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COMPARISON BETWEEN $T(r, f)$ AND $\log M(r, f)$*)

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1. Introduction. Let $f(z)$ be a transcendental entire function and let

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|$$

be the maximum modulus of $f(z)$ on $|z|=r$ and

$$T(r) = T(r, f) = (1/2\pi) \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

the characteristic function of $f(z)$, where $\log^+|x| = \max(\log|x|, 0)$.

We define the order ρ and lower order λ of $f(z)$ as follows;

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}, \quad \lambda = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Paley [6] proved that for each $\rho \, (0 \leqq \rho \leqq \infty)$, there is an entire function of order ρ for which

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{T(r, f)} = \infty.$$

On the other hand, it is conjectured that

$$C_f = \liminf_{r\to\infty} \frac{\log M(r, f)}{T(r, f)} \leqq \pi\rho$$

for $1/2 < \rho < \infty$ (see [4, 6]), and it is known that

$$C_f \leqq \pi\rho/\sin \pi\rho$$

for $0 \leqq \rho \leqq 1/2$, and this is the best possible estimate (see [9, 11]).

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Further, we know the following results.

[I] For $0 \leqq \rho < 1$, $C_f \leqq \pi\rho/\sin \pi\rho$. ([9, 11])

[II] For $0 \leqq \rho < \infty$, $C_f \leqq C(\rho)$ ([4, 6]) and $C(\rho) \sim 2e\rho$ (see [6]),

where $C(\rho)$ is a constant depending only on ρ.

[III] For $1/2 \leqq \rho < \infty$, if there exists a θ such that

$$\log |f(re^{i\theta})| \sim \log M(r, f),$$

then

$$C_f \leqq \pi\rho. \qquad ([2])$$

[IV] For $0 \leqq \lambda < 1$, $C_f \leqq \pi\lambda/\sin \pi\lambda$. (See [1, 5].)

[V] For $1/2 \leqq \lambda < \infty$, if there is a θ such that

$$\log |f(re^{i\theta})| \sim \log M(r, f),$$

then

$$C_f \leqq \pi\lambda. \qquad ([7])$$

In this note, we prove that for $0 \leqq \lambda < \infty$ there is a constant $C(\lambda)$ depending only on λ such that $C_f \leqq C(\lambda)$ and $C(\lambda) \sim 2e\lambda$.

2. Lemmas.

We give here some lemmas which we use in the next section.

LEMMA 1. *For any positive r and R such that $r<R<\infty$, it holds that*

$$T(r, f) \leqq \log M(r, f) \leqq \frac{R+r}{R-r} T(R, f).$$

From these inequalities, we obtain

$$\rho = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \lambda = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} \qquad \text{(see [3])}.$$

LEMMA 2. *Let $f(z)$ be an entire function of lower order λ $(0 \leqq \lambda < \infty)$. Then there exists a function $\lambda(r)$ having the following properties:*

(1) *$\lambda(r)$ is a non-negative continuous function of r for $r \geqq r_0 > 0$,*

(2) *$\lambda(r)$ is differentiable for $r > r_0$ except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist,*

(3) $\lim_{r\to\infty} r\lambda'(r)\log r = 0$,

(4) $\lim_{r\to\infty} \lambda(r) = \lambda$,

(5) $r^{\lambda(r)} \leqq \log M(r,f)$ *and* $\liminf_{r\to\infty} \dfrac{\log M(r,f)}{r^{\lambda(r)}} = 1.$ (See [8].)

We call this function $\lambda(r)$ *a lower proximate order for* $f(z)$.

LEMMA 3. *Let* $U(r)=r^{\lambda(r)}$ $(r \geqq r_0)$. *Then for* $k>1$

$$\lim_{r\to\infty} \frac{U(kr)}{U(r)} = k^{\lambda}. \qquad (\textit{See } [10].)$$

PROOF. By a simple calculation, we have

$$\frac{rU'(r)}{U(r)} = r\lambda'(r)\log r+\lambda(r).$$

Therefore, using the properties (3) and (4) of Lemma 2, we see that for any $\varepsilon > 0$, there is an r_1 such that for every $r \geqq r_1$,

$$\frac{\lambda-\varepsilon}{r} < \frac{U'(r)}{U(r)} < \frac{\lambda+\varepsilon}{r}.$$

Integrating the above inequalities from r to kr, we have

$$(\lambda-\varepsilon)\log k < \log \frac{U(kr)}{U(r)} < (\lambda+\varepsilon)\log k,$$

so that

$$\lim_{r\to\infty} \frac{U(kr)}{U(r)} = k^{\lambda}.$$

3. Theorem. Now, we can prove the following theorem.

THEOREM. *Let* $f(z)$ *be an entire function of lower order* λ $(0 \leqq \lambda < \infty)$. *Then*

$$C_f \begin{cases} \leqq (\lambda+\sqrt{\lambda^2+1})\left(\dfrac{1+\sqrt{\lambda^2+1}}{\lambda}\right)^{\lambda} & (\lambda > 0), \\ \leqq 1 & (\lambda = 0). \end{cases}$$

PROOF. Let $R = r(1+x)$, $x > 0$. Then from Lemma 1,

$$\log M(r) \leqq \frac{x+2}{x} T((1+x)r).$$

Dividing each side by $U(r)$ of Lemma 3 and taking the inferior limit, we have

$$1 = \liminf_{r\to\infty} \frac{\log M(r)}{U(r)} \leqq \frac{x+2}{x} \liminf_{r\to\infty} \frac{T((1+x)r)}{U(r)}.$$

Consequently

$$\frac{x}{x+2} \leqq \liminf_{r\to\infty} \frac{T((1+x)r)}{U(r)}.$$

Here

$$\frac{T((1+x)r)}{U(r)} = \frac{T((1+x)r)}{U((1+x)r)} \cdot \frac{U((1+x)r)}{U(r)}$$

so that

$$\liminf_{r\to\infty} \frac{T((1+x)r)}{U(r)} \leqq \liminf_{r\to\infty} \frac{T((1+x)r)}{U((1+x)r)} \cdot \limsup_{r\to\infty} \frac{U((1+x)r)}{U(r)}$$
$$= \liminf_{r\to\infty} \frac{T(r)}{U(r)} \cdot (1+x)^{\lambda}$$

by Lemma 3. Using this inequality and from the equality

$$\frac{\log M(r)}{T(r)} = \frac{\log M(r)}{U(r)} \cdot \frac{U(r)}{T(r)},$$

we get

$$C_f = \liminf_{r\to\infty} \frac{\log M(r)}{T(r)} \leqq \liminf_{r\to\infty} \frac{\log M(r)}{U(r)} \cdot \limsup_{r\to\infty} \frac{U(r)}{T(r)}$$
$$= 1 \cdot \frac{1}{\liminf_{r\to\infty} \frac{T(r)}{U(r)}} \begin{cases} \leqq \frac{x+2}{x}(1+x)^{\lambda}, & \lambda > 0, \\ \leqq \frac{x+2}{x}, & \lambda = 0. \end{cases}$$

Put

$$K(x) = \begin{cases} \frac{x+2}{x}(1+x)^{\lambda}, & \lambda > 0, \\ \frac{x+2}{x}, & \lambda = 0. \end{cases}$$

Then $K(x)$ takes the minimum value

$$C(\lambda) = (\lambda+\sqrt{\lambda^2+1})\left(\frac{1+\sqrt{\lambda^2+1}}{\lambda}\right)^{\lambda},$$

being $\sim 2e\lambda\ (\lambda \to \infty)$, for $x = \dfrac{1-\lambda+\sqrt{\lambda^2+1}}{\lambda}$ if $\lambda>0$, and $K(x)$ decreases monotonously to 1 as $x \to \infty$ if $\lambda=0$. From this fact, we have

$$C_f \leqq C(\lambda), \quad (\lambda \geqq 0),$$

where

$$C(\lambda) = \begin{cases} (\lambda+\sqrt{\lambda^2+1})\left(\dfrac{1+\sqrt{\lambda^2+1}}{\lambda}\right)^{\lambda}, & \lambda > 0, \\ 1, \quad \lambda = 0. \end{cases}$$

Clearly $C(\lambda) \sim 2e\lambda$ as λ tends to infinity and $C(\lambda) \leqq (2\lambda+1)e$ for any λ $(0 \leqq \lambda < \infty)$.

REMARK. Thus the best estimate of C_f which we have known is as follows.

Let $0 < \xi < 1$ be the root of the equation

$$\frac{\pi x}{\sin \pi x} = C(x).$$

Then

$$C_f \leqq \pi\lambda/\sin \pi\lambda \quad \text{in } 0 \leqq \lambda \leqq \xi$$

and

$$C_f \leqq C(\lambda) \quad \text{in } \xi < \lambda < \infty.$$

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Added in proof : Recently Petrenko has stated a positive answer for Paley's conjecture without proof in Dokl. Akad. Nauk SSSR 184-5(1969).