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COMPACT ORIENTABLE SUBMANIFOLD OF CODIMENSION 2 IN AN ODD DIMENSIONAL SPHERE

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Introduction. It has been proved by H. Liebmann [3] that the only ovaloid with constant mean curvature in Euclidean space E^3 is a sphere. The analogous theorem for a convex m-dimensional hypersurface in E^{m+1} has been proved by W. Süss [6]. Recently Y. Katsurada [1], [2] and K. Yano [9] have generalized the above theorem to an m-dimensional hypersurface in an Einstein space admitting one-parameter groups of conformal transformations or of homothetic transformations.

Thus we may expect an analogous theorem for a submanifold of codimension greater than 1 in a certain Riemannian manifold. On the other hand the present author studied, in the previous paper [4], a certain hypersurface in an odd dimensional sphere S^{2n+1} and found that the natural contact structure of S^{2n+1} plays an important role in the study of the hypersurface of S^{2n+1}.

This fact suggests that, using the natural contact structure of S^{2n+1}, we can solve the problem similar to the Liebmann-Süss problem for a submanifold of codimension 2 in an odd dimensional sphere.

The purpose of the paper is to prove the analogue of the Liebmann-Süss theorem for a submanifold of codimension 2 in S^{2n+1}. For this purpose, we give in §1, some properties of the contact structure of S^{2n+1} and in §2 some formulas in the theory of submanifold of codimension 2. In §3, we study a submanifold of codimension 2 in an odd dimensional sphere and introduce some quantities for later use.

In §4 some integral formulas for a submanifold of codimension 2 in an odd dimensional sphere are derived and under certain conditions the theorem mentioned above is proved. However an umbilical submanifold of codimension 2 in $(2n+1)$-dimensional sphere does not necessarily satisfy the conditions of our theorem. So in §5 we show an example of umbilical submanifold which satisfies our conditions.

1. Contact Riemannian structure on an odd dimensional sphere.

A $(2n+1)$-dimensional differentiable manifold M is said to have a contact structure and to be a contact manifold if there exists on M a 1-form $\eta = \eta_\lambda dx^\lambda$ such that

$$\eta \wedge (d\eta)^n \neq 0 \tag{1.1}$$

everywhere on M, where $\wedge$ denotes the exterior multiplication and $d\eta$ the exterior derivative of η. η is called a contact form on M.

Since (1. 1) means that the 2-form $d\eta$ is of rank $2n$ everywhere on M, we can find a unique vector field ξ^λ on M satisfying

$$\eta_\lambda \xi^\lambda = 1\,, \quad (d\eta)_{\mu\lambda}\xi^\lambda = 0\,. \tag{1.2}$$

Let S^{2n+1} be an odd dimensional sphere which is represented by the equation

$$\sum_{A=1}^{2n+2} (X^A)^2 = 1\,, \tag{1.3}$$

in a $(2n+2)$-dimensional Euclidean space E^{2n+2} with rectangular coordinates X^A $(A=1, 2, \cdots, 2n+2)$. We put

$$\eta = \frac{1}{2}\sum_{\alpha=1}^{n+1}(X^{n+1+\alpha}dX^\alpha - X^\alpha dX^{n+1+\alpha})\,, \tag{1.4}$$

then the 1-form η defines a contact form on S^{2n+1} and consequently we can find a vector field ξ^λ on S^{2n+1} satisfying (1. 2).

The Riemannian metric tensor $G_{\lambda\kappa}$ on S^{2n+1} is naturally induced from the Euclidean space E^{2n+2} in such a way that

$$G_{\lambda\kappa} = \delta_{\lambda\kappa} + \frac{X^\lambda X^\kappa}{(X^{2n+2})^2}\,, \quad G^{\lambda\kappa} = \delta^{\lambda\kappa} - X^\lambda X^\kappa\,. \tag{1.5}$$

With respect to this Riemannian metric, the Riemannian curvature tensor $R_{\nu\mu\lambda}{}^\kappa$ of S^{2n+1} satisfies

$$R_{\nu\mu\lambda}{}^\kappa = G_{\mu\lambda}\delta_\nu^\kappa - G_{\nu\lambda}\delta_\mu^\kappa\,. \tag{1.6}$$

We define a linear transformation $F_\lambda{}^\kappa$: $T(S^{2n+1}) \rightarrow T(S^{2n+1})$ by

$$F_\lambda{}^\kappa = \frac{1}{2}G^{\mu\kappa}(d\eta)_{\lambda\mu} = \frac{1}{2}G^{\mu\kappa}(\partial_\lambda \eta_\mu - \partial_\mu \eta_\lambda)\,. \tag{1.7}$$

Then the set ($F_\lambda{}^\kappa$, ξ^κ, η_λ, $G_{\lambda\kappa}$) satisfies[1)]

$$G_{\kappa\lambda}\xi^\lambda = \eta_\kappa \,, \tag{1.8}$$

$$G_{\lambda\kappa}F_\nu{}^\lambda F_\mu{}^\kappa = G_{\nu\mu} - \eta_\nu\eta_\mu \,, \tag{1.9}$$

and consequently we have

$$\eta_\kappa F_\lambda{}^\kappa = 0 \,, \tag{1.10}$$

$$F_\mu{}^\lambda F_\lambda{}^\kappa = -\delta_\mu^\kappa + \eta_\mu\xi^\kappa \,. \tag{1.11}$$

In general, the set ($F_\lambda{}^\kappa$, ξ^κ, η_λ, $G_{\lambda\kappa}$) satisfying (1.1), (1.2), (1.7), (1.8) and (1.9) is called a contact Riemannian (or metric) structure.

It is known[2)] that if the contact Riemannian structure on S^{2n+1} is the one which is defined by (1.4), (1.5) and (1.7), the structure satisfies further

$$\frac{1}{2}\tilde{\nabla}_\mu(d\eta)_{\lambda\kappa} = \eta_\lambda G_{\mu\kappa} - \eta_\kappa G_{\mu\lambda} \,, \tag{1.12}$$

and

$$\tilde{\nabla}_\lambda\xi^\kappa = F_\lambda{}^\kappa \,, \tag{1.13}$$

where $\tilde{\nabla}$ denotes the covariant derivative with respect to the Riemannian metric $G_{\lambda\kappa}$.

2. Submanifolds of codimension 2 in a Riemannian manifold. Let $\tilde{M}^{m+2}$ be a Riemannian manifold of dimension $m+2$ with local coordinates $\{X^\kappa\}$ and $G_{\lambda\kappa}$ be the Riemannian metric tensor of $\tilde{M}^{m+2}$. We denote by M^m a differentiable submanifold of codimension 2 in $\tilde{M}^{m+2}$ and by $\{x^i\}$ the local coordinates of M^m. Then the immersion $\iota: M^m \to \tilde{M}^{m+2}$ is locally represented by $X^\kappa = X^\kappa(x^1, x^2, \cdots, x^m)$, $\kappa = 1, 2, \cdots, m+2$.

Assuming that manifolds M^m and $\tilde{M}^{m+2}$ are both orientable, we put $B_i{}^\kappa = \partial_i X^\kappa$ ($\partial_i = \partial/\partial x^i$). Then m vectors $B_i{}^\kappa$ span the tangent plane of M^m at each point of M^m and the Riemannian metric tensor g_{ji} of M^m is given by

$$g_{ji} = G_{\lambda\kappa}B_j{}^\lambda B_i{}^\kappa \,. \tag{2.1}$$

We assume that $B_i{}^\kappa$ ($i = 1, 2, \cdots, m$) give the positive direction in M^m and choose the mutually orthogonal unit normal vectors C^κ and D^κ to M^m in such a way that $B_i{}^\kappa$, C^κ, D^κ give the positive direction in $\tilde{M}^{m+2}$.

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1) S. Sasaki and Y. Hatakeyama [5].
2) S. Sasaki and Y. Hatakeyama [5].

In the sequel we always consider a coordinate neighborhood of M^m in which there exist such two fields of unit normal vectors to M^m. We denote by $(B^i{}_\kappa, C_\kappa, D_\kappa)$ the dual basis of $(B_i{}^\kappa, C^\kappa, D^\kappa)$.

The van der Wearden-Bortolotti covariant derivatives $\nabla_j B_i{}^\kappa$, $\nabla_j C^\kappa$ and $\nabla_j D^\kappa$ of $B_i{}^\kappa$, C^κ and D^κ are respectively given by

(2.2) $$\nabla_j B_i{}^\kappa = \partial_j B_i{}^\kappa - \left\{ {h \atop j\ i} \right\} B_h{}^\kappa + \left\{ {\kappa \atop \mu\ \lambda} \right\} B_j{}^\mu B_i{}^\lambda ,$$

(2.3) $$\nabla_j C^\kappa = \partial_j C^\kappa + \left\{ {\kappa \atop \mu\ \lambda} \right\} B_j{}^\mu C^\lambda ,$$

and

(2.4) $$\nabla_j D^\kappa = \partial_j D^\kappa + \left\{ {\kappa \atop \mu\ \lambda} \right\} B_j{}^\mu D^\lambda .$$

Let H_{ji}, K_{ji} be the second fundamental tensors of M^m with respect to the normals C^κ, D^κ and L_i the third fundamental tensor of M^m. Then we have the following Gauss and Weingarten equations:

(2.5) $$\nabla_j B_i{}^\kappa = H_{ji} C^\kappa + K_{ji} D^\kappa ,$$

(2.6) $$\nabla_j C^\kappa = - H_j{}^i B_i{}^\kappa + L_j D^\kappa , \qquad \nabla_j D^\kappa = -K_j{}^i B_i{}^\kappa - L_j C^\kappa ,$$

where $H_j{}^i = g^{ih} H_{jh}$ and $K_j{}^i = g^{ih} K_{jh}$.

The mean curvature vector field H^κ of M^m in $\widetilde{M}^{m+2}$ is defind by

(2.7) $$H^\kappa = \frac{1}{2n}(H_i{}^i C^\kappa + K_i{}^i D^\kappa).$$

We know that H^κ is independent of the choice of mutually orthogonal unit normal vectors C^κ and D^κ and consequently that H^κ is a globally defined vector field over $\widetilde{M}^{m+2}$.

LEMMA 2.1. *Let M^m be a submanifold of a Riemannian manifold $\widetilde{M}^{m+2}$. In order that the covariant derivative $\nabla_j H^\kappa$ of the mean curvature vector field is tangent to M^m, it is necessary and sufficient that*

(2.8) $$\nabla_j H_r{}^r = K_r{}^r L_j ,$$

and

(2.9) $$\nabla_j K_r{}^r = - H_r{}^r L_j .$$

PROOF. Differentiating (2.7) covariantly, we have

$$2n\nabla_j H^\kappa = (\nabla_j H_i^{\ i} - K_i^{\ i} L_j) C^\kappa + (\nabla_j K_i^{\ i} + H_i^{\ i} L_j) D^\kappa - (H_i^{\ i} H_j^{\ h} + K_i^{\ i} K_j^{\ h}) B_h^{\ \kappa}.$$

This proves the assertion of Lemma 2.1.

When the second fundamental tensors are at each point of the submanifold M^m of the form

$$H_{ji} = H g_{ji}, \quad K_{ji} = K g_{ji}, \tag{2.10}$$

we call the submanifold a totally umbilical submanifold. Moreover if the both functions H and K vanish identically, we call it a totally geodesic submanifold.

LEMMA 2.2. *A necessary and sufficient condition for a submanifold of codimension* 2 *to be umbilical is that the following equations are satisfied.*

$$H_{ji} H^{ji} = \frac{1}{m}(H_i^{\ i})^2, \quad K_{ji} K^{ji} = \frac{1}{m}(K_i^{\ i})^2. \tag{2.11}$$

PROOF. This follows from the identities

$$\left(H_{ji} - \frac{1}{m} H_k^{\ k} g_{ji}\right)\left(H^{ji} - \frac{1}{m} H_k^{\ k} g^{ji}\right) = H_{ji} H^{ji} - \frac{1}{m}(H_i^{\ i})^2,$$

$$\left(K_{ji} - \frac{1}{m} K_k^{\ k} g_{ji}\right)\left(K^{ji} - \frac{1}{m} K_k^{\ k} g^{ji}\right) = K_{ji} K^{ji} - \frac{1}{m}(K_i^{\ i})^2,$$

and the positive definiteness of the Riemannian metric g_{ji}.

We now write the equations of Gauss, Mainardi-Codazzi and Ricci-Kühne:

$$\widetilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \nu} B_j^{\ \mu} B_i^{\ \lambda} B_h^{\ \kappa} = R_{kjih} - (H_{ji}H_{kh} - H_{ki}H_{jh}) - (K_{ji}K_{kh} - K_{ki}K_{jh}), \tag{2.12}$$

$$\begin{cases} \widetilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \nu} B_j^{\ \mu} B_i^{\ \lambda} C^\kappa = \nabla_k H_{ji} - \nabla_j H_{ki} - L_k K_{ji} + L_j K_{ki}, \\ \widetilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \nu} B_j^{\ \mu} B_i^{\ \lambda} D^\kappa = \nabla_k K_{ji} - \nabla_j K_{ki} + L_k H_{ji} - L_j H_{ki}, \end{cases} \tag{2.13}$$

and

$$\widetilde{R}_{\nu\mu\lambda\kappa} B_k^{\ \nu} B_j^{\ \mu} C^\lambda D^\kappa = \nabla_k L_j - \nabla_j L_k - K_{ki} H_j^{\ i} + K_{ji} H_k^{\ i}. \tag{2.14}$$

If $\widetilde{M}^{m+2}$ has the curvature tensor of the form (1. 6), equations (2. 12), (2. 13) and (2. 14) can be rewritten respectively as

$$R_{kjih} = g_{ji}g_{kh} - g_{ki}g_{jh} + H_{ji}H_{kh} - H_{ki}H_{jh} + K_{ji}K_{kh} - K_{ki}K_{jh}\,, \tag{2. 15}$$

$$\begin{cases} \nabla_k H_{ji} - \nabla_j H_{ki} = L_k K_{ji} - L_j K_{ki}\,, \\ \nabla_k K_{ji} - \nabla_j K_{ki} = -L_k H_{ji} + L_j H_{ki}\,, \end{cases} \tag{2. 16}$$

and

$$\nabla_k L_j - \nabla_j L_k = K_{ki} H_j{}^i - K_{ji} H_k{}^i\,. \tag{2. 17}$$

LEMMA 2.3. *Let M^m be a submanifold with non-vanishing mean curvature vector field in a Riemannian manifold of constant curvature. If the covariant derivative of the mean curvature vector field is tangent to M^m, we have*

$$K_{ki} H_j{}^i = K_{ji} H_k{}^i\,. \tag{2. 18}$$

PROOF. As a consequence of Lemma 2.1 we have (2. 8) and (2. 9). By virtue of the assumption we can suppose that $K_i{}^i \neq 0$ without loss of generality. Differentiating (2. 8) covariantly, we get

$$\nabla_k \nabla_j H_i{}^i = -H_i{}^i L_k L_j + K_i{}^i \nabla_k L_j\,,$$

from which

$$K_i{}^i(\nabla_k L_j - \nabla_j L_k) = 0\,.$$

This, together with (2. 17), implies (2. 18). This completes the proof.

3. Submanifolds of codimension 2 in an odd dimensional sphere. We consider a submanifold of codimension 2 in an odd dimensional sphere S^{2n+1}. In the following discussions we always regard S^{2n+1} as a contact manifold with the contact Riemannian structure $(F_\lambda{}^\kappa, \xi^\kappa, \eta_\lambda, G_{\lambda\kappa})$ defined by (1. 4), (1. 5) and (1. 7).

The transform $F_\lambda{}^\kappa B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\kappa$ can be expressed as a linear combination of $B_i{}^\kappa$, C^κ and D^κ. So we can put

$$F_\lambda{}^\kappa B_i{}^\lambda = f_i{}^h B_h{}^\kappa + f_i C^\kappa + g_i D^\kappa\,, \tag{3. 1}$$

which implies that

$$f_i{}^h = B^h{}_\kappa F_\lambda{}^\kappa B_i{}^\lambda\,, \tag{3. 2}$$

$$f_i = F_\lambda{}^\kappa B_i{}^\lambda C_\kappa , \tag{3.3}$$

and

$$g_i = F_\lambda{}^\kappa B_i{}^\lambda D_\kappa . \tag{3.4}$$

Since $F_{\lambda\kappa}$ is skew symmetric with respect to its indices we can easily see that $f_{ji} = g_{jh} f_i{}^h$ is also skew symmetric with respect to its indices.

The transform $F_\lambda{}^\kappa C^\lambda$ of C^λ by $F_\lambda{}^\kappa$ is perpendicular to C^κ and consequently we can put

$$F_\lambda{}^\kappa C^\lambda = -f^h B_h{}^\kappa + r D^\kappa \tag{3.5}$$

from which we have $f^h = g^{hi} f_i$ and

$$r = F_\lambda{}^\kappa C^\lambda D_\kappa = F_{\lambda\kappa} C^\lambda D^\kappa . \tag{3.6}$$

In exactly the same way we have

$$F_\lambda{}^\kappa D^\lambda = - g^h B_h{}^\kappa - r C^\kappa , \tag{3.7}$$

where $g^h = g^{hi} g_i$. The function r seems to be dependent of the choice of the mutually orthogonal unit normal vectors C^κ and D^κ. However we can verify that r is independent of the choice of these vectors[3]. Consequently we see that r is a globally defined function on M.

Since the vector field ξ^κ is tangent to S^{2n+1} it is represented as a linear combination of $B_i{}^\kappa$, C^κ and D^κ. Consequently we put

$$\xi^\kappa = u^h B_h{}^\kappa + a C^\kappa + b D^\kappa , \tag{3.8}$$

which implies that

$$u^h = \xi^\kappa B^h{}_\kappa = g^{hi} B_i{}^\kappa \eta_\kappa , \tag{3.9}$$

$$a = C^\kappa \eta_\kappa , \quad b = D^\kappa \eta_\kappa . \tag{3.10}$$

Differentiating (3.3) and (3.4) covariantly and making use of (1.12), (2.2), (2.3) and (2.4), we get

$$\nabla_j f_h = - a g_{jh} - r K_{jh} - f_h{}^i H_{ji} + g_h L_j , \tag{3.11}$$

$$\nabla_j g_h = - b g_{jh} + r H_{jh} - f_h{}^i K_{ji} - f_h L_j . \tag{3.12}$$

3) Y. Watanabe [7].

We also have

$$\nabla_j r = K_{ji} f^i - H_{ji} g^i , \tag{3.13}$$

$$\nabla_j a = f_j - u^i H_{ji} + b L_j , \tag{3.14}$$

$$\nabla_j b = g_j - u^i K_{ji} - a L_j . \tag{3.15}$$

4. Integral formulas. In this paragraph we construct some integral formulas which are valid in a submanifold of codimension 2 in an odd dimensional sphere. For this purpose we first state the Green-Stokes' theorem which plays an important role in our discussion.

GREEN-STOKES' THEOREM.[4] *Let M be a compact orientable Riemannian manifold. Then for an arbitrary vector field v^i, we have*

$$\int_M \nabla_i v^i \, dM = 0 , \tag{4.1}$$

where dM is the volume element of M.

In the following we always suppose that the submanifold M^{2n-1} of codimension 2 in S^{2n+1} be compact and orientable.

Differentiating (3.13) covariantly and making use of (3.11) and (3.12), we have

$$\begin{aligned}\nabla_k \nabla_j r = (\nabla_k K_{ji} + H_{ji} L_k) f^i - (\nabla_k H_{ji} - K_{ji} L_k) g^i + bH_{jk} - aK_{jk} \\ - r(H_j{}^i H_{ki} + K_j{}^i K_{ki}) + f_i{}^r (H_j{}^i K_{kr} - K_j{}^i H_{kr}) ,\end{aligned}$$

from which, together with Lemma 2.3, we have

$$\begin{aligned}\nabla_j \nabla^j r = (\nabla^r K_{ri} + H_i{}^r L_r) f^i - (\nabla^r H_{ri} - K_i{}^r L_r) g^i + bH_r{}^r - aK_r{}^r \\ - r(H_{ji} H^{ji} + K_{ji} K^{ji}) .\end{aligned}$$

As we have mentioned in §3 r is a globally defined function on M^{2n-1}, and so $\nabla_j \nabla^j r$ is also a globally defined function over M^{2n-1}. Hence we have

$$\begin{aligned}\int_M [(\nabla^r K_{ri} + H_i{}^r L_r) f^i - (\nabla^r H_{ri} - K_i{}^r L_r) g^i + bH_r{}^r - aK_r{}^r \\ - r(H_{ji} H^{ji} + K_{ji} K^{ji})] \, dM = 0 ,\end{aligned} \tag{4.2}$$

4) For example K. Yano and S. Bochner [10].

because of Green-Stokes' theorem.

Next we try to construct another integral formula for the later use. We put

$$w_i = H_r{}^r g_i - K_r{}^r f_i . \tag{4.3}$$

f_i and g_i both depend on the choice of the mutually orthogonal unit normal vectors C^κ and D^κ. However we have the

LEMMA 4.1. *w_i is independent of the choice of the mutually orthogonal unit normal vectors C^κ and D^κ. Consequently it defines a vector field on M^{2n-1}.*

PROOF. Let $'C^\kappa$ and $'D^\kappa$ be mutually orthogonal unit normal vectors to M^{2n-1} at $p \in M^{2n-1}$. Since M^{2n-1} and S^{2n+1} are both orientable we can find following relations between a pair of unit normal vectors (C^κ, D^κ) and $('C^\kappa, 'D^\kappa)$ that

$$'C^\kappa = C^\kappa \cos\theta - D^\kappa \sin\theta , \quad 'D^\kappa = C^\kappa \sin\theta + D^\kappa \cos\theta , \tag{4.4}$$

for some function θ defined on M^{2n-1}. Then the second fundamental tensors $'H_{ji}$ and $'K_{ji}$ with respect to $'C^\kappa$ and $'D^\kappa$ are defined by

$$\nabla_j B_i{}^\kappa = 'H_{ji}{}'C^\kappa + 'K_{ji}{}'D^\kappa . \tag{4.5}$$

From these two equations we easily see that

$$'H_{ji} = H_{ji} \cos\theta - K_{ji} \sin\theta , \qquad 'K_{ji} = H_{ji} \sin\theta + K_{ji} \cos\theta ,$$

which imply that

$$'H_i{}^i = H_i{}^i \cos\theta - K_i{}^i \sin\theta , \qquad 'K_i{}^i = H_i{}^i \sin\theta + K_i{}^i \cos\theta . \tag{4.6}$$

Substituting (4. 4) into $'f_i = F_\lambda{}^\kappa B_i{}^\lambda{}'C^\kappa$ and $'g_i = F_\lambda{}^\kappa B_i{}^\lambda{}'D^\kappa$, we also have

$$'f_i = f_i \cos\theta - g_i \sin\theta , \qquad 'g_i = f_i \sin\theta + g_i \cos\theta . \tag{4.7}$$

Consequently we have

$$'w_i = 'H_r{}^r{}'g_i - 'K_r{}^r{}'f_i = H_r{}^r g_i - K_r{}^r f_i = w_i .$$

This shows that w_i is independent of the choice of the mutually orthogonal unit normal vectors C^κ and D^κ. This completes the proof.

Differentiating w_i covariantly and making use of (3.11) and (3.12), we have

$$\begin{aligned}\nabla_j w_i = \nabla_j(H_r{}^r g_i - K_r{}^r f_i) = (\nabla_j H_r{}^r - K_r{}^r L_j)\, g_i - (\nabla_j K_r{}^r + H_r{}^r L_j) f_i \\ - (bH_r{}^r - aK_r{}^r)\, g_{ji} + r(H_r{}^r H_{ji} + K_r{}^r K_{ji}) \\ - f_i{}^s(K_{js} H_r{}^r - H_{js} K_r{}^r)\,,\end{aligned}$$

from which

$$\begin{aligned}\nabla_i w^i = (\nabla_i H_r{}^r - K_r{}^r L_i)\, g^i - (\nabla_i K_r{}^r + H_r{}^r L_i) f^i - (2n-1)(bH_r{}^r - aK_r{}^r) \\ + r[(H_r{}^r)^2 + (K_r{}^r)^2]\,,\end{aligned}$$

because of skew symmetric property of f_{ji}. Consequently we have

$$\begin{aligned}(4.8)\quad \int_M [(\nabla_i H_r{}^r - K_r{}^r L_i)\, g^i - (\nabla_i K_r{}^r + H_r{}^r L_i) f^i - (2n-1)(bH_r{}^r - aK_r{}^r) \\ + r\{(H_r{}^r)^2 + (K_r{}^r)^2\}]\, dM = 0\,,\end{aligned}$$

by means of Green-Stokes' theorem.

Suppose that the covariant derivative of the mean curvature vector field H^κ of M^{2n-1} in S^{2n+1} is tangent to M^{2n-1}. Then (2.16) and Lemma 2.1 show that

$$(4.9)\qquad \nabla^r H_{jr} = K_j{}^r L_r\,, \qquad \nabla^r K_{jr} = -H_j{}^r L_r\,.$$

Consequently we have

$$(4.10)\qquad \int_M [r(H_{ji}H^{ji} + K_{ji}K^{ji}) - (bH_r{}^r - aK_r{}^r)]\, dM = 0\,,$$

because of (4.2).

On the other hand (4.8) and Lemma 2.1 show that

$$(4.11)\qquad \int_M [r\{(H_r{}^r)^2 + (K_r{}^r)^2\} - (2n-1)(bH_r{}^r - aK_r{}^r)]\, dM = 0\,.$$

Eliminating $\int_M (bH_r{}^r - aK_r{}^r)\, dM$ from the above two equations, we find

$$(4.12)\qquad \int_M r\left[\left(H_{ji}H^{ji} - \frac{(H_r{}^r)^2}{2n-1}\right) + \left(K_{ji}K^{ji} - \frac{(K_r{}^r)^2}{2n-1}\right)\right] dM = 0\,.$$

From this, together with Lemma 2.2, we have the

THEOREM 4.2. *Let M be a compact orientable submanifold of codimension* 2 *in an odd dimensional sphere. We suppose that the covariant derivative of the mean curvature vector field H^κ is tangent to M and that the inner product of $F_\lambda{}^\kappa C^\lambda$ and D^κ is an almost everywhere non-zero valued function on M and does not change the sign. Then M is a totally umbilical submanifold of codimension* 2 *and consequently a* $(2n-1)$-*dimensional sphere.*

5. An example. In Theorem 4.2 we have assumed that the function $r \equiv F_{\lambda\kappa} C^\lambda D^\kappa$ does not change the sign and is almost everywhere non-zero valued. On the other hand, Y. Watanabe [7] has proved that, if, in a contact manifold with the contact Riemannian structure $(F_\lambda{}^\kappa, \xi^\kappa, \eta_\lambda, G_{\lambda\kappa})$ satisfying (1.12) and (1.13), the submanifold of codimension 2 is totally umbilical the function r is a solution of the partial differential equation

$$\nabla_j \nabla_i r = -\{(1+H^2+K^2)r + c\}\, g_{ji}\,,$$

where $H = \frac{1}{2n-1} H_r{}^r$, $K = \frac{1}{2n-1} K_r{}^r$ and c is a constant. These facts show that if the function r is a non-zero constant in the submanifold the umbilical submanifold satisfies the condition of Theorem 4.2.

Now it is natural to ask whether we can find a totally umbilical submanifold satisfying $r =$ constant. In this paragraph we give an example of such a submanifold.

The exterior derivative of the contact form η given by (1.4) becomes

$$d\eta = \frac{1}{2} \sum_{\alpha=1}^{n+1} (dX^{n+1+\alpha} \wedge dX^\alpha - dX^\alpha \wedge dX^{n+1+\alpha})\,. \tag{5.1}$$

Since S^{2n+1} is defined by (1.3), we have

$$X^{2n+2} dX^{2n+2} = -\sum_{A=1}^{2n+1} X^A dX^A\,. \tag{5.2}$$

From these two relations we have

$$(5.3)\quad (2F_{\lambda\kappa}) = (d\eta)_{\lambda\kappa} = \begin{pmatrix}
0 & & & -\frac{X^1}{X^{2n+2}} & -1 & & 0 \\
 & \ddots & & \vdots & & \ddots & \\
 & & 0 & -\frac{X^n}{X^{2n+2}} & 0 & & -1 \\
\frac{X^1}{X^{2n+2}} & \cdots & \frac{X^n}{X^{2n+2}} & 0 & \frac{X^{n+2}}{X^{2n+2}} & \cdots & \frac{X^{2n+1}}{X^{2n+2}} \\
1 & & 0 & -\frac{X^{n+2}}{X^{2n+2}} & 0 & & \\
 & \ddots & & \vdots & & \ddots & \\
0 & & 1 & -\frac{X^{2n+1}}{X^{2n+2}} & & & 0
\end{pmatrix},$$

which implies that

$$(5.4)\quad 2F_{\lambda\kappa}C^\lambda D^\kappa = \sum_{\alpha=1}^{n}\left\{\frac{X^\alpha}{X^{2n+2}}(C^\alpha D^{n+1} - C^{n+1}D^\alpha) \right.$$
$$\left. + \frac{X^{n+1+\alpha}}{X^{2n+2}}(C^{n+1+\alpha}D^{n+1} - C^{n+1}D^{n+1+\alpha}) + C^\alpha D^{n+1+\alpha} - C^{n+1+\alpha}D^\alpha\right\}.$$

Now we consider a submanifold of S^{2n+1} whose local representation is given by

$$(5.5)\quad \begin{cases} X^A = x^A \ (A = 1, \cdots, n, n+2, \cdots, 2n),\ X^{n+1} = 0, \\ (X^{2n+1})^2 = t - \sum_{\alpha=1}^{n}(x^\alpha)^2 - \sum_{\alpha=n+2}^{2n}(x^\alpha)^2, \quad 0 < t < 1, \\ X^{2n+2} = \sqrt{1-t}\ . \end{cases}$$

Then the submanifold is compact and we have

$$(5.6)\quad B_i^{\ \kappa} = \delta_i^{\ \kappa}\ (\kappa = 1, \cdots, n, n+2, \cdots, 2n),\ B_i^{\ n+1} = 0,\ B_i^{2n+1} = -\frac{X^i}{X^{2n+1}}.$$

We put

$$(5.7)\quad (C^\kappa) = (\overbrace{0, \cdots 0}^{n}, 1, \overbrace{0, \cdots, 0}^{n}),\quad (D^\kappa) = (X^1, \cdots, X^n, 0, X^{n+2}, \cdots, X^{2n+1}).$$

Then C^κ and D^κ are mutually orthogonal unit normal vectors to the submanifold defined by (5.5).

The submanifold is, as is easily seen, totally umbilical submanifold of codimension 3 in E^{2n+2}. Since S^{2n+1} is a totally umbilical submanifold of E^{2n+2}, we have, by Yano's formula[5], the submanifold defined by (5. 5) is totally umbilical in S^{2n+1}. Since, in a Riemannian manifold with the curvature tensor of the form (1. 6), any totally umbilical submanifold M satisfies that $\nabla_j H^\kappa \in T(M)$, we have only to examine if r is constant.

Substituting (5. 7) into (5. 4), we find

$$2r = 2F_{\lambda\kappa} C^\lambda D^\kappa = -\frac{1}{X^{2n+2}} \sum_{\alpha=1}^{n} \{(X^\alpha)^2 + (X^{n+1+\alpha})^2\},$$

from which, we have

$$r = -\frac{t}{2\sqrt{1-t}}$$

because of (5. 5). This shows that our example is a desired one.

BIBLIOGRAPHY

[1] Y. KATSURADA, Generalized Minkowski formulas for closed hypersurfaces in a Riemannian space, Ann. Mat. Pura. Appl., 57(1962), 283-294.

[2] Y. KATSURADA, On a certain property of closed hypersurfaces in an Einstein space, Comm. Math. Helv., 38(1964), 165-171.

[3] H. LIEBMANN, Über die Verbiegung der geschlossenen Flächen positiver Krümmung, Math. Ann., 53(1900), 91-112.

[4] M. OKUMURA, Certain hypersurfaces of an odd dimensional sphere, Tôhoku Math. Journ., 19(1967), 381-395.

[5] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with contact metric structures, Journ. of Math. Soc. of Japan, 14(1962), 249-271.

[6] W. SÜSS, Zur relativen Differentialgeometrie V, Tôhoku Math. Journ., 31(1929), 202-209.

[7] Y. WATANABE, Totally umbilical surfaces in normal contact Riemannian manifolds, Kōdai Math. Sem. Rep., 19 (1967), 474-487.

[8] K. YANO, Sur quelques propriétés conformes de V_l dans V_m dans V_n, Proc. of the Imperial Acad., 16(1940), 173-177.

[9] K. YANO, Closed hypersurfaces with constant mean curvature in a Riemannian manifold. Journ. of Math. Soc. of Japan, 17(1965), 333-340.

[10] K. YANO AND S. BOCHNER, Curvature and Betti numbers, Ann. of Math. Studies, No. 32, 1953.

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5) K. Yano [8].