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ALMOST AUTOMORPHIC AND ALMOST PERIODIC SOLUTIONS WHICH MINIMIZE FUNCTIONALS

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1. Introduction. In this note we give a sufficient condition for the existence of an almost periodic solution of a system of equations. Most of the sufficient conditions in the literature are either in terms of stability or are uniqueness theorems that use a result of Amerio [1]. Our condition is that certain solutions minimize a functional. An early example of such a condition goes back to Favard. [2]. In the proof of Theorem 2 he shows that a certain linear nonhomogeneous system has a unique solution with minimum norm. This solution turns out to be almost periodic. We show how to systematize this argument to get generalizations of some of Favard's results.

2. Definitions and preliminary results. If $\{\alpha'_n\}$ is a sequence we write it as α'. If $\alpha = \{\alpha_n\}$ is a subsequence of α' we write $\alpha \subset \alpha'$. For vector functions of a real variable, the symbol $T_\alpha f(t) = \lim_{n\to\infty} f(t+\alpha_n)$ and it is used only if the indicated limit exists. The sense in which the limit exists will always be indicated. Bochner [3] was the first to notice that almost periodic functions (=a.p.) are precisely those functions f for which given α', β' sequences, there exist $\alpha \subset \alpha'$, $\beta \subset \beta'$ such that $T_{\alpha+\beta} f = T_\alpha(T_\beta f) = T_\alpha T_\beta f$ pointwise, where $\alpha+\beta = \{\alpha_n+\beta_n\}$. It is this characterization that we use in this note.

In the same paper, Bochner introduces the notation of an almost automorphic (=a.a.) function. A bounded function f is a.a. if for every sequence α' there exists $\alpha \subset \alpha'$ such that $T_\alpha f = g$ and $T_{-\alpha} g = f$ exist pointwise. Here $-\alpha = \{-\alpha_n\}$.

We introduce a slightly stronger concept. A bounded function f is compact almost automorphic (compact a.a.) if the above limits are required to exist uniformly on every compact subset of the reals.

We shall consider the differential equation

$$(1)\qquad x' = F(t, x),$$

where F is compact a.a. in t uniformly in $x \in K$, i.e., given α' there exists $\alpha \subset \alpha'$ such that $T_\alpha F(t, x) = \lim_n F(t+\alpha_n, x) = G(t, x)$ and $T_{-\alpha}G(t, x) = \lim_n G(t-\alpha_n, x) = F(t, x)$ exist uniformly on $I \times K$ where I is an arbitrary compact subset of R. For our purposes, we shall assume that K is compact in R^n and F is continuous on $R \times K$. For the vectors $x \in R^n$, $|x|$ denotes the Euclidean norm and $\|x\| = \sup_{t \in R} |x(t)|$, when x is a function on R into R^n.

The hull of F is the collection of functions G such that $T_\alpha F = G$ for some α and the limit is uniform on compact sets.

The key to all our arguments is the following.

LEMMA 1. *Let F be a continuous compact a.a. function in t uniformly in $x \in K$, K compact, and $\varphi(t)$ a solution of* (1) *with $\varphi(t) \in K$ for all t. Then given α', there exists $\alpha \subset \alpha'$ such that $T_\alpha F = G$, $T_{-\alpha}G = F$, $T_\alpha \varphi = \psi$, $T_{-\alpha}\psi = \varphi_1$ all exist uniformly on compact sets and ψ is a solution of $x' = G(t, x)$, with φ_1 a solution of $x' = F(t, x)$.*

PROOF. If α' is a given sequence, we first take a subsequence $\beta \subset \alpha'$ such that $T_\beta F = G$ and $T_{-\beta}G = F$ uniformly on compact sets. If $I_N = [-N, N]$ then $F(t, x)$ and $F(t+\beta_n, x)$ are uniformly bounded on $I_N \times K$ so that $\varphi'(t)$ and $\varphi'(t+\beta_n)$ are uniformly bounded. By Ascoli's Theorem and then the familiar diagonalization argument one gets a subsequence $\gamma \subset \beta$ such that $T_\gamma \varphi = \psi$ exists uniformly on compact sets. Consequently ψ is a solution of $x' = G(t, x)$. In the same way, there is a subsequence $\alpha \subset \gamma$ such that $T_{-\alpha}\psi = \varphi_1$ exists and φ_1 is a solution of $x' = F(t, x)$.

LEMMA 2. (Amerio) *Let F be a continuous compact a.a. function in t uniformly in $x \in K$, K compact. If there is a solution $\varphi(t)$ of $x' = F$ with $\varphi(t) \in K$ and φ is defined on $[t_0, \infty)$ for some t_0, then every equation in the hull has a solution defined on R with values in K.*

PROOF. If $\varphi(t)$ is a solution of $x' = F(t, x)$ on $[t_0, \infty)$, let $\alpha'_n = n$. Then $\varphi(t+n)$ is a solution of $x' = F(t+n, x)$ on $[t_0 - n, \infty)$. Now one applies the argument of Lemma 1 to get a subsequence $\alpha \subset \alpha'$ such that $T_\alpha F = G$, $T_{-\alpha}G = F$ and $T_\alpha \varphi = \psi$ exists such that $\psi' = G(t, \psi)$. Clearly ψ will be defined on R and hence $T_{-\alpha}\psi$ will be a solution of $x' = F(t, x)$ defined on R and $T_{-\alpha}\psi(t) \in K$ for all t. One gets a solution to the other equations in the hull by the argument of Lemma 1.

Coppel [4] shows that if G is in the hull of F, then the hull of G is contained in the hull of F. Now suppose $T_\alpha\varphi$ exists uniformly on compact sets, for φ a solution of $x'=F$. Then $T_\alpha\varphi$ is a solution of some equation in the hull of F. In fact, there is a $\beta\subset\alpha$ such that $T_\beta F=G$ exists uniformly on compact sets, then $T_\beta\varphi=T_\alpha\varphi$ is a solution of $x'=G$. Henceforth $T_\alpha\varphi$ means that the limit exists at least uniformly on compact sets.

Let λ be a mapping from the solutions in K of the equations $x'=G(t,x)$ in the hull of F to the real numbers. This functional is said to be subvariant if $\lambda(T_\alpha\varphi)\leqq\lambda(\varphi)$. In view of Lemma 1, and the above discussion $T_\alpha\varphi$ is in the domain of λ whenever φ is. An example of a subvariant functional is $\lambda(\varphi)=\|\varphi\|$.

3. **Main results**. The idea now is, that if λ is a subvariant functional for the solutions in some compact set, then $\lambda(T_{-\alpha}T_\alpha\varphi)\leqq\lambda(\varphi)$ and hence if λ is minimized by only one solution φ of a particular equation, then φ is compact a.a. This leads to a sufficient condition for the existence of a.p. solutions. By a solution we shall mean that the domain is all of R.

THEOREM 1. *Suppose that F is continuous and compact a.a. in t uniformly in $x\in K$, K compact and that* (1) *has a solution with values in K. If there exists a subvariant functional λ such that $\mu=\min\{\lambda(\varphi)\mid\varphi$ is a solution of $x'=F(t,x),\varphi(t)\in K\}$ exists and is attained by a unique solution, then that minimizing solution is compact a.a.*

PROOF. Let $\varphi(t)$ be the unique solution of $x'=F(t,x)$, with $\varphi(t)\in K$, that minimizes λ. From Lemma 1, given α', there exists $\alpha\subset\alpha'$ such that $T_{-\alpha}T_\alpha\varphi$ is a solution of $x'=F(t,x)$. Now by the subvariance of λ, $\lambda(T_{-\alpha}T_\alpha\varphi)\leqq\lambda(T_\alpha\varphi)\leqq\lambda(\varphi)$. Since the minimizing solution is unique $T_{-\alpha}T_\alpha\varphi=\varphi$, and the convergence is uniform on compact sets.

THEOREM 2. *Suppose F is a.p. in t uniformly for $x\in K$, K compact, and that for some G in the hull of F, there is a solution in K. If there is a subvariant functional λ such that $\mu_G=\min\{\lambda(\varphi)\mid\varphi$ is a solution $x'=G(t,x),\varphi(t)\in K\}$ exists for each G in the hull of F and is attained by a unique solution, then every equation in the hull has an a.p. solution.*

PROOF. By the preceding Theorem, the minimizing solution φ of $x'=F(t,x)$ is a.a. and $\lambda(T_\alpha\varphi)=\lambda(\varphi)$ whenever $T_\alpha\varphi$ exists. We claim further, that $T_\alpha\varphi$ is the minimizing solution of $T_\alpha F=G$. If $\lambda(\psi)<\lambda(T_\alpha\varphi)=\lambda(\varphi)$

for $\psi' = G(t, \psi)$, then there is a subsequence $\beta \subset \alpha$ such that $T_{-\beta}G = F$ and $T_{-\beta}\psi$ is a solution of $x' = F(t, x)$. Then $\lambda(T_{-\beta}\psi) \leqq \lambda(\psi) < \lambda(\varphi)$ is a contradiction. Now let α' and β' be given. Extract subsequences by Lemma 1 such that $T_\alpha\varphi$, $T_\beta T_\alpha\varphi$, $T_{\alpha+\beta}\varphi$, $T_\alpha F$, $T_\beta T_\alpha F$ and $T_{\alpha+\beta}F$ all exist, the first three uniformly on compact sets, the last three uniformly on $R \times K$. By Bochner's Theorem we may assume $T_{\alpha+\beta}F = T_\beta T_\alpha F$. Thus $T_\beta T_\alpha\varphi$ and $T_{\alpha+\beta}\varphi$ are solutions of the same equation which by the above are both minimizing solutions. Hence $T_{\alpha+\beta}\varphi = T_\beta T_\alpha\varphi$ and by Bochner's Theorem φ is a.p. Clearly $T_\alpha\varphi$ is an a.p. solution of $x' = T_\alpha F$.

COROLLARY 1. *If F is continuous and compact a.a. in t uniformly for $x \in K$, K compact and F has a unique solution $\varphi(t) \in K$, then φ is compact a.a. If further F is a.p. and each equation in the hull of F has at most one solution in K, then φ is a.p.*

PROOF. Take $\lambda(\varphi) = 1$.

It is an intriguing question whether Corollary 1 is true in the following form. If F is a.p. and has a unique solution in K, then that solution is a.p. In Fink and Seifert [5], it is shown that if that solution is uniformly stable, then it is a.p. A related statement is also possible. If every equation in the hull of F has a unique solution in K and F is compact a.a., is the unique solution of $x' = F$ a.p.? This seems to be unknown. Corollary 2 is also an immediate consequence of Amerio's Theorem.

COROLLARY 2. (Favard). *If $f(t, x) = A(t)x + g(t)$ where A is an a.p. $n \times n$ matrix and g is an a.p. vector function and if every equation $x' = L(t)x$ where L is in the hull of A has the property that every bounded nontrivial solution φ satisfies $|\varphi(t)| \geqq \varepsilon > 0$ for some $\varepsilon(\varphi)$, then $x' = f(t, x)$ has an a.p. solution if and only if it has a bounded solution.*

PROOF. An a.p. solution is bounded. For the converse, let a bounded solution φ_1 exist. Let K be a compact set containing the range of φ_1. We take $\lambda(y) = \|y\|$. Clearly λ is subvariant. Let $a_L = \inf\{\lambda(y) \mid y' = Ly + h, y \in K\}$ where $Lx + h$ is in the hull of $Ax + g$. Let $\lambda(\varphi_n) \to a_L$. Then $\{\varphi_n'\}$ is uniformly bounded, hence by Ascoli's Theorem we have a subsequence which converges uniformly on compact sets to a solution φ of $x' = Lx + h$. Clearly $\lambda(\varphi) = a_L$. We now show that the minimum is attained only by φ. If ψ is another solution with $\lambda(\psi) = a_L$ then $\varphi - \psi$ is a solution of $x' = Lx$ and hence there exists $\varepsilon > 0$ such that $|\varphi - \psi|(t) \geqq \varepsilon$. This implies that $2\varphi(t) \cdot \psi(t) \leqq |\varphi(t)|^2 + |\psi(t)|^2 - \varepsilon^2$. Now $(\varphi + \psi)/2$ is a solution of $x' = Lx + h$ and the above inequality yields $|(\varphi + \psi)/2|^2(t) = [|\varphi(t)|^2 + |\psi(t)|^2]/4 + \varphi(t) \cdot \psi(t)/2$

$\leqq [\|\varphi\|^2+\|\psi\|^2]/4 + [\|\varphi(t)\|^2+\|\psi(t)\|^2-\varepsilon^2]/4 \leqq a_L^2 - \varepsilon^2/4 < a_L^2$. This contradiction shows the uniqueness. Now Theorem 2 gives the result.

Notice that in Theorem 2, λ is not required to be non-negative nor even continuous. We give an example of a second order system where $\lambda(y) = \sup y - \inf y$ is minimized by a unique solution. Specifically we consider the equation

$$(2) \qquad y'' = f(t, y, y')$$

with f continuous on $R\times K$, where K is some compact subset of R^2. We say that (2) satisfies the maximum principle if for every $a < b$ and two solutions y and z, $y(a)-z(a) \leqq M$, $y(b)-z(b) \leqq M$, and $y(t)-z(t) \geqq 0$ on $[a, b]$ imply that $y(t)-z(t) \leqq M$ on $[a, b]$.

LEMMA 3. *Suppose that f in equation* (2) *is continuous and suppose that a solution exists with values in K. Then there exists a solution φ such that $\lambda(\varphi) \leqq \lambda(\psi)$ for all solutions ψ in K where $\lambda(\varphi) = \sup\varphi - \inf\varphi$.*

PROOF. Let $\lambda(\varphi_n) \to \inf\{\lambda(\varphi) | \varphi$ a solution of (2) in $K\} = a$, where φ_n are solutions of (2) in K. Since for any N, f is bounded on $[-N, N]\times K$, we have φ_n, φ_n', and φ_n'' uniformly bounded so by the Ascoli and diagonalization argument we get a subsequence converging uniformly on compact sets to a solution φ. If $\lambda(\varphi) > a$ then $\sup_{t\in I}\varphi(t) - \inf_{t\in I}\varphi(t) > a$ for some compact I. But this quantity is a continuous function under uniform convergence on compact sets, so that $\sup_{t\in I}\varphi - \inf_{t\in I}\varphi = \lim_n[\sup_{t\in I}\varphi_n - \inf_{t\in I}\varphi_n] \leqq a$. Hence $\lambda(\varphi) = a$.

THEOREM 3. *Let f in* (2) *be a.p. in t uniformly for $(y, z)\in K$ and suppose that every equation $y''=g(t, y, y')$ in the hull of f satisfies the maximum principle. If one of these equations has a solution in K, then every equation has an a.p. solution in K. Here K is compact.*

PROOF. Let $\lambda(y) = \sup y - \inf y$. By Lemma 3, there is a solution φ_g which minimizes λ over the solutions in K of $y''=g(t, y, y')$. Let $a_g = \lambda(\varphi_g)$. We claim that for all g in the hull of f, $a_g = a_f$. Choose α so that $T_\alpha g = f$ and $T_\alpha\varphi_g$ is a solution of $y'' = f$. Since λ is subvariant, $a_f \leqq \lambda(T_\alpha\varphi_g) \leqq a_g$. By a symmetric argument $a_g \leqq a_f$. Now φ_g may not be unique for solutions in K, however we show that it is on a suitably chosen compact subset K_0. Suppose φ_f is one solution of (2) such that $\lambda(\varphi_f) = a_f$. Let $K_0 = [a, b]\times H$, where $a = \inf\varphi_f$, $b = \sup\varphi_f$ and H is a compact interval containing the range of $\varphi_f'(t)$. By Lemma 1, each equation $y'' = g(t, y, y')$ in the hull of f has a solution in K_0. Furthermore, $a_g = a_f$ implies that every solution φ of $y'' = g$ with φ in K_0 satisfies $b = \sup \varphi$ and $a = \inf \varphi$. As a matter of fact, a stronger statement is true. For any t_0, $\sup\{\varphi(t) \mid t \geqq t_0\} = \sup\{\varphi(t) \mid t \leqq t_0\} = b$ and a similar statement holds for $\inf \varphi(t)$. For example, if $\sup\{\varphi(t) \mid t \geqq t_0\} \leqq b - \varepsilon$ for some $\varepsilon > 0$, then consider the sequence $\alpha' = \{n\}$. Extract a subsequence $\alpha \subset \alpha'$ such that $T_\alpha g = h$ and $T_\alpha \varphi = \psi$ exist uniformly on compact sets. Now ψ is a solution to $y'' = h$ and since $\alpha_n \to \infty$, $t_1 + \alpha_n \geqq t_0$ for large n, and we have $\psi(t_1) = \lim \varphi(t_1 + \alpha_n) \leqq b - \varepsilon$. Thus $\sup \psi \leqq b - \varepsilon$, which is a contradiction.

The maximum principle is now applied to show that each equation $y'' = g(t, y, y')$ has only one solution in K_0. If φ and ψ are distinct solutions in K_0, then let $h = \varphi - \psi$. Suppose there is a t_0 such that $\varepsilon = h(t_0) > 0$ and $h'(t_0) > 0$. Then $h(t) \geqq h(t_0)$ for all $t \geqq t_0$, for if $h(t_1) < h(t_0)$ for some t_1, then there are points $t_0 < t_2 < t_3$ such that $h(t_3) = h(t_0)$, $h(t_2) > h(t_0)$ and $h(t) > 0$ on $[t_0, t_3]$. This contradicts the maximum principle on $[t_0, t_3]$. There is a sequence $t_n \to \infty$ such that $\lim_n \psi(t_n) = b$, but then $\varphi(t_n) \geqq \psi(t_n) + \varepsilon$ for large n implies that $\sup \varphi \geqq b + \varepsilon > b$, so that φ is not in K_0. If $h'(t_0) < 0$, one shows that $h(t) \geqq h(t_0)$ for all $t \leqq t_0$. Now by choice of notation either one of the above occurs, or $h'(t) \equiv 0$ in which case the argument is clear. Hence there is exactly one solution of each equation $y'' = g$ in K_0. This solution is a.p. by either Theorem 2 or Corollary 1.

Note that the above argument also shows that if f is compact a.a., then $x'' = f$ has a compact a.a. solution.

COROLLARY 3. *Let f be compact a.a. in t uniformly for $(y, z) \in K$, K compact, and suppose every equation $y'' = g(t, y, y')$ in the hull of f satisfies the maximum principle. If $y'' = f$ has a solution in K, then it has a compact a.a. solution in K.*

It would be appropriate at this point to give some sufficient conditions on f so that the hypotheses of Theorem 2 are satisfied. Suppose for example, that $f(t, y, y')$ is strictly increasing as a function y for each fixed t and y'. Let $h(t)$ be the difference of two solutions of $y'' = f(t, y, y')$. Then it is easy to show that $h(t_0) \neq 0$ and $h'(t_0) = 0$ imply that $h(t_0) h''(t_0) > 0$. The maximum principle follows from this. This is the condition of Lemma 3.3 in Jackson [5]. We then have the following corollary.

COROLLARY 4. *Let $f(t, y, z)$ be a.p. in t uniformly for $(y, z) \in K$, K compact, and suppose that every function in the hull of f is strictly* *increasing in y. Then* (2) *has an a.p. solution in K if and only if it has a solution in K.*

Note that the existence of an a.p. solution reduces to finding a solution φ on a ray $[a, \infty)$ such that both φ and φ' are bounded on this ray. See Lemma 2. See Schrader [7] or Schmitt [8] for interesting sufficient conditions. We note, however, that Corollary 4 yields a much better result than Theorem 4 of Fink [9]. We reproduce the statement here as

COROLLARY 5. *Suppose that $f(t, y, z)$ satisfies*

(i) *there exist $a < b$ such that $f(t, a, 0) \leqq 0 \leqq f(t, b, 0)$ for all $t \in R$;*
(ii) *there exist $c < 0 < d$ such that $f(t, x, c)$ and $f(t, x, d)$ do not change sign for $t \in R$ and $x \in [a, b]$;*
(iii) *f is uniformly a.p. in t on $[a, b] \times [c, d]$;*
(iv) *each function in the hull of f is strictly increasing in the second variable.*

Then there is an a.p. solution φ of (2) with $(\varphi(t), \varphi'(t)) \in [a, b] \times [c, d]$.

In the particular case when $f(t, y, z) = g(y, z) + h(t)$ for h a.p., then the above conditions are particularly simple since the increasing in y of g and the existence of the four constants a, b, c, d is all that is needed for the existence of an a.p. solution.

We also may note that the uniqueness of a solution in a compact set implies that the module of the solution is contained in the module of the function f, so that Theorem 3 and its corollaries also have this containment as conclusions; see [1]. This is not necessarily true for the general case of Theorem 2.

The argument of Theorem 3 can be extended to second order systems. Suppose that in equation (2), y y' and f are n-vectors. Suppose we have a maximum principle for systems which says that the ordinary scalar maximum principle holds for each component. Call this the vector maximum principle. Then we have

THEOREM 4. *Let f be a vector function a.p. in t uniformly for $y \in K_1$, $y' \in K_2$, where K_1 and K_2 are compact, and $K_1 = \times_{i=1}^{n} [a_i, b_i]$. Suppose that there is a solution of* (2) *with $y(t) \in K_1$ and $y'(t) \in K_2$ for all t and that every equation in the hull of f satisfies the vector maximum principle, then* (2) *has an a.p. solution in $K_1 \times K_2$.*

PROOF. Define $\lambda_1(y) = \sup y_1 - \inf y_1$ where y_1 is the first component. One goes through the argument of Theorem 3 to get $[c_1, d_1] \subset [a_1, b_1]$ such that every solution of a fixed equation $y'' = g$ in the hull of f that remains in $[c_1, d_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \times K_2$ has the same first component. Let $K^1 = [c_1, d_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Then for $\lambda_2(y) = \sup y_2 - \inf y_2$ we repeat the argument for solutions which are required to be in $K^1 \times K_2$. In n steps we get a set $K^n = [c_1, d_1] \times \cdots \times [c_n, d_n]$ such that precisely one solution of each equation in the hull of f lies in $K^n \times K_2$. Now apply Theorem 2.

Two sufficient conditions for the vector maximum principle are given in Heimes [10], both of which essentially demand strictly increasing component functions in maching variable, i.e., f_k is increasing in y_k. We refer the interested reader to that paper.

4. Concluding remarks. Theorem 2 of this paper, through Corollary 1, proves a special case of Amerio's Theorem. As a matter of fact, almost all, if not all, of the uses of Amerio's Theorem in the literature only use the special case when uniqueness is involved. In any case, the hypotheses of Amerio's Theorem are that in $x' = F(t, x)$, F is uniformly a.p. on K and that every equation in the hull $x' = G(t, x)$ has the property that for each solution in K, there is a constant $\rho(\varphi)$ such that $0 < \rho(\varphi) \leqq |\varphi(t) - \psi(t)|$, for ψ any other solution in K. One can show that ρ may be picked independent of φ and that each equation has only a finite number of solutions in K. These seem to be rather strong hypotheses.

It may be of interest to note that neither hypothesis of Theorem 2 nor those of Amerio's Theorem are consequences of the other. To show that an equation may have only separated solutions but not minimize a subvariant functional, consider the following example given in Seifert [11].

Let $z_1' = -z_1$ and $z_2' = z_2(1 - z_2^2)$. Write this as a vector system $z' = f(z)$ where $f(z) = (-z_1, z_2(1 - z_2^2))^T$. Now, if $x = A(t)z$, where

$$A(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$

then $x' = F(t, x)$, where $F(t, x) = A'(t)A^T(t)x + A(t)f(A^T(t)x)$. F is periodic of period π. There are two periodic solutions $x_1 = (-\sin t, \cos t)^T$ and $x_2 = (\sin t, -\cos t)^T$. Furthermore, these are the only solutions that satisfy $|x(t)| = 1$ for all t. Thus for $K = \{x \mid |x| = 1\}$ we have exactly these two solutions which are separated. But $x_1(t + \pi) = x_2(t)$ so that for any subvariant functional λ two aplications of translation by π yield $\lambda(x_2) \leqq \lambda(x_1) \leqq \lambda(x_2)$ so that neither x_1 nor x_2 can be a unique minimizing solution. This also shows that the minimizing condition is not necessary for an a.p. solution.

Conversely, a system which minimizes a functional but does not have separated solutions, consider $y''+y=f(t)$, where f is a.p. and such that a bounded solution exists, say $f(t)=\sin\sqrt{2}\,t+\sin\sqrt{3}\,t$ then there is a bounded solution and the difference of two solutions has constant norm. Thus by Corollary 2, there is a solution which minimizes the norm. The solution is not separated because the norm of differences, though a constant for each pair, can be arbitrarily small. For the above f, no solution is periodic so it is not clear how to get separated solutions by restricting the compact set K in E^2.

The minimizing conditions of Theorem 1 and 2 are used first to get compact a.a. solutions and then to show that these a.a. solutions are a.p. We have used the condition $T_{\alpha+\beta}\varphi=T_\alpha T_\beta\varphi$ to make this last step. Veech [12] has shown that φ is a.p. if and only if φ is a.a. and $T_\alpha\varphi$ is a.a. whenever this limit exists pointwise. This condition does not seem to be of use in the present context.

Finally, it would be of interest to know the connection between the existence of subvariant functionals with minimizing solutions and the stability conditions which yield Lyapunov functions.

George Seifert has pointed out that the condition that $f(t,y,y')$ be strictly increasing in y is almost necessary for the maximum principle to hold. In fact, let $t_0, y_1>y_2$, and z be given and assume that f is continuous. Let $y_i(t)$ be solutions to the initial value problems $y_i(t_0)=y_i$, $y_i'(t_0)=z$, $i=1,2$. By the maximum principle on $[t_0-\delta, t_0+\delta]$, for δ sufficiently small, t_0 cannot be an interior maximum of $y_1(t)-y_2(t)$. Hence $y_1''(t_0)-y_2''(t_0)\geqq 0$. This implies that $f(t_0,y_1,z)\geqq f(t_0,y_2,z)$. That is, a necessary condition for the maximum principle is that f be nondecreasing in y.

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