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HYPERSURFACES SATISFYING A CERTAIN CONDITION ON THE RICCI TENSOR

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1. **Introduction**. The Riemannian curvature tensor R of a locally symmetric Riemannian manifold (M,g) satisfies

$$(*) \qquad R(X, Y)\cdot R = 0 \quad \text{for any tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. A result of K. Nomizu [2] tells us that the converse is affirmative in the case where M is a certain hypersurface in a Euclidean space. That is:

Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x) \geqq 3$ at least at one point x. If M satisfies condition $(*)$, then it is of the form $M=S^k \times E^{m-k}$, where S^k is a hypersphere in a Euclidean subspace E^{k+1} of E^{m+1} and E^{m-k} is a Euclidean subspace orthogonal to E^{k+1}.

Let R_1 be the Ricci tensor of (M,g). Then condition $(*)$ implies in particular

$$(**) \qquad R(X, Y)\cdot R_1 = 0 \quad \text{for any tangent vectors } X \text{ and } Y.$$

First we have

THEOREM A. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in a Euclidean space E^{m+1} so that the type number $k(x) \geqq 3$ at least at one point x. If M satisfies condition $(**)$ and has the positive scalar curvature, then it is of the form $M = S^k \times E^{m-k}$.*

This theorem says that, under the circumstance, condition (**) implies that R_1 is parallel and, in fact, M is symmetric.

If M is compact, then $k(x) \geqq 3$ is replaced by $m \geqq 3$, and we have

THEOREM B. *Let M be an m-dimensional, connected and compact Riemannian manifold which is isometrically immersed in E^{m+1}, where $m \geqq 3$. If M satisfies condition* (**) *and has the positive scalar curvature, then it is a hypersphere.*

For the case where $k(x) = 2$ we have

THEOREM C. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in E^{m+1} so that the type number $k(x) = 2$ at least at one point x. If M satisfies condition* (**) *and the scalar curvature is a positive constant, then $M = S^2 \times E^{m-2}$.*

If the Ricci tensor R_1 is parallel, then M satisfies condition (**). Hence, we can show

THEOREM D. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in E^{m+1}. If the Ricci tensor is parallel and the scalar curvature is positive, then it is of the form $M = S^k \times E^{m-k}$.*

The condition on the type number $k(x)$ at a point x is replaced by the rank $r(x)$ of the Ricci tensor at the point, provided that $r(x)$ is greater than 1. Namely we have

COROLLARY. *Let M be an m-dimensional, connected and complete Riemannian manifold which is isometrically immersed in E^{m+1}. Assume that M satisfies condition* (**) *and the scalar curvature S is positive. And suppose one of the following conditions is satisfied:*

(i) *the Ricci tensor has the rank $r(x) \geqq 3$ at some point x,*

(ii) *the Ricci tensor has the rank $r(x) = 2$ at some point x and S is constant.*

Then M is of the form $S^k \times E^{m-k}$, $k = r(x)$.

Proofs are given by modifications of the arguments in [2] and by applying results of P. Hartman [1] and T. Y. Thomas [3].

2. Reduction of condition (**). Let M be a connected hypersurface in a Euclidean space E^{m+1} and let g be the induced metric on M. Let U be a neighborhood of a point x_0 of M on which we can choose a unit vector field ξ normal to M. For local vector fields X and Y on U tangent to M, we have the formulas of Gauss and Weingarten:

$$D_XY = \nabla_XY + h(X, Y)\xi, \tag{2.1}$$

$$D_X\xi = -AX, \tag{2.2}$$

where D_X and ∇_X denote covariant differentiations for the Euclidean connection of E^{m+1} and the Riemannian connection on M, respectively. h is the second fundamental form and A is a symmetric endomorphism satisfying $h(X, Y) = g(AX, Y)$. Then the equation of Gauss is

$$R(X, Y) = AX \wedge AY, \tag{2.3}$$

where, in general, $X \wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X - g(Z, X)Y$. The type number $k(x)$ at a point x is, by definition, the rank of A at x. For a point x of M, take an orthonormal basis $(e_1, \cdots, e_m)$ of the tangent space $T_x(M)$ such that $Ae_h = \lambda_h e_h$, $1 \leqq h \leqq m$. Then (2.3) is written as

$$R(e_i, e_j) = \lambda_i\lambda_j e_i \wedge e_j. \tag{2.4}$$

Now by condition (**) and

$$[R(e_i, e_j)\cdot R_1](e_k, e_h) = -R_1(R(e_i, e_j)e_k, e_h) - R_1(e_k, R(e_i, e_j)e_h),$$

we have

$$\lambda_i\lambda_j(\delta_{jk}R_{ih} - \delta_{ik}R_{jh} + \delta_{jh}R_{ik} - \delta_{ih}R_{jk}) = 0,$$

where R_{jk} are the components of R_1 with respect to the frame $\{e_h\}$. If we put $h = i \neq j = k$, then we get

$$\lambda_i\lambda_j(R_{ii} - R_{jj}) = 0. \tag{2.5}$$

Next, since $R(e_i, e_j)e_k = \lambda_i\lambda_j(\delta_{jk}e_i - \delta_{ik}e_j)$, R has the following components

$$R^h{}_{kij} = \lambda_i\lambda_j(\delta_{jk}\delta_i^h - \delta_{ik}\delta_j^h). \tag{2.6}$$

Contracting in h and i, we have

$$R_{jk}=\delta_{jk}\lambda_j(\Sigma_{i\neq j}\lambda_i). \tag{2.7}$$

Therefore R_1 is diagonal, and by (2.5), R_1 has at most two eigenvalues 0 and γ. If $\gamma\neq 0$, then the multiplicity of γ is equal to the type number $k(x)$. And the scalar curvature S of M is given by $S=k(x)\gamma$. We denote the mean curvature of M in E^{m+1} by $K=m^{-1}\Sigma\lambda_h$. Then, putting $j=k$ in (2.7), we see that λ_h is a solution of the equation

$$\lambda_h^2-mK\lambda_h+\gamma=0. \tag{2.8}$$

Consequently, we have a number $s\,(0\leqq s\leqq k(x))$ such that

$$\lambda_1=\lambda_2=\cdots=\lambda_s=\lambda,$$

$$\lambda_{s+1}=\cdots=\lambda_{k(x)}=mK-\lambda=\mu,$$

$$\lambda_{k(x)+1}=\cdots=\lambda_m=0,$$

by interchanging the order in $\{e_h\}$.

3. Proofs of theorems. Let $f:M\to E^{m+1}$ be an isometric immersion of an m-dimensional, connected and complete Riemannian manifold M with property (**). The scalar curvature S of M is assumed to be positive. Since the conclusion of our theorem is $M=S^k\times E^{m-k}$, in the proofs we can assume that M is oriented, and hence k is globally defined.

LEMMA 3.1. *The scalar curvature $S>0$ implies that the type number is a constant $k\geqq 2$.*

PROOF. By $S=k(x)\gamma$ at x, we have $\gamma>0$ on M. Since R_1 has at most two different eigenvalues 0 and γ, the inequality $\gamma>0$ on M tells us that the multiplicity of γ is constant on M. On the other hand, $\gamma\neq 0$ at x implies that $k(x)$ is the multiplicity of γ, and hence $k(x)=k$ a constant on M. Suppose that $k=1$. Then we have $mK=\lambda$ and $\gamma=0$ by (2.8). This is a contradiction, and we have $k\geqq 2$.

LEMMA 3.2. *Every sectional curvature is non-negative.*

PROOF. Let x be an arbitrary point of M and let $\{e_h\}$ be an orthonormal basis of $T_x(M)$ such that $Ae_i=\lambda e_i$ for $1\leqq i\leqq s$, $Ae_u=\mu e_u$ for $s+1\leqq u\leqq k$, and $Ae_t=0$ for $k+1\leqq t\leqq m$. Take an arbitrary 2-plane in $T_x(M)$. Then we have two vectors X and Y which span the 2-plane:

$$X = \Sigma_{i=1}^{s} a_i e_i + \Sigma_{u=s+1}^{k} b_u e_u + \Sigma_{t=k+1}^{m} c_t e_t,$$

$$Y = \Sigma_{j=1}^{s} a'_j e_j + \Sigma_{v=s+1}^{k} b'_v e_v + \Sigma_{t=k+1}^{m} c'_t e_t,$$

where we can assume that $a_i, b_u, c_t, a'_j, b'_v$, and c'_t are non-negative (by changing some $e_h \to -e_h$, if necessary). By (2.3) we have

$$R(X, Y) = \lambda^2 \Sigma_{i,j} a_i a'_j e_i \wedge e_j + \lambda\mu \Sigma_{i,v} a_i b'_v e_i \wedge e_v$$
$$+ \lambda\mu \Sigma_{u,j} b_u a'_j e_u \wedge e_j + \mu^2 \Sigma_{u,v} b_u b'_v e_u \wedge e_v.$$

After a simple calculation we have

$$- g(R(X, Y)X, Y) = [(\Sigma a_i^2)(\Sigma a_j'^2) - (\Sigma a_i a'_i)^2]\lambda^2$$
$$+ [(\Sigma a_i^2)(\Sigma b_v'^2) + (\Sigma b_u^2)(\Sigma a_j'^2) - 2(\Sigma a_i a'_i)(\Sigma b_u b'_u)]\lambda\mu$$
$$+ [(\Sigma b_u^2)(\Sigma b_v'^2) - (\Sigma b_u b'_u)^2]\mu^2.$$

The right hand side of the above equation is equal to

$$([(\Sigma a_i^2)(\Sigma a_j'^2)]^{1/2}\lambda + [(\Sigma b_u^2)(\Sigma b_v'^2)]^{1/2}\mu)^2$$
$$- ((\Sigma a_i a'_i)\lambda + (\Sigma b_u b'_u)\mu)^2$$
$$+ ((\Sigma a_i^2)(\Sigma b_v'^2) + (\Sigma b_u^2)(\Sigma a_j'^2)$$
$$- 2[(\Sigma a_i^2)(\Sigma a_j'^2)(\Sigma b_u^2)(\Sigma b_v'^2)]^{1/2})\lambda\mu.$$

Since $\gamma = \lambda\mu$ and $k\gamma = S$, we can assume that λ and μ are positive. Then, by well known inequalities we have $-g(R(X, Y)X, Y) \geqq 0$ and thereby every sectional curvature is non-negative.

LEMMA 3.3. (P. Hartman [1]) *Let M be an m-dimensional, connected and complete Riemannian manifold such that all 2-dimensional sections have non-negative curvatures. If $f: M \to E^{m+\delta}$, $\delta > 0$, is an isometric immersion such that the relative nullity function ν is a positive constant, then fM is ν-cylindrical.*

By Lemmas 3.1 and 3.2, we can apply Lemma 3.3 for $\delta = 1$, $\nu = m - k$ and we get the Riemannian product $M = M^k \times E^{m-k}$, where M^k is a k-dimensional, connected and complete Riemannian manifold and E^{m-k} is an $(m-k)$-dimensional Euclidean space. Furthermore the restriction f' of f to

fM^k is an isometric immersion of M^k into a $(k+1)$-dimensional Euclidean subspace E^{k+1} which is orthogonal to a Euclidean subspace E^{m-k} in E^{m+1}.

Let $\{e_h\}$ be an orthonormal basis at a point (x,y) of M, $x \in M^k$ and $y \in E^{m-k}$, such that the first $e_1, \cdots, e_k$ are tangent to M^k at x and $e_{k+1}, \cdots, e_m$ are tangent to E^{m-k} at y. Then the Ricci tensor R_1' of M^k and the Ricci tensor R_1 of M have the same value $R_1'(e_i, e_j) = R_1(e_i, e_j)$ for $1 \leqq i,\ j \leqq k$. On the other hand, we see that $R_1(e_i, e_j) = \gamma g(e_i, e_j)$ for $1 \leqq i,\ j \leqq k$. Hence, M^k is an Einstein space. By a theorem of (E. Cartan and) T. Y. Thomas [3], fM^k is a hypersphere in E^{k+1}. This completes the proofs of Theorems A and C.

If M is compact, then the type number $k(x)$ at some point x is equal to m (cf. [2], p.57). So we see that $S = m\gamma$, namely, M is an Einstein space. Hence, fM is a hypersphere and we have Theorem B.

If the Ricci tensor is parallel, then the scalar curvature is constant. Therefore to prove Theorem D it suffices to notice the Ricci identity

$$(\nabla\nabla R_1)(Z, W;\ X, Y) - (\nabla\nabla R_1)(Z, W;\ Y, X) = (R(X, Y)\cdot R_1)(Z, W). \tag{3.1}$$

4. Remarks.

REMARK 1. *If* $\dim M = 2$, *then condition* (**) *is trivial.*

In fact, we have $R_1 = ag$ for some differentiable function a on M. Then the Ricci identity (3.1) shows that condition (**) is satisfied always.

REMARK 2. *If* $\dim M = 3$, *then condition* (**) *is equivalent to condition* (*).

In fact, if $\dim M = 3$, then we have

$$R(X, Y) = R^1X \wedge Y + X \wedge R^1Y - (1/2)SX \wedge Y, \tag{4.1}$$

where S is the scalar curvature and R^1 is defined by $R_1(X, Y) = g(R^1X, Y)$. If we take an orthonormal basis $\{e_h\}$ such that $R^1e_h = \gamma_h e_h$, $1 \leqq h \leqq 2$, then condition (*) is equivalent to

$$(\gamma_i - \gamma_j)(2(\gamma_i + \gamma_j) - S) = 0, \tag{4.2}$$

which is also equivalent to condition (**).

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