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ISOMETRIC IMBEDDINGS OF COMPACT SYMMETRIC SPACES

SHOSHICHI KOBAYASHI*)

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1. Introduction. The purpose of this paper is to construct an equivariant isometric imbedding of each of the following symmetric spaces into a Euclidean space in a unified manner:

(i) Grassmann manifolds;
(ii) Classical groups;
(iii) $SO(2n)/U(n)$, $U(2n)/Sp(n)$;
(iv) $U(n)/O(n)$, $Sp(n)/U(n)$;
(v) $E_6/\mathrm{Spin}(10)\times T$, $F_4/\mathrm{Spin}(9)$, $E_7/E_6\times T$, E_6/F_4.

For (i), (ii), (iii) and (iv) our constructions are explicit and are given in §§2—5. The general principle of our imbeddings is given in §6. The general principle can be applied to the exceptional case (v). But no explicit constructions will be given for (v). For the hermitian symmetric spaces, our imbeddings coincide with those obtained in [3]. For the real, complex and quaternionic projective spaces and for the Cayley projective plane $F_4/\mathrm{Spin}(9)$, our imbeddings coincide with those constructed in [5]. For the classical groups, our imbeddings are nothing but the natural ones into the spaces of matrices of the same size.

For the compact hermitian symmetric spaces, our imbeddings are minimal (and substantial) in the sense of total curvature, [1]. The same is true for the real, complex and quaternionic projective spaces and the Cayley projective plane, [5]. We conjecture that this is the case for all spaces given above[1].

2. Grassmann manifolds. Let F denote the field $\boldsymbol{R}$ of real numbers, the field $\boldsymbol{C}$ of complex numbers or the field $\boldsymbol{Q}$ of quaternions. Let $U(n;F)$ be the group of unitary matrices of degree n over F, i.e.,

$$U(n;F)=\{A\,;{}^t\bar{A}A=I_n\}$$

so that $U(n;R)=O(n)$, $U(n;\boldsymbol{C})=U(n)$ and $U(n;\boldsymbol{Q})=\mathrm{Sp}(n)$. Let $H(n;F)$

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1) ADDED IN PROOF: This conjecture has been verified by Takeuchi and the author. These imbeddings give also minimal submanifolds of spheres. See M. Takeuchi & S. Kobayashi, Minimal imbeddings of R-spaces, to appear in J. Diff. Geometry.

denote the space of hermitian matrices of degree n over F, i.e.,

$$H(n;F) = \{A\,;{}^t\overline{A} = A\}\,.$$

We shall define an imbedding of $U(p+q;F)/U(p;F)\times U(q;F)$ into $H(p+q;F)$. Denote by E the following matrix:

$$E = \begin{pmatrix} aI_p & 0 \\ 0 & bI_q \end{pmatrix}, \quad a=-q/(p+q), \quad b=p/(p+q)\,.$$

Then the mapping $A\in U(p+q;F) \longrightarrow AE{}^t\overline{A}\in H(p+q;F)$ induces an imbedding of $U(p+q;F)/U(p;F)\times U(q;F)$ into $H(p+q;F)$. The fact that this imbedding (as well as those constructed in §§3—5) is equivariant and isometric will be explained in §6 in a unified manner.

3. Classical groups. Let $U(n;F)$ be as in §2. Denote by $M(n;F)$ the space of matrices of degree n over F. Set

$$E = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Then the mapping

$$A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in U(n;F)\times U(n;F) \longrightarrow AE{}^t\overline{A} = \begin{pmatrix} 0 & X{}^t\overline{Y} \\ Y{}^t\overline{X} & 0 \end{pmatrix} \in M(2n;F)$$

induces an imbedding of $U(n;F)=(U(n;F)\times U(n;F))/\Delta$ (where Δ is the diagonal) into $M(n;F)$ which sends $X\in U(n;F)$ into $X\in M(n;F)$.

4. $SO(2n)/U(n)$, $U(2n)/Sp(n)$. Set

$$E = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and denote by $\mathrm{Skew}(2n;F)$ the space of skew-symmetric matrices of degree $2n$ over F. Then the mapping

$$A\in SO(2n) \longrightarrow AE{}^tA \in \mathrm{Skew}(2n;\boldsymbol{R})$$

induces an imbedding of $SO(2n)/U(n)$ into $\mathrm{Skew}(2n;\boldsymbol{R})$.

Similarly, the mapping

$$A\in U(2n) \longrightarrow AE{}^tA \in \mathrm{Skew}(2n;\boldsymbol{C})$$

induces an imbedding of $U(2n)/Sp(n)$ into Skew$(2n\,;\boldsymbol{C})$ and hence an imbedding of $SU(2n)/Sp(n)$ into Skew$(2n\,;\boldsymbol{C})$.

5. $\boldsymbol{U(n)/O(n)}$, $\boldsymbol{Sp(n)/U(n)}$. For any quaternion $x = x_0+x_1i+x_2j+x_3k$, we set

$$\tilde{x} = x_0 + x_1 i - x_2 j + x_3 k\,.$$

If x is real or complex, then $\tilde{x} = x$. Denote by Sym$(n\,;F)$ the space of matrices degree n over F defined by

$$\mathrm{Sym}(n\,;F) = \{A \in M(n\,;F)\,;\, {}^t\tilde{A} = A\}\,.$$

Let $U(n\,;F)$ be as in §2. Then the mapping

$$A \in U(n\,;F) \longrightarrow A\,{}^t\tilde{A} \in \mathrm{Sym}(n\,;F)$$

induces an imbedding $U(n\,;F)/H \longrightarrow \mathrm{Sym}(n\,;F)$, where H is the subgroup of $U(n\,;F)$ given by

$$H = \{A \in U(n\,;F)\,;\, \bar{A} = \tilde{A}\}\,.$$

If we write $A \in U(n\,;F)$ in the following form

$$A = A_0 + A_1 i + A_2 j + A_3 k, \quad (A_0, A_1, A_2, A_3 \text{ are real})\,,$$

then A is in H if and only if $A_1=A_3=0$. If $F=\boldsymbol{C}$, then $U(n\,;\boldsymbol{C})=U(n)$ and $H=SO(n)$ and we obtain an imbedding $U(n)/SO(n) \longrightarrow \mathrm{Sym}(n\,;\boldsymbol{C})$ and hence also an imbedding $SU(n)/SO(n) \longrightarrow \mathrm{Sym}(n\,;\boldsymbol{C})$. If $F=\boldsymbol{Q}$, then $U(n\,;\boldsymbol{Q}) = Sp(n)$ and $H=U(n)$ and we obtain an imbedding $Sp(n)/U(n) \longrightarrow Sym(n\,;\boldsymbol{Q})$.

6. General principle. Let L be a connected real noncompact simple Lie group of finite center and $\boldsymbol{L}$ its Lie algebra. Let K be a maximal compact subgroup of L and $\boldsymbol{K}$ its Lie algebra. Let

$$\boldsymbol{L} = \boldsymbol{K} + \boldsymbol{P}$$

be the Cartan decomposition. Assume that there exists an element $E \in \boldsymbol{P}$ such that $\mathrm{ad}\,E$ has eigen-values -1, 0, 1. Let H be the subgroup of K defined by

$$H = \{h \in K;\ \ (\mathrm{ad}\ h)E = E\}\,.$$

Then the mapping $k \in K \longrightarrow (\mathrm{ad}\ k)\ E \in \boldsymbol{P}$ induces an imbedding $K/H \longrightarrow \boldsymbol{P}$. The imbedding is obviously equivariant with respect to the adjoint action of K on $\boldsymbol{P}$. Moreover the Killing-Cartan form of $\boldsymbol{L}$ restricted to $\boldsymbol{P}$ is an inner product in $\boldsymbol{P}$ invariant by the adjoint action of K. Hence the imbedding induces another K-invariant metric on K/H. Since an invariant metric on an irreducible symmetric space is unique up to a constant factor, each irreducible factor of K/H is thus isometrically imbedded into $\boldsymbol{P}$.

$L=SL(p+q;\boldsymbol{R})$, $L=SL(p+q;\boldsymbol{C})$ and $L=SU^*(2p+2q)$ give Case (i). $L=SO(n,n)$, $L=SU(n,n)$ and $L=Sp(n,n)$ give case (ii). $L=SO(2n;\boldsymbol{C})$ and $L=SO^*(4n)$ give case (iii). $L=Sp(n;\boldsymbol{R})$ and $L=Sp(n;\boldsymbol{C})$ give case (iv). $L=E_6^c$, $L=E_6^1$, $L=E_7^c$ and $L=E_7^1$ give case (v). For the detail, see [2], [4], [6].

For (i), (ii), (iii) and (iv) it is an easy matter to verify directly that the imbedding constructed in §§2—5 are equivariant and isometric.

7. Table. Let $M=K/H$ be one of the compact symmetric spaces considered above and $M \longrightarrow \boldsymbol{R}^N$ the equivariant isometric imbedding constructed above. Then

$M=K/H$	$\dim M$	$\boldsymbol{R}^N$	N
$O(p+q)/O(p)\times O(q)$	pq	$H(p+q;\boldsymbol{R})$	$\frac{1}{2}(p+q)(p+q+1)$
$U(p+q)/U(p)\times U(q)$	$2pq$	$H(p+q;\boldsymbol{C})$	$(p+q)^2$
$Sp(p+q)/Sp(p)\times Sp(q)$	$4pq$	$H(p+q;\boldsymbol{Q})$	$2(p+q)^2-(p+q)$
$SO(n)$	$\frac{1}{2}n(n-1)$	$M(n;\boldsymbol{R})$	n^2
$U(n)$	n^2	$M(n;\boldsymbol{C})$	$2n^2$
$Sp(n)$	$2n^2+n$	$M(n;\boldsymbol{Q})$	$4n^2$
$SO(2n)/U(n)$	n^2-n	$Skew(2n;\boldsymbol{R})$	$n(2n-1)$
$U(2n)/Sp(n)$	$n(2n+1)$	$Skew(2n;\boldsymbol{C})$	$2n(2n-1)$
$U(n)/O(n)$	$\frac{1}{2}n(n+1)$	$Sym(n;\boldsymbol{C})$	$n(n+1)$
$Sp(n)/U(n)$	n^2+n	$Sym(n;\boldsymbol{Q})$	$n(2n+1)$
$E_6/Spin(10)\times T$ [2)]	32	E_6	78
$F_4/Spin(9)$	16	$H(3, Cay)$	26
$E_7/E_6\times T$ [2)]	54	E_7	133
E_6/F_4	26	?	54

We recall some of the notations used above.

$M(n;F)=$ the space of $n\times n$ matrices over F,

$H(n;F)=\{A\in M(n;F)\,;\,{}^t\bar{A}=A\}$,

$Sym(n;F)=\{A\in M(n;F)\,;\,{}^t\tilde{A}=A\}$,

$Skew(n;F)=\{A\in M(n;F)\,;\,{}^tA=-A\}$,

2) These two are infinitesimal expressions.

$H(3, Cay)$ = the space of 3×3 hermitian Cayley matrices.

If we denote by $H_0(n; F)$ the subspace of $H(n; F)$ consisting of matrices with trace equal to 0, then it is clear from our construction that the Grasmmann manifold $U(p+q; F)/U(p; F)\times U(q; F)$ is imbedded in $H_0(n; F)$. Similarly, Tai [5] proved that the Cayley projective plane is imbedded in the subspace $H_0(3; Cay)$ of $H(3, Cay)$ consisting of matrices with zero trace. When K/H is hermitian, our imbedding maps K/H into $P=\sqrt{-1}K$, [1], [2]. Hence the receiving spaces for $E_6/Spin(10)\times T$ and $E_7/E_6\times T$ can be identified with E_6 and E_7 respectively. The receiving space for E_6/F_4 is not clear. From its dimension we suspect that it may have something to do with the tangent space of $E_7/E_6\times T$.

It is of interest to note that $N\sim 2\dim M$ for many of the cases considered.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA, U. S. A.