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RINGS OF U-DOMINANT DIMENSION $\geqq 1$

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Dedicated to Professor Tadao Tannaka on his 60th birthday.

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Recently Tachikawa [8] has proved that, if R is a ring of dominant dimension $\geqq 1$, then domi. dim. $R_R \geqq 2$ if and only if the injective hull $E(R_R)$ of R_R has the double centralizer property. Our purpose here is to generalize this result which is also the origin of the present paper. We are mainly concerned with double centralizers of finitely-faithful injective modules and examine the double centralizer property of such modules. To this end, we introduce in Section 1 U-dominant dimension for modules, where U is a module. The U-dominant dimension is, roughly speaking, a relative dominant dimension with respect to a given module U (for a definition of dominant dimension, see Kato [3, §1] and Tachikawa [8, §1]). It is shown in Theorem 1 that the double centralizer of a finitely-faithful, injective module U_R over a ring R has always U-dominant dimension $\geqq 2$. On the other hand our main Theorem 2 states that a finitely-faithful, injective right R-module U_R has the double centralizer property if and only if U-domi. dim. $R_R \geqq 2$. The final Section 3 is devoted to the situation that U-domi. dim. $R_R = 1$. Let U-domi. dim. $R_R = 1$, where U_R is finitely-faithful and injective, and let Q be the double centralizer of U_R. Then $R \neq Q$ by Theorem 2. Now let $R \subset Q' \subset Q$ be an intermediate ring between R and Q. Then Theorem 3 states that U-domi. dim. $Q'_{Q'} = 1$ if and only if $Q' \neq Q$. These theorems yield interesting corollaries which generalize results of Mochizuki [5, Theorem 3.1], Tachikawa [7, Theorem 1.4] and Tachikawa [8, Theorem 1.4].

Throughout this paper, rings will have a unit element and modules will be unital. A_R will denote, as usual, the fact that A is a right module over a ring R. If A_R is a module over a ring R, $E(A_R)$ will denote the injective hull of A_R. We adopt the notation that homomorphisms of modules will be written on the side opposite to the scalars.

1. Introduction. Let R be a ring, and U_R a right R-module. A right R-module X_R is called U-torsionless in case $X_R \hookrightarrow \prod U_R$, where $\prod U_R$ is a direct product of copies of U_R. It is easy to see that X_R is U-torsionless if and only if, for each $0 \neq x \in X_R$, there exists $f \in \mathrm{Hom}(X_R, U_R)$ such that $f(x) \neq 0$. Now let

$$0 \longrightarrow A_R \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n$$

be a minimal injective resolution of a right R-module A_R. We shall say that A_R has U-dominant dimension $\geqq n$ if each X_i is U-torsionless (denoted by U-domi. dim. $A_R \geqq n$). U-domi. dim. $A_R = n$ if U-domi. dim. $A_R \geqq n$ and U-domi. dim. $A_R \ngeqq n+1$. In case $U_R = R_R$, R-domi. dim. A_R = domi. dim. A_R.

A right R-module U_R is called finitely-faithful if there exist $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ in U_R such that $\varepsilon_1 r = \varepsilon_2 r = \cdots = \varepsilon_n r = 0$, $r \in R$, implies $r=0$. It is then clear that U_R is finitely-faithful if and only if $R_R \hookrightarrow \overset{n}{\oplus} U_R$, where $\overset{n}{\oplus} U_R$ is the direct sum of n-copies of U_R. It is the Morita's observation that each faithful right module over a right Artinian ring is finitely-faithful (see Morita [6, Theorem 2]).

Let U_R be a faithful right R-module and let $S = \mathrm{Hom}(U_R, U_R)$ be the endomorphism ring of U_R. Then ${}_SU_R$ is (S, R)-bimodule and $R \subset \mathrm{Hom}({}_SU, {}_SU) = Q$ by virtue of the faithfulness of U_R. In this paper, we shall regard R as a subring of Q. The ring $Q = \mathrm{Hom}({}_SU, {}_SU)$ is called the double centralizer of U_R and we shall say that U_R has the double centralizer property whenever $R = Q$.

2. Rings of U-dominant dimension $\geqq 2$. Throughout this section, let R be a ring, U_R a finitely-faithful injective right R-module, $S = \mathrm{Hom}(U_R, U_R)$, and $Q = \mathrm{Hom}({}_SU, {}_SU)$ the double centralizer of U_R. Since U_R is finitely-faithful, there exists $(\varepsilon_i) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \overset{n}{\oplus} U_R$ such that $(\varepsilon_i)r = 0$, $r \in R$, implies $r=0$. The following lemma is fundamental in this paper.

LEMMA 1. *Let U_R, S, Q, and (ε_i) be as above. Let Q' be a ring between R and Q. Then*

(1) ${}_SU = S\varepsilon_1 + \cdots + S\varepsilon_n$.

(2) $Q_Q \hookrightarrow \overset{n}{\oplus} U_Q$ *by the canonical map* $q \rightsquigarrow (\varepsilon_i)q$, $q \in Q$.

(3) *Each R-map $f \colon A_{Q'} \longrightarrow U_{Q'}$ is necessarily a Q'-map, where $A_{Q'}$ is a right Q'-module.*

(4) *Let $A_{Q'}$ be a right Q'-module. If A_R is torsionless, then $A_{Q'}$ is torsionless.*

PROOF. Our method used here has its origin in Lambek's paper [4].

(1) For each $u \in U_R$, we have the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R_R & \xrightarrow{(\varepsilon_i)} & \overset{n}{\oplus} U_R \\ & & u \downarrow & \swarrow (s_i) & \\ & & U_R & & \end{array}$$

by virtue of the injectivity of U_R, where $(s_i)=(s_1, s_2, \cdots, s_n) \in \overset{n}{\oplus} \mathrm{Hom}(U_R, U_R) = \mathrm{Hom}(\overset{n}{\oplus} U_R, U_R)$. Hence $u = (s_i)(\varepsilon_i) = s_1\varepsilon_1 + \cdots + s_n\varepsilon_n \in S\varepsilon_1 + \cdots + S\varepsilon_n$.

(2) Let $(\varepsilon_i)q=0$, $q \in Q$. Then ${}_SU \cdot q = (S\varepsilon_1 + \cdots + S\varepsilon_n)q = S(\varepsilon_1 q) + \cdots + S(\varepsilon_n q) = 0$. Hence $q = 0$.

(3) For each $a \in A_{Q'}$, we set $f_a(q') = f(aq') - f(a)q'$ for $q' \in Q'$, then $f_a \in \mathrm{Hom}(Q'_R, U_R)$. Since U_R is injective, there exists a map $(s_i) \in \overset{n}{\oplus} \mathrm{Hom}(U_R, U_R) = \mathrm{Hom}(\overset{n}{\oplus} U_R, U_R)$ making

$$\begin{array}{ccccc} 0 & \longrightarrow & Q'_R & \xrightarrow{(\varepsilon_i)} & \overset{n}{\oplus} U_R \\ & & f_a \downarrow & \swarrow (s_i) & \\ & & U_R & & \end{array}$$

commutative, where the horizontal map (ε_i) is a monomorphism by (2). Hence $f_a(q') = (s_i)(\varepsilon_i q') = \sum_{i=1}^{n} s_i(\varepsilon_i q') = \sum_{i=1}^{n} (s_i\varepsilon_i)\, q' = \left(\sum_{i=1}^{n} s_i\, \varepsilon_i\right) q' = (s_i)(\varepsilon_i)\cdot q' = f_a(1)q' = 0$, concluding that $f_a=0$. Thus f is a Q'-map.

(4) Since A_R is torsionless, for each $0 \neq x \in A_R$, there exists $f \in \mathrm{Hom}(A_R, R_R)$ such that $f(x) \neq 0$. We show that f is a Q'-map. In fact, $\varepsilon_i f\colon A_{Q'} \longrightarrow U_{Q'}$ is an R-map, and hence it is also a Q'-map by the above (3). Therefore, $\varepsilon_i \cdot f(aq') = (\varepsilon_i f)(aq') = (\varepsilon_i f)a \cdot q' = \varepsilon_i f(a)\, q'$, for $a \in A$, $q' \in Q'$, consequently, $\varepsilon_i(f(aq') - f(a)q') = 0$ for $i = 1, 2, \cdots, n$. Hence, in view of (2), $f(aq') = f(a)q'$ for $a \in A$, $q' \in Q'$. It follows from this that $A_{Q'}$ is torsionless.

REMARK. It may be interesting to observe that each finitely-faithful, quasi-injective module is injective. To see this, use Lemma 1, (1) and Johnson and Wong [1, Theorem 1.1].

We are now ready to prove one of our main results.

THEOREM 1. *Let U_R be finitely-faithful and injective, and let Q be the double centralizer of U_R. Then* U-domi. dim. $Q_Q \geqq 2$.

PROOF. In view of Lemma 1, (3), the injectivity of U_R implies the injectivity of U_Q. Hence, by Lemma 1, (2), we have $E(Q_Q) \subsetneq \overset{n}{\oplus} U_Q$. Thus U-domi. dim. $Q_Q \geqq 1$. Next we show that $\overset{n}{\oplus} U_Q/(\varepsilon_i)Q$ is U_Q-torsionless. Let $(u_i) \in \overset{n}{\oplus} U_Q$, $(u_i) \notin (\varepsilon_i)Q$. Then there exists $(s_i) \in \overset{n}{\oplus} \mathrm{Hom}(U_R, U_R) = \mathrm{Hom}(\overset{n}{\oplus} U_R, U_R)$ such that $(s_i)(\varepsilon_i)=0, (s_i)(u_i) \neq 0$. To see this, suppose on the contrary that $(s_i)(\varepsilon_i)=0$, $(s_i) \in \overset{n}{\oplus} \mathrm{Hom}(U_R, U_R)$, implies $(s_i)(u_i)=0$. Then, using Lemma 1, (1), the map $q: \sum_{i=1}^{n} s_i \varepsilon_i \rightsquigarrow \sum_{i=1}^{n} s_i u_i$ of ${}_SU$ into ${}_SU$ is well-defined and hence $q \in Q = \mathrm{Hom}({}_SU, {}_SU)$. But then $\varepsilon_i q = u_i$ and $(u_i) = (\varepsilon_i q) \in (\varepsilon_i) Q$, contradicting the assumption that $(u_i) \notin (\varepsilon_i) Q$. Therefore the map (s_i) induces an R-map $f\colon \overset{n}{\oplus} U_Q/(\varepsilon_i)Q \longrightarrow U_Q$ such that $f((u_i)+(\varepsilon_i)Q) \neq 0$. But f is a Q-map by Lemma 1, (3). Thus $\overset{n}{\oplus} U_Q/(\varepsilon_i) Q \subsetneq \prod U_Q$. The commutative diagram

$$\begin{array}{ccc} Q_Q & \subset & E(Q_Q) \\ \wr\wr & & \cap \\ (\varepsilon_i)Q & \subset & \overset{n}{\oplus} U_Q \end{array}$$

implies that $E(Q_Q)/Q \subsetneq \overset{n}{\oplus} U_Q/(\varepsilon_i)Q \subsetneq \prod U_Q$. Hence $E(E(Q_Q)/Q) \subsetneq \prod U_Q$ by the injectivity of U_Q, and we come to a conclusion that U-domi. dim. $Q_Q \geqq 2$.

The following corollary is a generalization of Mochizuki [5, Theorem 3.1].

COROLLARY. *Let Q be the double centralizer of a finitely-faithful, injective, torsionless right R-module U_R. Then* domi. dim. $Q_Q \geqq 2$.

PROOF. Since U_R is torsionless, U_Q is torsionless by Lemma 1, (4). Combining this fact with Theorem 1, we have immediately domi. dim. $Q_Q \geqq 2$.

REMARK. The category $\mathfrak{M}_R$ of right R-modules has a finitely-faithful, injective, torsionless module if and only if domi. dim. $R_R \geqq 1$ (see Kato [2, Proposition 1]). The above corollary indicates that each ring of dominant dimension $\geqq 1$ can be imbedded in a ring of dominant dimension $\geqq 2$.

We are now ready to establish the double centralizer theorem for finitely-faithful, injective modules.

THEOREM 2. *Let U_R be finitely-faithful and injective. Then the following conditions are equivalent*:

(1) U-domi. dim. $R_R \geqq 2$.
(2) *U_R has the double centralizer property.*

PROOF. (1) implies (2). Let Q be the double centralizer of U_R. Then $\mathrm{Hom}(Q_R/R, U_R) = 0$. In fact, let $f \in \mathrm{Hom}(Q_R, U_R)$, $f(R) = 0$. Then f is a Q-map by Lemma 1, (3) and hence $f(q) = f(1q) = f(1)q = 0$ for $q \in Q$. Therefore $f=0$. On the other hand we have $(\varepsilon_i)R \subset E((\varepsilon_i)R) \subset \overset{n}{\oplus} U_R$ making use of the injectivity of $\overset{n}{\oplus} U_R$, and set $\overset{n}{\oplus} U_R = E((\varepsilon_i)R) \oplus V_R$ by virtue of the injectivity of $E((\varepsilon_i)R)$. Then

$$Q_R/R \hookrightarrow \overset{n}{\oplus} U_R/(\varepsilon_i)R \approx E((\varepsilon_i)R)/(\varepsilon_i)R \oplus V_R \hookrightarrow \prod U_R$$

since $E((\varepsilon_i)R)/(\varepsilon_i)R \approx E(R_R)/R$ is U-torsionless by (1). Now combining $Q_R/R \hookrightarrow \prod U_R$ with $\mathrm{Hom}(Q_R/R, U_R) = 0$, we have $Q = R$ as an immediate conclusion.

(2) implies (1) by Theorem 1.

The preceding theorem provides us a nice characterization of rings for which each finitely-faithful, injective right module has the double centralizer property.

COROLLARY. *The following conditions on a ring R are equivalent*:

(1) *R is its own ring of right quotients in the sense of Lambek* [4].

(2) $E(R_R)/R \hookrightarrow \prod E(R_R)$.

(3) *Each finitely-faithful, injective right R-module has the double centralizer property.*

PROOF. (1) implies (2). R is its own ring of right quotients in the sense of Lambek [4] if and only if $E(R_R)$ has the double centralizer property. By the theorem above, $E(R_R)$-domi. dim. $R_R \geqq 2$, or equivalently, $E(R_R)/R \hookrightarrow \prod E(R_R)$.

(2) implies (3). Let U_R be finitely-faithful and injective. Then both finitely-faithfulness and injectivity of U_R imply $R_R \subset E(R_R) \hookrightarrow \overset{n}{\oplus} U_R$. Hence

$$E(R_R)/R \hookrightarrow \prod E(R_R) \hookrightarrow \prod U_R\,.$$

Thus U-domi. dim. $R_R \geqq 2$ and hence U_R has the double centralizer property by Theorem 2.

(3) implies (1). This is clear since $E(R_R)$ is finitely-faithful and injective.

REMARK. It is straightforward to see that domi. dim. $R_R \geqq 2$ if and only if domi. dim. $R_R \geqq 1$, $E(R_R)/R \hookrightarrow \prod E(R_R)$. By the light of this fact, the above corollary is a generalization of Tachikawa [8, Theorem 1.4] which states that domi. dim. $R_R \geqq 2$ if and only if domi. dim. $R_R \geqq 1$ and R is its own right quotient ring in the sense of Lambek [4].

3. Rings of U-dominant dimension 1. In this section, let U_R be finitely-faithful and injective, Q the double centralizer of U_R, and Q' a ring between R and Q. Note that $R \neq Q$ if and only if U-domi. dim. $R_R = 1$ by Theorem 2.

THEOREM 3. *Let U_R, Q, and Q' be as above. Then* U-domi. dim. $Q'_{Q'} = 1$ *if and only if $Q' \neq Q$.*

PROOF. In view of Lemma 1, (3), the injectivity of U_R implies the injectivity of $U_{Q'}$. Hence $Q'_{Q'} \subset E(Q'_{Q'}) \subsetneq \overset{n}{\oplus} U_{Q'}$ making use of Lemma 1, (2). Thus U-domi. dim. $Q'_{Q'} \geqq 1$. Since $\mathrm{Hom}(U_{Q'}, U_{Q'}) = \mathrm{Hom}(U_R, U_R) = S$ by Lemma 1, (3), the double centralizer of $U_{Q'}$ is just the ring Q. Therefore U-domi. dim. $Q'_{Q'} = 1$ if and only if $Q' \neq Q$ by Theorem 2 ($U_{Q'}$ is finitely-faithful and injective!).

We close out this paper with an application of Theorem 3 to the theory of QF-3 rings.

COROLLARY. *Let R be a ring of dominant dimension* 1, *U_R a finitely-faithful, injective, torsionless module, and Q the double centralizer of U_R. Let Q', $R \subset Q' \subset Q$, be a ring such that $Q' \neq Q$. Then* domi. dim. $Q'_{Q'} = 1$.

PROOF. U-domi. dim. $Q'_{Q'} = 1$ by the above theorem. But $U_{Q'}$ is torsionless and $Q'_{Q'} \subsetneq \overset{n}{\oplus} U_{Q'}$ by Lemma 1. Therefore domi. dim. $Q'_{Q'} = 1$.

REMARK. It is the observation of Tachikawa that, if R is a ring of dominant dimension $\geqq 1$, Q the double centralizer of $E(R_R)$, and $R \subset Q' \subset Q$ is an intermediate ring, then domi. dim. $Q'_{Q'} \geqq 1$.

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