfies (3.6), and therefore it is a C-Killing form. Again from Lemma 3.2 it must be special C-Killing. Thus we proved the following

THEOREM 3.3. *A 1-form ξ on a compact Sasakian space is special C-Killing if and only if it satisfies the relation* (3.7).

Next we consider the relation between C-Killing and special C-Killing forms. Then

THEOREM 3.4. *For a C-Killing form ξ on a Sasakian space, a 1-form ζ defined by $\zeta=\xi-\xi'\eta$ ($\xi'=i(\eta)\xi$) is special C-Killing. Conversely for a special C-Killing form ζ and a scalar function f, a 1-form ξ defined by $\xi=\zeta+f\eta$ is C-Killing if and only if $\nabla_\eta f=0$.*

PROOF. The first half. Since $\zeta'=i(\eta)\zeta=0$, (3.6) coincides with (3.7) for ζ, hence we have only to show that ζ satisfies (3.7). Calculating directly, we get

$$\begin{aligned}\nabla_\lambda\zeta_\mu+\nabla_\mu\zeta_\lambda &= \nabla_\lambda\xi_\mu+\nabla_\mu\xi_\lambda-\nabla_\lambda\xi'\eta_\mu-\nabla_\mu\xi'\eta_\lambda-\xi'\varphi_{\lambda\mu}-\xi'\varphi_{\mu\lambda} \\ &= 2\xi^\rho(\varphi_{\rho\lambda}\eta_\mu+\varphi_{\rho\mu}\eta_\lambda) \\ &= 2\zeta^\rho(\varphi_{\rho\lambda}\eta_\mu+\varphi_{\rho\mu}\eta_\lambda).\end{aligned}$$

For the latter half, we look for the condition that the 1-form $f\eta$ to be C-Killing, and get

$$\begin{aligned}&\delta(f\eta)=-\nabla_\eta f, \\ &\theta(f\eta)(g_{\lambda\mu}-\eta_\lambda\eta_\mu)=0.\end{aligned}$$

Therefore if the 1-form ξ_λ is C-Killing, then $\nabla_\eta f=0$, and the converse is true.

4. Special *C*-Killing forms. In this section, we show that a C-harmonic p-form ($p\leqq(n-1)/2$) is invariant by a C-Killing form in a compact Sasakian space. Let ξ be a special C-Killing form. Then we have from (3.8) and (3.5)

$$\nabla^{\lambda}\nabla_{\lambda}\xi_{\mu} + R_{\mu\rho}\xi^{\rho} = -2(D\xi-(n+1)\xi')\eta_{\mu} - 4\xi_{\mu}, \tag{4.1}$$

where we put $\xi'=i(\eta)\xi$. Differentiating (3.7) by ∇_{ν} and adding cyclicly with respect to the subscript λ, μ, ν, we have

$$\begin{aligned}\nabla_{\lambda}\nabla_{\mu}\xi_{\nu} = R_{\mu\nu\lambda}{}^{\varepsilon}\xi_{\varepsilon} &+ \eta_{\lambda}(\nabla_{\mu}\xi^{\rho}\varphi_{\rho\nu} - \nabla_{\nu}\xi^{\rho}\varphi_{\rho\mu}) + \eta_{\mu}(\nabla_{\lambda}\xi^{\rho}\varphi_{\rho\nu} - \nabla_{\nu}\xi^{\rho}\varphi_{\rho\lambda}) \\ &+ \eta_{\nu}(\nabla_{\lambda}\xi^{\rho}\varphi_{\rho\mu} + \nabla_{\mu}\xi^{\rho}\varphi_{\rho\lambda}) + 2\xi^{\rho}(\varphi_{\rho\lambda}\varphi_{\mu\nu} + \varphi_{\rho\mu}\varphi_{\lambda\nu}) \\ &+ 2\xi' g_{\lambda\mu}\eta_{\nu} + 2(\eta_{\lambda}\eta_{\mu}\xi_{\nu} - \eta_{\mu}\eta_{\nu}\xi_{\lambda} - \eta_{\lambda}\eta_{\nu}\xi_{\mu}),\end{aligned} \tag{4.2}$$

which will be used in the proof of Lemma 4.2.

By virtue of (2.8) and (3.5) we obtain

$$\Phi\xi = -\nabla_{\eta}\xi(\equiv\bar{\xi}).$$

Then making use of (3.7), we have

$$\begin{aligned}(d\bar{\xi})_{\lambda\mu} &= \nabla_{\lambda}(\varphi_{\mu\rho}\xi^{\rho}) - \nabla_{\mu}(\varphi_{\lambda\rho}\xi^{\rho}) \\ &= (\eta_{\lambda}\xi_{\mu} - \eta_{\mu}\xi_{\lambda}) + (\varphi_{\lambda}^{\rho}\nabla_{\rho}\xi_{\mu} - \varphi_{\mu}^{\rho}\nabla_{\rho}\xi_{\lambda}) \\ &= (e(\eta)\xi)_{\lambda\mu} + (\Gamma\xi)_{\lambda\mu}.\end{aligned}$$

Thus we can get

$$d\bar{\xi} = e(\eta)\xi + \Gamma\xi. \tag{4.3}$$

LEMMA 4.1. *In a compact Sasakian space, for any* 1*-form satisfying* $\xi'\equiv i(\eta)\xi=$ *constant we have*

$$(\Gamma\xi, e(\eta)\xi) = -(e(\eta)\xi, e(\eta)\xi). \tag{4.4}$$

PROOF. In a Sasakian space, for any p-form u the following

$$\Gamma i(\eta)u + i(\eta)\Gamma u = -pu + e(\eta)i(\eta)u \tag{4.5}$$

holds good. In fact, owing to (2.2) we have

$$\begin{aligned}\Gamma i(\eta)u + i(\eta)\Gamma u &= di(\eta)\nabla_{\eta}u - \nabla_{\eta}di(\eta)u - (p-1)e(\eta)i(\eta)u \\ &+ i(\eta)d\nabla_{\eta}u - \nabla_{\eta}i(\eta)du - pi(\eta)e(\eta)u\end{aligned}$$

$$= \theta(\eta)\nabla_\eta u - \nabla_\eta \theta(\eta) u - pu + e(\eta)i(\eta)u$$
$$= - pu + e(\eta)i(\eta)u\,,$$

since ∇_η commutes with $i(\eta)$ and $\theta(\eta)$. Hence it follows for a 1-form ξ

$$\begin{aligned}(e(\eta)\xi, \Gamma\eta) &= (\xi, i(\eta)\Gamma\xi)\\ &= (\xi, -\Gamma i(\eta)\xi - \xi + e(\eta)i(\eta)\xi)\\ &= (\xi, -\Gamma i(\eta)\xi - i(\eta)e(\eta)\xi)\\ &= (\xi, \Gamma\xi') - (e(\eta)\xi, e(\eta)\xi).\end{aligned}$$

If ξ' is constant, then we have $\Gamma\xi'=0$. This proves the lemma.

LEMMA 4.2. *In a compact Sasakian space, we have for a special C-Killing form* ξ

$$(\Gamma\xi, \Gamma\xi) = (e(\eta)\xi, e(\eta)\xi)\,. \tag{4. 6}$$

PROOF. Calculating $D\Gamma\xi$ for a special C-Killing form ξ, we first see from (4.2)

$$\eta^\rho\eta^\sigma\nabla_\rho\nabla_\sigma\xi_\lambda = - \xi_\lambda + \xi'\eta_\lambda,$$
$$\varphi^{\rho\sigma}\nabla_\rho\nabla_\tau\xi_\sigma = (\varphi_\rho{}^\sigma R_{\sigma\tau} + (n + 2)\varphi_{\rho\tau})\xi^\rho.$$

Then making use of these relations and (4.1) we have

$$\begin{aligned}(D\Gamma\xi)_\lambda &= \varphi^{\rho\sigma}\nabla_\rho(\varphi_\sigma{}^\tau\nabla_\tau\xi_\lambda - \varphi_\lambda{}^\tau\nabla_\tau\xi_\sigma)\\ &= - \nabla^\rho\nabla_\rho\xi^\lambda + \eta^\rho\eta^\sigma\nabla_\rho\nabla_\sigma\xi_\lambda - D\xi\eta_\lambda + \varphi_\lambda{}^\rho\eta^\sigma\nabla_\rho\xi_\sigma - \varphi_\lambda^\tau\varphi^{\rho\sigma}\nabla_\rho\nabla_\tau\xi_\sigma\\ &= - \nabla^\rho\nabla_\rho\xi_\lambda - R_{\lambda\rho}\xi^\rho - D\xi\eta_\lambda - (n + 2)\xi_\lambda + (2n + 1)\xi'\eta_\lambda\\ &= D\xi\eta_\lambda - \xi'\eta_\lambda - (n-2)\xi_\lambda\,.\end{aligned}$$

On the other hand, since $\Gamma\xi'=0$ for a special C-Killing form ξ_λ, we have by virtue of (2.13)

$$(D\xi, \xi') - (n-1)(\xi', \xi')=0\,. \tag{4. 7}$$

Integrating $i(\xi)(D\Gamma\xi)$ on M^n, we have

$$(\xi, D\Gamma\xi) = (\xi', D\xi) - (\xi', \xi') - (n-2)(\xi, \xi)$$

$$=-(n-2)(e(\eta)\xi, e(\eta)\xi).$$

Taking account of (2.13) again and considering Lemma 4.1, it follows that

$$\begin{aligned}(\Gamma\xi, \Gamma\xi) &= (\xi, D\Gamma\xi)-(n-1)(e(\eta)\xi, \Gamma\xi)\\ &= -(n-2)(e(\eta)\xi, e(\eta)\xi) + (n-1)(e(\eta)\xi, e(\eta)\xi)\\ &= (e(\eta)\xi, e(\eta)\xi),\end{aligned}$$

and the lemma is proved.

THEOREM 4.3. *In a compact Sasakian space, $\bar{\xi}=\Phi\xi$ is a closed 1-form for any C-Killing form ξ.*

PROOF. By virtue of Theorem 3.4, a 1-form $\zeta=\xi-\xi'\eta$ is a special C-Killing form for a C-Killing form ξ. Moreover the relation

$$\Phi\xi = \Phi\zeta$$

holds good. Therefore it is sufficient to prove the theorem for a special C-Killing form ξ. From (4.3), (4.4) and (4.6), we have

$$\begin{aligned}(d\bar{\xi}, d\bar{\xi}) &= (\Gamma\xi, \Gamma\xi) + 2(\Gamma\xi, e(\eta)\xi) + (e(\eta)\xi, e(\eta)\xi)\\ &= (e(\eta)\xi, e(\eta)\xi)-2(e(\eta)\xi, e(\eta)\xi) + (e(\eta)\xi, e(\eta)\xi)\\ &= 0,\end{aligned}$$

which shows $d\bar{\xi}=0$.

From this Theorem 4.3 and (4.3), we have for a special C-Killing form ξ

$$\nabla_\lambda\bar{\xi}_\mu=\nabla_\mu\bar{\xi}_\lambda, \tag{4. 8}$$

$$\Gamma\xi=-e(\eta)\xi. \tag{4. 8}$$

Making use of (4.8), (4.2) becomes a simpler form as follows.

COROLLARY 4.4. *In a compact Sasakian space, we have for a special C-Killing form ξ*

$$\nabla_\lambda\nabla_\mu\xi_\nu=R_{\mu\nu\lambda}{}^\rho\xi_\rho-2\nabla_\lambda\bar{\xi}_\mu\eta_\nu + 2\xi^\rho(\varphi_{\rho\lambda}\varphi_{\mu\nu} + \varphi_{\rho\mu}\varphi_{\lambda\nu}). \tag{4.10}$$

COROLLARY 4.5. *In a compact Sasakian space, $d\xi$ is hybrid for a special C-Killing form ξ, that is, $d\xi$ satisfies the relations*

$$\eta^{\rho}(d\xi)_{\rho\lambda}=0, \qquad \varphi_{\lambda}{}^{\rho}\varphi_{\mu}{}^{\sigma}(d\xi)_{\rho\sigma}=(d\xi)_{\lambda\mu}.$$

PROOF. Since $i(\eta)d\xi=\theta(\eta)\xi-di(\eta)\xi=0$, the first relation is evident. Next, (4.9) is written explicitely as follows

$$\varphi_{\lambda}{}^{\sigma}\nabla_{\sigma}\xi_{\mu}-\varphi_{\mu}{}^{\sigma}\nabla_{\sigma}\xi_{\lambda}=-\eta_{\lambda}\xi_{\mu}+\eta_{\mu}\xi_{\lambda}\,.$$

Transvecting it with φ_{ν}^{λ}, we have

$$\varphi_{\lambda}{}^{\rho}\varphi_{\mu}{}^{\sigma}\nabla_{\sigma}\xi_{\rho}=(\nabla_{\mu}\xi_{\lambda}-\nabla_{\lambda}\xi_{\mu})/2.$$

Exchanging the indices λ and μ in this relation, and subtracting them, we get

$$\varphi_{\lambda}{}^{\rho}\varphi_{\mu}{}^{\sigma}(\nabla_{\sigma}\xi_{\rho}-\nabla_{\rho}\xi_{\sigma})=\nabla_{\mu}\xi_{\lambda}-\nabla_{\lambda}\xi_{\mu}.$$

This is equivalent to

$$\varphi_{\lambda}{}^{\rho}\varphi_{\mu}{}^{\sigma}(d\xi)_{\rho\sigma}=(d\xi)_{\lambda\mu}\,.$$

LEMMA 4.6. *In a Sasakian space, we have for a* 1-*form* ξ

$$i(\xi)L-Li(\xi)=-2e(\bar{\xi}),\quad \bar{\xi}=\Phi\xi, \tag{4.11}$$

$$i(\xi)\Lambda=\Lambda i(\xi),\quad i(\eta)\Lambda_{\xi}=\Lambda_{\xi}i(\eta). \tag{4.12}$$

PROOF. For any p-form $u=(u_{\lambda_1\cdots\lambda_p})$, we have

$$\begin{aligned}\frac{1}{2}(i(\xi)Lu)_{\lambda_0\cdots\lambda_p}=\xi^{\sigma}\Big(&\varphi_{\sigma\lambda_0}u_{\lambda_1\cdots\lambda_p}-\sum_i\varphi_{\sigma\lambda_i}u_{\lambda_1\cdots\overset{i}{\hat{\lambda}_0}\cdots\lambda_p}-\sum_i\varphi_{\lambda_i\lambda_0}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p}\\ &+\sum_{i<j}\varphi_{\lambda_i\lambda_j}u_{\lambda_1\cdots\overset{i}{\hat{\sigma}}\cdots\overset{j}{\hat{\lambda}_0}\cdots\lambda_p}\Big)=-\Big(\bar{\xi}_{\lambda_0}u_{\lambda_1\cdots\lambda_p}-\sum_i\bar{\xi}_{\lambda_i}u_{\lambda_1\cdots\overset{i}{\hat{\lambda}_0}\cdots\lambda_p}\Big)\\ &+\Big(\varphi_{\lambda_0\lambda_1}\xi^{\sigma}u_{\sigma\lambda_2\cdots\lambda_p}-\sum_{j\geqq 2}\varphi_{\lambda_0\lambda_j}\xi^{\sigma}u_{\sigma\lambda_2\cdots\overset{j}{\hat{\lambda}_1}\cdots\lambda_p}-\sum_{j\geqq 2}\varphi_{\lambda_j\lambda_1}\xi^{\sigma}u_{\sigma\lambda_2\cdots\overset{j}{\hat{\lambda}_0}\cdots\lambda_p}\\ &+\sum_{2\leqq i<j}\varphi_{\lambda_i\lambda_j}\xi^{\sigma}u_{\sigma\lambda_2\cdots\overset{i}{\hat{\lambda}_0}\cdots\overset{j}{\hat{\lambda}_1}\cdots\lambda_p}\Big)=-(e(\bar{\xi})u)_{\lambda_0\cdots\lambda_p}+\frac{1}{2}(Li(\xi)u)_{\lambda_0\cdots\lambda_p}.\end{aligned}$$

(4.12) is evident.

LEMMA 4.7. *In a Sasakian space, we have for a* 1-*form* ξ

(4.13) $$\triangle i(\xi)-i(\xi)\triangle=d\Lambda_\xi-\Lambda_\xi d+\delta\theta(\xi)-\theta(\xi)\delta.$$

This is easily obtained from (1.8).

LEMMA 4.8. *In a compact Sasakian space, we have for a special C-Killing form ξ and for any p-form u,*

(4.14) $$\begin{aligned}(\delta\theta(\xi)-\theta(\xi)\delta)u&=2e(\overline{\xi})\wedge u+2i(\overline{\xi})i(\eta)du-2di(\overline{\xi})u'\\&\quad+4\overline{\xi}^\rho\nabla_\rho u'+2(D\xi-(n-1)\xi')u',\end{aligned}$$

where we put $\overline{\xi}=\Phi\xi$, $\xi'=i(\eta)\xi$, $u'=i(\eta)u$,

PROOF. (4.14) is a result of a little complicated but straightforward calculation. We sketch the outline. For a p-form $u=(u_{\lambda_1\cdots\lambda_p})$, we have

$$\begin{aligned}(\delta\theta(\xi)u)_{\lambda_2\cdots\lambda_p}&=-\nabla^\rho\Big(\xi^\sigma\nabla_\sigma u_{\rho\lambda_2\cdots\lambda_p}+\nabla_\rho\xi^\sigma u_{\sigma\lambda_2\cdots\lambda_p}+\sum_{i\geq 2}\nabla_{\lambda_i}\xi^\sigma u_{\sigma\lambda_2\cdots\overset{i}{\hat{\sigma}}\cdots\lambda_p}\Big)\\&=A+B+C+D,\end{aligned}$$

where we put

$$A=-(\nabla^\rho\xi^\sigma+\nabla^\sigma\xi^\rho)\nabla_\rho u_{\sigma\lambda_2\cdots\lambda_p},$$

$$B=-\nabla^\rho\nabla_\rho\xi^\sigma u_{\sigma\lambda_2\cdots\lambda_p},\qquad C=-\sum_{j\geq 2}\nabla^\rho\nabla_{\lambda_j}\xi^\sigma u_{\rho\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p},$$

$$D=-\xi^\sigma\nabla^\rho\nabla_\sigma u_{\rho\lambda_2\cdots\lambda_p}-\sum_{j\geq 2}\nabla_{\lambda_j}\xi^\sigma\nabla^\rho\nabla_\rho u_{\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}.$$

Taking account of (3.7), we have

$$\begin{aligned}A&=-2\xi^\tau(\varphi_\tau{}^\rho\eta^\sigma+\varphi_\tau{}^\sigma\eta^\rho)\nabla_\rho u_{\sigma\lambda_2\cdots\lambda_p}\\&=2(i(\overline{\xi})\nabla_\eta u)_{\lambda_2\cdots\lambda_p}+2\overline{\xi}^\sigma\eta^\rho\nabla_\sigma u_{\rho\lambda_2\cdots\lambda_p}\\&=2(i(\overline{\xi})\nabla_\eta u)_{\lambda_2\cdots\lambda_p}+2\overline{\xi}^\rho\nabla_\rho u'_{\lambda_2\cdots\lambda_p}+2\xi'u'_{\lambda_2\cdots\lambda_p}-2\xi^\sigma u_{\sigma\lambda_2\cdots\lambda_p},\end{aligned}$$

and from (4.1) we obtain

$$B=R^\sigma_\rho\xi^\rho u_{\sigma\lambda_2\cdots\lambda_p}+2(D\xi-(n+1)\xi')u_{\lambda_2\cdots\lambda_p}+4\xi^\sigma u_{\sigma\lambda_2\cdots\lambda_p}.$$

By virtue of Corollary 4.4 and (4.8), C can be calculated as

$$C=-\sum_{j\geqq 2}\{R_{\lambda_j\sigma\rho}{}^{\varepsilon}\xi_{\varepsilon}-2\nabla_{\rho}\overline{\xi_{\lambda_j}}\eta_{\sigma}+2\xi^{\tau}(\varphi_{\tau\rho}\varphi_{\lambda_j\sigma}+\varphi_{\tau\lambda_j}\varphi_{\rho\sigma})\}u^{\rho}_{\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}$$

$$=\sum_{j\geqq 2}R^{\sigma\rho\varepsilon}_{\lambda_j}\,\xi_{\varepsilon}u_{\rho\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}+2\sum_{j\geqq 2}\overline{\xi_{\lambda_j}}\varphi^{\rho\sigma}u_{\rho\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}$$

$$+2\sum_{j\geqq 2}(\nabla_{\lambda_j}\overline{\xi}^{\rho}\eta^{\sigma}+\overline{\xi}^{\rho}\nabla_{\lambda_j}\eta^{\sigma})u_{\rho\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}$$

$$=\sum_{j\geqq 2}R^{\sigma}{}_{\lambda_j}{}^{\rho\varepsilon}\xi_{\varepsilon}u_{\rho\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}+2(e(\overline{\xi})\Lambda u)_{\lambda_2\cdots\lambda_p}+2(di(\overline{\xi})u')_{\lambda_2\cdots\lambda_p}$$

$$+2(i(\overline{\xi})i(\eta)du)_{\lambda_2\cdots\lambda_p}-2(i(\overline{\xi})\nabla_{\eta}u)_{\lambda_2\cdots\lambda_p}+2\eta^{\sigma}\overline{\xi}^{\rho}\nabla_{\rho}u_{\sigma\lambda_2\cdots\lambda_p}.$$

Lastly applying the Ricci's identity, we get

$$D=-\xi^{\sigma}(\nabla_{\sigma}\nabla^{\rho}u_{\rho\lambda_2\cdots\lambda_p}+R^{\varepsilon}_{\sigma}u_{\varepsilon\lambda_2\cdots\lambda_p}-\sum_{j\geqq 2}R^{\rho}{}_{\sigma\lambda_j}{}^{\varepsilon}u_{\rho\lambda_2\cdots\overset{j}{\hat{\varepsilon}}\cdots\lambda_p})+\sum_{j\geqq 2}\nabla_{\lambda_j}\xi^{\sigma}(\delta u)_{\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}$$

$$=(e(\xi)\delta u)_{\lambda_2\cdots\lambda_p}-\xi^{\sigma}R^{\varepsilon}_{\sigma}u_{\varepsilon\lambda_2\cdots\lambda_p}-\sum_{j\geqq 2}\xi_{\varepsilon}R^{\varepsilon\rho}{}_{\lambda_j}{}^{\sigma}u_{\rho\lambda_2\cdots\overset{j}{\hat{\sigma}}\cdots\lambda_p}.$$

Adding these four relations, we can obtain (4.14).

Let u be a C-harmonic p-form ($p\leqq(n-1)/2$), and ξ be a special C-Killing form. Suppose that our Sasakian space is compact. Then by virtue of Proposition 1.1 and (4.14) we have

$$(\delta\theta(\xi)-\theta(\xi)\delta)u=2e(\overline{\xi})\Lambda u.$$

Taking account of Proposition 1.3 and Lemma 4.6, we get

$$\begin{aligned}i(\xi)\triangle u&=i(\xi)L\Lambda u=Li(\xi)\Lambda u-2e(\overline{\xi})\Lambda u\\&=L\Lambda i(\xi)u-2e(\overline{\xi})\Lambda u.\end{aligned}$$

Since

$$\begin{aligned}\text{(4.15)}\qquad \delta i(\xi)u&=\Lambda_{\xi}u-i(\xi)e(\eta)\Lambda u\\&=\Lambda_{\xi}u-\xi'\Lambda u+e(\eta)i(\xi)\Lambda u\end{aligned}$$

is valid and ξ' is constant, we have

$$(i(\xi)u, d\Lambda_\xi u) = (\delta i(\xi)u, \Lambda_\xi u)$$
$$= (\Lambda_\xi u, \Lambda_\xi u) - \xi'(\Lambda u, \Lambda_\xi u).$$

Hence it follows from (4.13) that

$$(i(\xi)u, \triangle i(\xi)u) = (i(\xi)u, i(\xi)\triangle u) + (i(\xi)u, d\Lambda_\xi u) + (i(\xi)u, 2e(\bar{\xi})\Lambda u)$$
$$= (\Lambda i(\xi)u, \Lambda i(\xi)u) + (\Lambda_\xi u, \Lambda_\xi u) - \xi'(\Lambda u, \Lambda_\xi u).$$

On the other hand, Λu is also C-harmonic from Proposition 1.2 and therefore

$$d\Lambda u = 0, \quad i(\eta)\Lambda u = 0$$

hold good. Hence we have

$$(\Lambda u, \Lambda_\xi u) = (\Lambda u, \delta i(\xi)u + i(\xi)e(\eta)\Lambda u)$$
$$= (\Lambda u, \xi'\Lambda u - e(\eta)i(\xi)\Lambda u) = \xi'(\Lambda u, \Lambda u).$$

Using this relation and (4.15), it follows that

$$(\delta i(\xi)u, \delta i(\xi)u) = (\Lambda_\xi u, \Lambda_\xi u) + \xi'^2(\Lambda u, \Lambda u) + (\Lambda i(\xi)u, \Lambda i(\xi)u) - 2\xi'(\Lambda u, \Lambda_\xi u)$$
$$= (\Lambda_\xi u, \Lambda_\xi u) + (\Lambda i(\xi)u, \Lambda i(\xi)u) - \xi'(\Lambda u, \Lambda_\xi u).$$

Therefore we can obtain

$$(i(\xi)u, \triangle i(\xi)u) = (\delta i(\xi)u, \delta i(\xi)u),$$

which shows that $di(\xi)u=0$. Hence we have $\theta(\xi)u=0$.

For an arbitrary scalar function f, the Lie derivative of any C-harmonic p-form u ($p \leqq (n-1)/2$) with respect to $f\eta$ vanishes. In fact, we have

$$\theta(f\eta)u = f\theta(\eta)u + e(df)i(\eta)u = 0,$$

if $p \leqq (n-1)/2$. Taking account of Theorem 3.4, we decompose a C-Killing form ξ as the sum of a special C-Killing form ζ and $\xi'\eta$. Then we have

$$\theta(\xi)u = \theta(\zeta)u + \theta(\xi'\eta)u = 0$$

for any C-harmonic p-form u. Consequently we attained to the following

THEOREM 4.9. *In a compact n-dimensional Sasakian space, let u be a*

C-harmonic p form ($p \leqq (n-1)/2$) *and* ξ *be a C-Killing form, then we have*

$$\theta(\xi)u = 0.$$

Especially, a harmonic p-form ($p \leqq (n-1)/2$) is C-harmonic in compact case. Further the fundamental 2-form $d\eta$ is C-harmonic. Thus we have the following corollaries.

COROLLARY 4.10. *Let* u *be a harmonic p-form* ($p \leqq (n-1)/2$) *and* ξ *be a C-Killing form. Then we have*

$$\theta(\xi)u = 0.$$

COROLLARY 4.11. *Let* ξ *be a C-Killing form. Then it satisfies*

$$\theta(\xi)d\eta = 0.$$

5. **Regular Sasakian structure**. We consider the meaning of C-Killing forms on a compact Sasakian space which has regular structure. Let (M^n, p, B^{n-1}) be a fibration of Boothby-Wang, where $B^{n-1} = M^n/(\eta)$ is the base space and p: $M^n \to B^{n-1}$ is the projection. We denote by ∇, ∇' the covariant differentiations with respect to the Riemann metric $g_{\lambda\mu}$ on M^n and g'_{ab}[1] on B^{n-1} defined naturally by $g_{\lambda\mu}$. We fix a point $x \in M^n$ and set $y = p(x)$. Taking local coordinates systems (x^λ) and (y^a) arround x and y, we represent the projection mapping as

$$y^a = p^a(x^1, \cdots, x^n).$$

We put $p^a{}_\lambda = \partial p^a/\partial x^\lambda$, and $\bar{u} = p^* u = (\bar{u}_{\lambda_1 \cdots \lambda_p})$ for a p-form $u = (u_{a_1 \cdots a_p})$ on B^{n-1}. Then we know that the following relation

$$\nabla_\mu \bar{u}_{\lambda_1 \cdots \lambda_p} = p^{a_1}_{\lambda_1} \cdots p^{a_p}_{\lambda_p} p^b{}_\mu \nabla'_b u_{a_1 \cdots a_p} - \sum_i (\varphi_\mu{}^\rho \eta_{\lambda_i} + \varphi_{\lambda_i}{}^\rho \eta_\mu) \bar{u}_{\lambda_1 \cdots \overset{i}{\rho} \cdots \lambda_p} \tag{5. 1}$$

is valid [4]. Applying (5.1) to a 1-form u_a, we have

$$\nabla_\mu \bar{u}_\lambda = p^a{}_\lambda p^b{}_\mu \nabla'_b u_a - (\varphi_\mu{}^\rho \eta_\lambda + \varphi_\lambda{}^\rho \eta_\mu) \bar{u}_\rho. \tag{5. 2}$$

Now suppose that the 1-form u_a is a Killing form on B^{n-1}. Then (5.2) shows

1) The Latin indices a, b, ..., run from 1 to n-1.

$$\nabla_\mu \bar{u}_\lambda + \nabla_\lambda \bar{u}_\mu = p^a{}_\lambda p^b{}_\mu (\nabla'_a u_b + \nabla'_b u_a) - 2(\varphi_\mu{}^\rho \eta_\lambda + \varphi_\lambda{}^\rho \eta_\mu)\bar{u}_\rho$$
$$= 2\bar{u}^\rho (\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda) .$$

Thus the form $\bar{u} = p^* u$ satisfies (3.7) and as the space is compact, u is a special C-Killing form.

Conversely we consider a C-Killing form ξ which satisfies $i(\eta)\xi = 0$. Since $\theta(\eta)\xi_\lambda = 0$ from Lemma 3.1, there exists a 1-form u on B^{n-1} such that $\xi = p^* u$. Then we have by virtue of (3.7) and (5.2)

$$p^b{}_\mu p^a{}_\lambda (\nabla'_a u_b + \nabla'_b u_a) = \nabla_\mu \xi_\lambda + \nabla_\lambda \xi_\mu - 2\xi^\rho (\varphi_{\rho\lambda}\eta_\mu + \varphi_{\rho\mu}\eta_\lambda)$$
$$= 0 .$$

Therefore we see

$$\nabla'_b u_a + \nabla'_a u_b = 0 ,$$

which shows that u_a is a Killing form on B^{n-1}. Thus we have the following

THEOREM 5.1. *Let M^n be a compact regular Sasakian space, and B^{n-1} be the base space of the fibration of Boothby-Wang. Then the vector space of Killing 1-forms on B^{n-1} is isomorphic to the vectcr space of C-Killing forms on M^n which are orthogonal to η.*

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