Tôhoku Math. Journ.
21(1969), 249-270.

ON THE ALGEBRA OF MEASURABLE OPERATORS FOR A GENERAL AW^*-ALGEBRA

KAZUYUKI SAITÔ

(Received November 19, 1968)

1. Introduction. It is an interesting problem in the non-commutative integration theory to construct a "measurable operator" without using unbounded linear operators. From this point of view, we shall extend Berberian's result on "The regular ring of a finite AW^*-algebra" to general AW^*-algebras. S. K. Berberian defined a "closed operator" for a finite AW^*-algebra in algebraic fashion and studied the structure of the "closed operators" [1].

The plan of this paper is as follows. Section 3 is devoted to formulate the notions of "strongly dense domains" and "measurable operators" with respect to a given AW^*-algebra M. Our definitions are closely related to that of [1]. Along the same lines with [1], we shall construct the algebra $\mathcal{C}$ of "measurable operators" for the general AW^*-algebras and study some preliminary algebraic properties of $\mathcal{C}$. Section 5 deals with the spectral theorem for "self-adjoint measurable operators" using the Cayley transform. Theorem 5.1 gives the necessary and sufficient condition for a unitary element in M to be the Cayley transform of some "self-adjoint element" of $\mathcal{C}$. In particular, Lemma 4.1 and Theorem 5.1 play essential rôles in our discussions. In section 6, Theorem 6.2 gives an alternative proof of ([5] Theorem): If $\mathcal{C}$ is regular ([10], Definition 2.2), then M is finite. Theorem 6.3 concerns with the polar decomposition of a "measurable operator" which is one of the main theorems in this paper. Moreover, we shall show that $\mathcal{C}$ is a Baer*-ring in the sense of [6].

Before going into discussions, the author wishes to express his gratitude to Prof. M. Takesaki for calling his attention to the reference [1], and he is also grateful to Prof. J. Tomiyama for useful conversations with him.

2. Notations and Definitions. An AW^*-algebra M is a C^*-algebra satisfying the following two conditions:

(a) In the set of projections any collection of orthogonal projections has a least upper bound.

(b) Any maximal commutative self-adjoint subalgebra is generated by its projections.

Denote the set of all self-adjoint elements, projections, partial isometries and unitary elements in M by M_{sa}, M_p, M_{pi} and M_u, respectively.

Let $\mathfrak{M}$ be the two sided ideal generated by all finite projections in M, then $\mathfrak{M}_p$ contains only finite projections.

If $\{e_n\}$ is a sequence in M_p, $e_n \uparrow$ means $e_n \leqq e_{n+1}$; if moreover sup $\{e_n, n \geqq 1\} = e$, we write $e_n \uparrow e$. The notations $e_n \downarrow$ and $e_n \downarrow e$ have the dual meanings.

The right projection of an element $x \in M$ is $RP(x)$, $LP(x)$ is the left projection; the relation $RP(x) \sim RP(x)$ will be needed. For a subset $S \subset M$, S' is the set of all elements of M which commute with each element of S. If S is a self-adjoint subset, then S' is an AW^*-subalgebra of M(that is, S' is itself an AW^*-algebra and the least upper bound of orthogonal projections computed in S' is the same as computed in M). If S consists of a single unitary element u, S' is an AW^*-subalgebra of M and S'' is a commutative AW^*-subalgebra of M.

3. Strongly dense domains and Measurable operators.

DEFINITION 3.1. ([1], p.228). A sequence $\{e_n\}$ in M_p is a strongly dense domain (SDD), in case $e_n \uparrow 1$ and $1-e_n \in \mathfrak{M}$.

An essentially measurable operator (EMO) is a pair of sequences $\{x_n, e_n\}$ with $x_n \in M$, $\{e_n\}$ an SDD, and such that $m<n$ implies $x_n e_m = x_m e_m$ and $(x_n)^* e_m = (x_m)^* e_m$.

For example if $x \in M$, we can take $x_n = x$ and $e_n = 1$ for all n; $\{x_n, e_n\}$ is an EMO, written briefly $\{x, 1\}$.

To introduce the algebraic operations in EMO, we need the following definition and lemma.

DEFINITION 3.2. If $x \in M$, and $e \in M_p$, we denote the largest projection right-annihilating $(1-e)x$ by $x^{-1}[e]$; that is, $1-x^{-1}[e]$ is the right projection of $(1-e)x$.

LEMMA 3.1. *Let* $\{e_n\}$, $\{f_n\}, \cdots \{g_n\}$ *be* SDD, *and* x *be any element of* M, *then* $\{e_n \wedge f_n \wedge \cdots \wedge g_n\}$ *and* $\{x^{-1}[e_n]\}$ *are* SDD.

PROOF. It is sufficient to consider the case of two SDD $\{e_n\}$ and $\{f_n\}$. Putting $g_n = e_n \wedge f_n$, $g = \sup\{g_n, n \geqq 1\}$, $h_n = x^{-1}[e_n]$ and $h = \sup\{h_n, n \geqq 1\}$; evidently $g_n \uparrow g$. Since $1-g \leqq 1-g_n = (1-e_n) \vee (1-f_n)$, and $1-e_n$, $1-f_n \in \mathfrak{M}$, by ([3], Theorem 6.2), we have $(1-e_n) \vee (1-f_n) \in \mathfrak{M}$, $1-g_n$ and $1-g \in \mathfrak{M}$. By Definition 3.2, $(1-e_k)h_k = 0$ and h_k is the largest such projection. If $m<n$, then $(1-e_n)xh_m = (1-e_n)(1-e_m)xh_m = 0$, hence $h_m \leqq h_n$. Since $1-h_n = 1-x^{-1}[e_n]$

$=RP((1-e_n)x) \sim LP((1-e_n)x) \leqq 1-e_n$, $1-x^{-1}[e_n] \in \mathfrak{M}$ for all n. Noting that $\{1-e_n,\ 1-f_n,\ 1-g_n,\ 1-h_n,\ 1-g,\ 1-h;\ n=1,2,\cdots\} \subset ((1-e_1)\vee(1-f_1)\vee(1-h_1))M((1-e_1)\vee(1-f_1)\vee(1-h_1))$ (Note that this is a finite AW^*-algebra), by ([3], p.248), for the unique normalized center-valued dimension function $D(\cdot)$ of $((1-e_1)\vee(1-f_1)\vee(1-h_1))M((1-e_1)\vee(1-f_1)\vee(1-h_1))$, we have

$$D(1-h_n) \leqq D(1-e_n),$$

and

$$D(1-g) \leqq D(1-g_n) \leqq D(1-e_n) + D(1-f_n);$$

$D(1-h) = D(1-g) = 0$ result from $D(1-e_n) \downarrow 0$ and $D(1-f_n) \downarrow 0$. This completes the proof of Lemma 3.1.

Suggested by ([9], Corollary 5.1), we introduce an equivalence relation in the set of all EMO:

DEFINITION 3.3. ([1], Definition 2.2) Two EMO $\{x_n, e_n\}$ and $\{y_n, f_n\}$ are equivalent, denoted *by* $\{x_n, e_n\} \equiv \{y_n, f_n\}$, if there exists an SDD $\{g_n\}$ such that $x_n g_n = y_n g_n$, $(x_n)^* g_n = (y_n)^* g_n$ for all n. The SDD $\{g_n\}$ implements the equivalence.

It is immediate that the relation just defined is indeed an equivalence relation. The next remarks, which are easy to verify, will be used frequently.

REMARK. If $\{x_n, e_n\}$ is an EMO and $\{f_n\}$ is any SDD, then $\{x_n, e_n \wedge f_n\}$ is an EMO, and $\{x_n, e_n\} \equiv \{x_n, e_n \wedge f_n\}$. If an SDD $\{g_n\}$ implements $\{x_n, e_n\} \equiv \{y_n, f_n\}$, and $h_n = e_n \wedge f_n \wedge g_n$, then $\{x_n, h_n\}$ and $\{y_n, h_n\}$ are EMO, and SDD $\{h_n\}$ implements $\{x_n, h_n\} \equiv \{y_n, h_n\}$.

DEFINITION 3.4. ([1], Definition 2.3) Let $\{x_n, e_n\}$ be an EMO and $[x_n, e_n]$ be its equivalence class. $[x_n, e_n]$ is said a "measurable operator" (MO). Denote the set of all MO by C and we use letters $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \cdots$ for the elements of C.

After suitable operations are defined, C is the Baer*-ring promised in the introduction, and $x \to [x, 1]$ is the imbedding of M in C.

Now we are in the position to define the operations in C. If $\{x_n, e_n\}$ and $\{y_n, f_n\}$ are EMO, and λ is a complex number, we define $\lambda\{x_n, e_n\} = \{\lambda x_n, e_n\}$, $\{x_n, e_n\} + \{y_n, f_n\} = \{x_n + y_n, e_n \wedge f_n\}$ and $\{x_n, e_n\}^* = \{(x_n)^*, e_n\}$; the right-hand members of these definitions are easily seen to be EMO. Set $g_n = e_n \wedge f_n \wedge ((y_n)^{-1}[e_n]) \wedge (((x_n)^*)^{-1}[f_n])$; it is straightforward to verify that $\{g_n\}$ is an

SDD, and that if $m < n$, then $(x_n y_n) g_m = (x_m y_m) g_m$ and $((y_n)^*(x_n)^*) g_m = ((y_m)^*(x_m)^*) g_m$, that is, $(x_n y_n)^* g_m = (x_m y_m)^* g_m$. This implies that $\{x_n y_n, g_n\}$ is an EMO, and this is our definition for $\{x_n, e_n\}\{y_n, f_n\}$. Mereover, if $\{x_n, e_n\} \equiv \{x'_n, e'_n\}$ and $\{y_n, f_n\} \equiv \{y'_n, f'_n\}$, then $\lambda\{x_n, e_n\} \equiv \lambda\{x'_n, e'_n\}$, $\{x_n, e_n\} + \{y_n, f_n\} \equiv \{x'_n, e'_n\} + \{y'_n, f'_n\}$, $\{x_n, e_n\}^* \equiv \{x'_n, e'_n\}^*$, and $\{x_n, e_n\}\{y_n, f_n\} \equiv \{x'_n, e'_n\}\{y'_n, f'_n\}$. Thus if $\boldsymbol{x} = [x_n, e_n]$ and $\boldsymbol{y} = [y_n, f_n]$, the definitions $\lambda\boldsymbol{x} = [\lambda x_n, e_n]$, $\boldsymbol{x} + \boldsymbol{y} = [x_n + y_n, e_n \wedge f_n]$, $\boldsymbol{x}^* = [(x_n)^*, e_n]$, and $\boldsymbol{xy} = [x_n y_n, g_n]$, are unambiguous. With these definitions, C becomes an associative algebra over the complex numbers, with involution $*$: $\boldsymbol{x}^{**} = \boldsymbol{x}$, $(\boldsymbol{x} + \boldsymbol{y})^* = \boldsymbol{x}^* + \boldsymbol{y}^*$, $(\lambda\boldsymbol{x})^* = \bar{\lambda}\boldsymbol{x}^*$ and $(\boldsymbol{xy})^* = \boldsymbol{y}^*\boldsymbol{x}^*$. If $x, y \in M$, and λ is a complex number, clearly $\{x, 1\} + \{y, 1\} \equiv \{x + y, 1\}$, $\lambda\{x, 1\} \equiv \{\lambda x, 1\}$, $\{x, 1\}^* \equiv \{x^*, 1\}$, and $\{x, 1\}\{y, 1\} \equiv \{xy, 1\}$; passing from $\{\cdot, \cdot\}$ to $[\cdot, \cdot]$, $[x, 1] + [y, 1] = [x + y, 1]$, $\lambda[x, 1] = [\lambda x, 1]$, $[x, 1]^* = [x^*, 1]$, and $[x, 1][y, 1] = [xy, 1]$, thus the mapping $x \to [x, 1]\,(x \in M)$ is a $*$-isomorphism of M into C; for if $[x, 1] = [y, 1]$, then $\{x, 1\} \equiv \{y, 1\}$, so there exists an SDD $\{e_n\}$ such that $(x - y)e_n = 0$ for all n. The result follows from ([3], Lemma 2.2).

Summarizing the above results, we have

THEOREM 3.1. *The set C of all* MO *is an associative algebra over the complex numbers, with involution $*$, with respect to the operations*

$$[x_n, e_n] + [y_n, f_n] = [x_n + y_n, e_n \wedge f_n],$$

$$\lambda[x_n, e_n] = [\lambda x_n, e_n],$$

$$[x_n, e_n]^* = [(x_n)^*, e_n]$$

and

$$[x_n, e_n]\,[y_n, f_n] = [x_n y_n, g_n],$$

where $\{g_n\}$ is the SDD *such that* $g_n = e_n \wedge f_n \wedge ((y_n)^{-1}[e_n]) \wedge (((x_n)^*)^{-1}[f_n])$. *The mapping $x\ (x \in M) \to [x, 1]$ is a $*$-isomorphism of M into C, and* $[1, 1]$ *is a unit element for C.*

To simplify the notations, we shall denote $[x, 1]$ by $\bar{x}$; then $\bar{1}$ is the unit element of C, which we condense further to 1. $\bar{M}$ is the image of M in C.

REMARK. Let $\boldsymbol{x} = [x_n, e_n]$ be in C: for any fixed index m, $[x_n, e_n]\overline{e_m} = \overline{x_m e_m}$. For $(e_m)^{-1}[e_n]$ is the largest projection right-annihilating $(1 - e_n)e_m$, noting that $(1 - e_n)e_m e_n = (1 - e_n)e_n e_m = 0$, we have $(e_m)^{-1}[e_n] \geqq e_n$, $\{x_n, e_n\}\{e_m, 1\} \equiv \{x_n e_m, e_n\}$. On the other hand, by Definition 3.1, we have for $n > m$,

$$x_n e_m e_n = x_m e_m e_n,$$

$$e_m(x_n)^*e_n = (x_ne_m)^*e_n = (x_me_m)^*e_n = e_m(x_m)^*e_n,$$

and for $n \leqq m$,

$$x_ne_me_n = x_ne_n = x_me_n = x_me_me_n,$$

$$e_m(x_n)^*e_n = e_m(x_m)^*e_n.$$

This implies that the SDD $\{e_n\}$ implements the equivalence $\{x_n, e_n\}\{e_m, 1\} \equiv \{x_me_m, 1\}$. It follows that if $[x_n, e_n]=[y_n, f_n]$, then $x_m(e_m \wedge f_m)=y_m(e_m \wedge f_m)$ for all m, thus the equivalent "linear operators" $\{x_n, e_n\}$ and $\{y_n, f_n\}$ agree, so to speak, on their largest possible common domain.

If M is a W^*-algebra ([8]), it is easy to see from ([9], Corollaries 5.1 and 5.3) that the $*$-algebra C just constructed is $*$-isomorphic with the $*$-algebra of measurable operators in the sense of [9], in such a way as to preserve the elements of M. Because of the inherent nature of the above construction, we have as an immediate corollary a theorem of Ogasawara and Yoshinaga:

THEOREM 3.2 ([1], [7]). *Let M and N be AW^*algebras, C_M, C_N, their $*$-algebras of measurable operators. There exists a one to one correspondence between the $*$-isomorphisms Φ: $C_M \to C_N$ and the $*$-isomorphisms ϕ: $M \to N$ and the correspondence $\Phi \to \phi$ is obtained by restricting Φ to M.*

PROOF. We may suppose M (resp. N) to be a self-adjoint subalgebra of C_M (resp. C_N). By Lemma 5.3, any $*$-homomorphism Φ: $C_M \to C_N$ necessarily maps M into N. On the other hand for ϕ preserves the finiteness of projections, any $*$-isomorphism ϕ: $M \to N$ can be lifted to a $*$-isomorphism Φ: $C_M \to C_N$; Φ is the mapping $[x_n, e_n] \to [\phi(x_n), \phi(e_n)]$. This induced Φ is unique. For, given any $\boldsymbol{x} \in C_M$, we can find an SDD $\{e_n\}$ in M such that $\boldsymbol{x}e_n \in M$ for all n; then $\Phi(\boldsymbol{x}e_n) = \Phi(\boldsymbol{x}) \cdot \Phi(e_n)$, $\phi(\boldsymbol{x}e_n) = \Phi(\boldsymbol{x})\phi(e_n)$, and by Lemma 4.5, we see that Φ is determined by its values on M. This completes the proof of Theorem 3.2.

Next we investigate the connection between subalgebra eMe ($e \in M_p$) of M and subalgebras of C. Noting that for any $e \in M_p$, eMe is also an AW^*-algebra ([3]), we have

THEOREM 3.3. *For any projection e in M, the algebra of all measurable operators for eMe is $*$-isomorphic to $\bar{e}C\bar{e}$.*

PROOF. We write $\{x_n,e_n\}_e$ to indicate an EMO with respect to eMe; in particular $x_n \in eMe$, $e_n \uparrow e$ and $e-e_n \in \mathfrak{M}$. Setting $e_n' = e_n+1-e$, we have $e_n' \uparrow 1$ and $1-e_n'=e-e_n \in \mathfrak{M}$, and it is easy to verify that the mappig $[x_n, e_n]_e$

$\to [x_n, e_n']$ is a $*$-isomorphism of the algebra of measurable operators for eMe into $\bar{e}C\bar{e}$. It is sufficient to show that this mapping is onto. Suppose $\boldsymbol{y}$ is a self-adjoint element of $\bar{e}C\bar{e}$, $\bar{u}$ its Cayley transform (Section 5, Lemma 5.1) and $\boldsymbol{y}=[y_n, f_n]$ with $y_n, f_n \in \{u\}''$ (Theorem. 5.2). Since $\bar{e}$ commutes with $\boldsymbol{y}$, e commutes with u (Remark following Lemma 5.1), hence e, y_n, f_n mutually commute. If we set $x_n = y_n e$, $e_n = f_n e$, then $e_n \uparrow e$ and $e - e_n \leqq 1 - f_n \in \mathfrak{M}$, so $\{e_n\}$ is an SDD in eMe. Moreover, an easy calculation shows that $\{x_n, e_n\}_e$ is an EMO in eMe and $[x_n, e_n'] = \boldsymbol{y}$. This completes proof of the theorem.

4. Preliminary algebraic properties of C.

LEMMA 4.1. *If* $\boldsymbol{x} = [x_n, e_n]$ $(\boldsymbol{x} \in C)$ *and all the* x_n *are invertible, then* $\boldsymbol{x}$ *is invertible, and* $\boldsymbol{x}^{-1}=[(x_n)^{-1}, h_n]$ *for a suitable* SDD $\{h_n\}$.

To prove this, we need the following lemma:

LEMMA. *For any* e *in* M_p *and any invertible element* s *in* M,

$$((s^*)^{-1})^{-1}[1 - e] = 1 - s^{-1}[e],$$

and if $1 - e \in \mathfrak{M}$, *then* $s^{-1}[1 - e]$ *is also in* $\mathfrak{M}$.

PROOF. By Definition 3.2, the right annihilator of $e(s^*)^{-1}$ $(RA(e(s^*)^{-1}))$ $=(((s^*)^{-1})^{-1}[1-e])M$, and the right annihilator of $(1-e)s$ $(RA(1-e)s))=$ $(s^{-1}[e])M$. Since $(1 - e)ss^{-1} e = 0$, we have

$$s^{-1} e \in RA((1 - e)s),$$

and

$$(e(s^*)^{-1})(1 - (s^{-1}[e])) = 0,$$

thus we have

$$1 - s^{-1}[e] \leqq ((s^*)^{-1})^{-1}[1 - e].$$

On the other hand,

$$\begin{aligned}(1 - e)s(s^{-1}[e]) &= 0,\\ s(s^{-1}[e]) &= es(s^{-1}[e]),\\ s^{-1}[e] &= (s^{-1})es(s^{-1}[e]),\\ s^{-1}[e] &= (s^{-1}[e])s^*e(s^*)^{-1}.\end{aligned}$$

Hence we have

$$s^{-1}[e]((s^*)^{-1})^{-1}[1 - e]) = (s^{-1}[e])s^*e(s^*)^{-1})(((s^*)^{-1})^{-1}[1 - e]) = 0$$

$$((s^*)^{-1})^{-1}[1 - e] \leqq 1 - s^{-1}[e].$$

The lemma follows.

PROOF OF LEMMA 4.1. Let f_n be the left projection of x_ne_n; we show that $\{f_n\}$ is an SDD. If $m<n$, then $f_n(x_me_m)=f_nx_ne_m=f_nx_ne_ne_m=x_ne_ne_m=x_me_m$ shows that $1-f_n\leqq 1-f_m$, that is, $f_m\leqq f_n$. Since the invertibility of x_n implies that by the above lemma, $1-f_n=1-RP(e_n(x_n)^*)=((x_n)^*)^{-1}[1-e_n]=1-((x_n)^{-1})^{-1}[e_n]\precsim 1-e_n$, by the same way as that used in the proof of Lemma 3.1, we have $1-f_n\in\mathfrak{M}$ and $f_n\uparrow 1$. Putting $y_n=(x_n)^{-1}$, *if* $m<n$, then $f_ny_m=y_mf_m$; for

$$x_me_m=x_ne_m,$$

$$y_nx_me_m=y_nx_ne_m=e_ne_m=y_mx_me_m,$$

$$(y_n-y_m)x_me_m=0,$$

$$(y_n-y_m)f_m=0.$$

Similarly on putting $g_n=LP((x_n)^*e_n)$, we have that $\{g_n\}$ is an SDD and $(y_n)^*g_m=(y_m)^*g_m$ when $m<n$; hence if $h_n=f_n\wedge g_n$, then $\{y_n, h_n\}$ is an EMO, and it is evident that $\boldsymbol{y}=[y_n, h_n]$ satisfies $\boldsymbol{xy}=\boldsymbol{yx}=1$. This completes the proof.

LEMMA 4.2. *If* $\boldsymbol{x}^*=\boldsymbol{x}$, *then we may write* $\boldsymbol{x}=[x_n, e_n]$ *with* $(x_n)^*=x_n$.

PROOF. If $\boldsymbol{x}=[y_n, f_n]$, then $\boldsymbol{x}=(1/2)(\boldsymbol{x}+\boldsymbol{x}^*)=[((x_n)^*+x_n)/2, f_n]$.

COROLLARY 4.1. *If* $\boldsymbol{x}^*=\boldsymbol{x}$, *then* $\boldsymbol{x}+i1$ *is invertible.*

PROOF. Let $\boldsymbol{x}=[x_n, e_n]$ with $(x_n)^*=x_n$; then $\boldsymbol{x}+i1=[x_n+i1, e_n]$ and each x_n+i1 is invertible. The assertion is clear from Lemma 4.1.

LEMMA 4.3. *Let* $\boldsymbol{u}=[u_n, e_n]$, *with* $u_n\in M_u$ *for all* n; *then there is a unique unitary element* $u\in M$ *such that* $\boldsymbol{u}=\bar{u}$.

PROOF. The proof is the same as that of ([1], Lemma 3.3). But for the sake of completeness, we sketch it. Put $w_n=u_ne_n$: since $(w_n)^*w_n=e_n$, w_n is a partial isometry, so $f_n=w_n(w_n)^*=u_ne_n(u_n)^*$ is the left projection of w_n. As shown in the proof of Lemma 4.1, $\{f_n\}$ is an SDD. Set $v_n=w_n-w_{n-1}=u_ne_n-u_{n-1}e_{n-1}=u_ne_n-u_ne_{n-1}=u_n(e_n-e_{n-1})$, where $u_0=w_0=e_0=0$; v_n is a partial isometry with initial projection e_n-e_{n-1}, and the final projection is $u_n(e_n-e_{n-1})(u_n)^*=u_ne_n(u_n)^*-u_{n-1}e_{n-1}(u_{n-1})^*=f_n-f_{n-1}$, where $f_0=0$. Since the v_n have orthogonal initial projections and orthogonal final projections, by ([4], Lemma 20) there is an element $u\in M_{pi}$ such that

$u^*u = \sup\left\{\sum_{i=1}^{n}(e_i - e_{i-1}),\ n \geqq 1\right\} = 1$ $uu^* = 1$, and $u(e_n - e_{n-1}) = v_n = u_n(e_n - e_{n-1})$. By mathematical induction, $ue_n = u_n e_n$ for all n. Then $e_n u e_n = e_n u_n e_n$, for fixed m, $n > m$ implies $e_m(e_n u e_n) = e_m(e_n u_n e_n)$, $e_m u e_n = e_m u_m e_n$, $(e_m u - e_m u_m)e_n = 0$, hence $u^* e_m = (u_m)^* e_m$, that is, $\{u, 1\} \equiv \{u_n, e_n\}$. The Lemma follows.

LEMMA 4.4. *If* $\boldsymbol{x}, \boldsymbol{y}, \cdots, \boldsymbol{z} \in C$ *and* $\boldsymbol{x}^*\boldsymbol{x} + \boldsymbol{y}^*\boldsymbol{y} + \cdots + \boldsymbol{z}^*\boldsymbol{z} = 0$, *then* $\boldsymbol{x} = \boldsymbol{y} = \cdots = \boldsymbol{z} = 0$.

PROOF. If $\boldsymbol{x} = [x_n, e_n]$, $\boldsymbol{y} = [y_n, f_n], \cdots$ and $\boldsymbol{z} = [z_n, g_n]$, then there is an SDD $\{h_n\}$ such that $e_n \wedge f_n \cdots \wedge g_n \geqq h_n$ and $((x_n)^* x_n + (y_n)^* y_n + \cdots + (z_n)^* z_n)\, h_n = 0\, h_n$, $h_n((x_n)^* x_n + (y_n)^* y_n + \cdots + (z_n)^* z_n)\, h_n = 0$, $x_n h_n = y_n h_n = \cdots = z_n h_n = 0$. Then, for fixed m, $n > m$ implies $h_m x_n h_n = h_m x_m h_n = 0$, $h_m x_m = 0$, $(x_m)^* h_m = 0$. Similarly $(y_m)^* h_m = \cdots = (z_m)^* h_m = 0$, $\boldsymbol{x} = \boldsymbol{y} = \cdots = \boldsymbol{z} = 0$.

LEMMA 4.5. *Let* $\boldsymbol{x} = [x_n, f_n] \in C$ *and for some* SDD $\{e_n\}$ $\boldsymbol{x}\bar{e}_n = 0$ *for all* n; *then* $\boldsymbol{x} = 0$.

PROOF. By the Remark following Theorem 3.1, we have $\boldsymbol{x}\overline{(e_n \wedge f_n)}$ $\overline{x_n(e_n \wedge f_n)} = \boldsymbol{x}\overline{e_n(e_n \wedge f_n)} = 0$. Thus $x_n(e_n \wedge f_n) = 0$ for all n. For fixed m, $n > m$, implies $(e_m \wedge f_m)\, x_n(e_n \wedge f_n) = (e_m \wedge f_m)\, x_m(e_n \wedge f_n) = 0$, and $(e_m \wedge f_m)\, x_m = 0$, that is, $(x_m)^*(e_m \wedge f_m) = 0$. This implies $\boldsymbol{x} = 0$. The lemma follows.

5. Spectral theory for C. The next lemma is elementary:

LEMMA 5.1. ([1], Lemma 4.1.) *Let* $\mathcal{B}$ *be an associative algebra with unit* 1 *over the complex numbers, with involution* $*$, *and such that* $x + i1$ *is invertible if* $x^* = x$. *Then the formulae*

$$u = (x - i1)(x + i1)^{-1}$$

$$x = i(1 + u)(1 - u)^{-1}$$

define mutually inverse one to one correspondences between the self-adjoint elements $x\,(x^* = x)$, *and the unitary elements* $u\,(u^*u = uu^* = 1)$ *such that* $1 - u$ *is invertible.*

If x, u are related as in Lemma 5.1, we call u the Cayley transform of x; it is evident that an element of $\mathcal{B}$ will commute with x if and only if it commutes with u. We can apply Lemma 4.1 to the algebra C (Corollary 4.1), as well as to the algebra M. Then we have the following:

THEOREM 5.1. *The formulae*

$$\boldsymbol{u} = (\boldsymbol{x} - i1)(\boldsymbol{x} + i1)^{-1}$$

$$\boldsymbol{x} = i(1 + \boldsymbol{u})(1 - \boldsymbol{u})^{-1}$$

define mutually inverse one to one correspondences between the self-adjoint elements $\boldsymbol{x} \in C$, and the unitary elements $\boldsymbol{u} \in C$ such that $1 - \boldsymbol{u}$ is invertible. The unitary elements $\boldsymbol{u}$ which so occur are those of the form $\boldsymbol{u} = \overline{u}$ for some $u \in M_u$. Moreover let $u \in M_u$, write $\{u\}'' = C(\Omega)$ with Ω a Stone space ([2]), *and let Ω_0 be the open set $\Omega_0 = \{\omega ; \omega \in \Omega, u(\omega) \neq 1\}$. Then $1 - \overline{u}$ is invertible if and only if Ω_0 is dense in Ω and there exist clopen (open and closed) sets Ω_n such that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega_0$, and the characteristic functions of $(\Omega_n)^c$ (the complement of Ω_n) are in $\mathfrak{M}$.*

PROOF. If $\boldsymbol{x}^* = \boldsymbol{x} \in C$, we can write $\boldsymbol{x} = [x_n, e_n]$ with $(x_n)^* = x_n$; then the Cayley transform of $\boldsymbol{x}$ is $\boldsymbol{u} = [(x_n - i1)(x_n + i1)^{-1}, f_n]$ where $\{f_n\}$ is a suitable SDD. As each $u_n = (x_n - i1)(x_n + i1)^{-1}$ is unitary, by Lemma 4.3, we get $\boldsymbol{u} = \overline{u}$ for some $u \in M_u$. Conversely if $\boldsymbol{u} \in C$ is unitary and $1 - \boldsymbol{u}$ is invertible, then we can define $\boldsymbol{x} = i(1 + \boldsymbol{u})(1 - \boldsymbol{u})^{-1}$; since $\boldsymbol{u}$ is the Cayley transform of $\boldsymbol{x}$, by the above argument we have that $\boldsymbol{u} = \overline{u}$ for some $u \in M_u$.

Next we suppose Ω_0 is dense in Ω and there are clopen sets Ω_n such that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega_0$, and the characteristic functions of $(\Omega_n)^c$ are in $\mathfrak{M}$; since we may suppose Ω_n increasing, if e_n is the characteristic function of Ω_n, then $1 - e_n \in \mathfrak{M}$ and the density shows $e_n \uparrow 1$, thus $\{e_n\}$ is an SDD. Define numerical function $G(\omega) = (1 - u(\omega))^{-1} (\omega \in \Omega_0)$; G is continuous on Ω_0. Setting $y_n = Ge_n$, we have clearly $y_n \in \{u\}''$ and $\{y_n, e_n\}$ is an EMO. As $(1-u)y_n = e_n = 1e_n$, $[y_n, e_n]$ is the inverse of $1 - \overline{u}$. Conversely, if $1 - \overline{u}$ is invertible, then $\overline{u}$ is the Cayley transform of the self-adjoint element $\boldsymbol{x} = i(1+\overline{u})(1-\overline{u})^{-1} (\in C)$, and we can write $\boldsymbol{x} = [x_n, e_n]$ with $(x_n)^* = x_n$ and $\boldsymbol{u} = [(x_n - i1)(x_n + i1)^{-1}, e_n]$. Taking an increasing sequence $\{r_n\}$ of positive numbers satisfying $\|x_n\| < r_n$ and $r_n \uparrow \infty$ $(n \uparrow \infty)$, we define clopen set $\Omega_n = \{\omega ; |u(\omega) - 1| > 2/((r_n)^2 + 1)^{1/2}\}^-$ (where A^- is the closure of a set A) ([2]). Noting that $2/((r_n)^2 + 1)^{1/2} \downarrow 0 \, (n \uparrow \infty)$ and

$$\{\omega ; |u(\omega) - 1| > 2/((r_n)^2 + 1)^{1/2}\} \subset \{\omega ; |u(\omega) - 1| > 2/((r_n)^2 + 1)^{1/2}\}^-$$

$$\subset \{\omega ; |u(\omega) - 1| \geqq 2/((r_n)^2 + 1)^{1/2}\} \subset \{\omega ; |u(\omega) - 1| > 2/((r_{n+1})^2 + 1)^{1/2}\},$$

we have $\Omega_n \uparrow$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega_0$. If Ω_0 is not dense, $\Omega - \Omega_0^-$ is a non-empty clopen set, whose characteristic function e is a non-zero projection. Since $u(\omega)=1$ for $\omega\in\Omega-\Omega_0^-$, we have $ue=e$, that is, $(1-\bar{u})\bar{e}=0$, contradicting the invertibility of $1-\bar{u}$. Let f_n be the characteristic function of $(\Omega_n)^c$. We show that $e_n\wedge f_n=0$. If the contrary holds,

$$\begin{aligned}\|(1-u)(f_n\wedge e_n)\| &= \|(1-u)f_n(f_n\wedge e_n)\|\\ &\leqq \|(1-u)f_n\| \leqq 2/((r_n)^2+1)^{1/2},\end{aligned}$$

while by Lemma 4. 3,

$$\begin{aligned}(1-u)(f_n\wedge e_n) &= (1-u)\,e_n(f_n\wedge e_n)\\ &= \{1-(x_n-i1)(x_n+i1)^{-1}\}\,e_n(f_n\wedge e_n)\end{aligned}$$

and noting that the numerical function $f(\eta)=4/(\eta^2+1)$ is strictly monotone decreasing for $\eta\geqq 0$, we have

$$\begin{aligned}4(e_n\wedge f_n) &\geqq (e_n\wedge f_n)\{1-(x_n-i1)(x+i1)^{-1}\}^*\{1-(x_n-i1)(x_n+i1)^{-1}\}(e_n\wedge f_n)\\ &\geqq 4/(\|x_n\|^2+1)(e_n\wedge f_n).\end{aligned}$$

This implies that

$$\begin{aligned}\|(1-u)(e_n\wedge f_n)\| &= \|\{1-(x_n-i1)(x_n+i1)^{-1}\}(e_n\wedge f_n)\|\\ &\geqq 2/(\|x_n\|^2+1)^{1/2}>2/((r_n)^2+1)^{1/2}.\end{aligned}$$

Hence this is a contradiction. By ([3], Theorem 5. 4), we have $f_n=f_n-e_n\wedge f_n \sim e_n\vee f_n-e_n\leqq 1-e_n\in\mathfrak{M}$, as desired.

REMARK. In finite case, as Berberian showed in ([1], Lemma 4. 2), it is sufficient for $1-u$ to be invertible that Ω_0 is dense in Ω, but in infinite case, as the following example shows, we cannot drop the last condition: there exist clopen sets Ω_n such that $\bigcup_{n=1}^{\infty}\Omega_n=\Omega_0$, and the characteristic function of $(\Omega_n)^c$ is in $\mathfrak{M}$. Let $\mathfrak{H}$ be an infinite dimensional separable Hilbert space, $\{\xi_i\}_{i=1}^{\infty}$ an orthonormal basis for it, and $\boldsymbol{M}$ be the full operator algebra on $\mathfrak{H}$. Then we know that $\mathfrak{M}_p$ is the set of all projections of finite rank. For a sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers $(\lambda_i\uparrow\infty\ (i\uparrow\infty))$, setting $\mathfrak{D}(T)=\{\xi\,;\ \sum_{i=1}^{\infty}(\lambda_i)^2|(\xi,\xi_i)|^2<\infty\}$, then $\mathfrak{D}(T)$ is a dense linear manifold in $\mathfrak{H}$. Define linear operators T on $\mathfrak{D}(T)$ and E_λ $(-\infty<\lambda<\infty)$ on $\mathfrak{H}$ by ;

$$T\xi = \sum_{i=1}^{\infty} \lambda_i(\xi, \xi_i)\xi_i \qquad \xi \in \mathfrak{D}(T),$$

and

$$E_\lambda \xi = P_{[\xi_1, \xi_2, \cdots, \xi_{n-1}]}\xi \qquad \xi \in \mathfrak{H}$$

(where n is the minimal n such that $\lambda_n \geqq \lambda$, $\xi_0 = 0$, and $P_{[\xi_1, \cdots, \xi_{n-1}]}$ is the orthogonal projection on the linear manifold $[\xi_1, \cdots, \xi_{n-1}]$), then T is a densely defined self-adjoint operator and $\{E_\lambda\}_{-\infty<\lambda<\infty}$ is the resolution of unity for T. If T is measurable in the sense of [9], then there exists a projection $P \in \boldsymbol{M}$ such that TP is bounded and $1-P \in \mathfrak{M}$. Let $\|TP\| < \lambda_0$, we have that $P \wedge (1-E_{\lambda_0}) = 0$. If otherwise, there is a non-zero $\xi \in \mathfrak{H}$ with $(P \wedge (1-E_{\lambda_0}))\xi = \xi$. $\|T\xi\| = \|TP\xi\| < \lambda_0\|\xi\|$, while $\|T\xi\| = \|T(1-E_{\lambda_0})\xi\| \geqq \lambda_0\|\xi\|$. This is a contradiction. Since for every projection $Q, R \in \boldsymbol{M}$, $Q - Q \wedge R \sim Q \vee R - R$, we have $1-E_{\lambda_0} = (1-E_{\lambda_0}) - P \wedge (1-E_{\lambda_0}) \sim P \vee (1-E_{\lambda_0}) - P \leqq 1-P \in \mathfrak{M}$, contradicting the definition of E_{λ_0}, that is, T is a non-measurable self-adjoint operator. Let U be the Cayley transform of T, $\{U\}'' = C(\Omega)$ with Ω a Stone space, and Ω_0 be the set $\{\omega ;\ U(\omega) \neq 1\}$. For $1-U$ is one to one, we have that Ω_0 is dense in Ω. But $1-U$ is not invertible in C (The preceding Remark of Theorem 3.2). For if $1-U$ is invertible in C, then $T = i(1+U)(1-U)^{-1}$ is in C, contradicting the above argument.

The rest of our discussions in this section is the slight modifications of ([1], sections 4, 5 and 6), but for the sake of completeness, we sketch them.

As a spectral theorem for a self-adjoint MO, we have:

THEOREM 5.2. *Let $\boldsymbol{x}$ be a self-adjoint element of C, $\boldsymbol{u} = \bar{\boldsymbol{u}}$ its Cayley transform. We can write $\boldsymbol{x} = [x_n, e_n]$ with $x_n, e_n \in \{u\}''$, $(x_n)^* = x_n$, $x_n e_n = x_n$ and $(x_n)^2 \uparrow$.*

PROOF. Write $\{u\}'' = C(\Omega)$, where Ω is a Stone space, by Theorem 5.1, there exists an increasing family of clopen sets $\{\Omega_n\}$ such that $\bigcup_{n=1}^{\infty} \Omega_n = \{\omega ;\ u(\omega) \neq 1\} (\equiv \Omega_0)$, $\left(\bigcup_{n=1}^{\infty} \Omega_n\right)^- = \Omega$, and the characteristic function of $(\Omega_n)^c$ is in $\mathfrak{M}$, thus the family $\{e_n\}$ of the characteristic functions of Ω_n is an SDD. Let F and G be the numerical functions defined for $\omega \in \Omega_0$ by

$$G(\omega) = (1-u(\omega))^{-1}$$

$$F(\omega) = i(1+u(\omega))(1-u(\omega))^{-1};$$

it is clear that F is real valued. Put $x_n = Fe_n$, and $y_n = Ge_n$, then $(x_n)^* = x_n$

$= x_n e_n$, $\{x_n, e_n\}$ and $\{y_n, e_n\}$ are EMO, and $[y_n, e_n]$ is the inverse of $1-\boldsymbol{u}$. As $x_n = Fe_n = i(1+u)Ge_n = i(1+u)y_n$, we have $[x_n, e_n] = i(1+\bar{\boldsymbol{u}})(1-\bar{u})^{-1} = \boldsymbol{x}$. If $m<n$, then $(x_m)^2 = (x_n e_m)^2 = (x_n)^* e_m x_n \leqq (x_n)^*(x_n) = (x_n)^2$. This completes the proof of Theorem 5.2.

Next, we characterize $\bar{M}$ as a subalgebra of $\mathcal{C}$, in terms of the algebraic structure of $\mathcal{C}$.

THEOREM 5.3. *If* $\boldsymbol{x} = [x_n, e_n]$, *with* $\|x_n\| \leqq k$ *for all* n, *then there is a unique element* $x \in M$ ($\|x\| \leqq k$) *such that* $\boldsymbol{x} = \bar{x}$.

PROOF. Considering that $\|(1/2)((x_n)^* + x_n)\| \leqq k$, we may assume $\boldsymbol{x}^* = \boldsymbol{x}$. If $\boldsymbol{u} = \bar{u}$ is the Cayley transform of $\boldsymbol{x}$, then by Theorem 5.2, we can write $\boldsymbol{x} = [y_n, f_n]$ with $y_n, f_n \in \{u\}''$, $(y_n)^* = y_n$ and $(y_n)^2 \uparrow$. Now we show that $\|y_n\| \leqq k$ for all n. Since $\{y_n, f_n\} \equiv \{x_n, e_n\}$, there exists an SDD$\{g_n\}$ such that $y_n g_n = x_n g_n$ for all n; then also $g_n (y_n)^2 g_n = g_n (x_n)^* x_n g_n$. The assumption $(x_n)^* x_n \leqq k^2 \cdot 1$ implies $g_n (x_n)^* x_n g_n \leqq k^2 g_n$, and then $g_n (y_n)^2 g_n \leqq k^2 g_n$. For fixed m, $n>m$ implies $(y_m)^2 \leqq (y_n)^2$, $g_n (y_m)^2 g_n \leqq g_n (y_n)^2 g_n \leqq k^2 g_n$, $g_n(k^2 \cdot 1 - (y_m)^2) g_n \geqq 0$; we may write $\{k^2 \cdot 1 - (y_m)^2\}''$ as the algebra $C(\Gamma)$ of continuous complex-valued functions on a Stone space Γ ([2], section 4). Assume that $(k^2 \cdot 1 - (y_m)^2)(\gamma) < 0$ for some $\gamma \in \Gamma$; choose a non-zero projection $g \in \{k^2 \cdot 1 - (y_m)^2\}''$, and a real number $\delta < 0$ such that $g(k^2 \cdot 1 - (y_m)^2) \leqq \delta g$. Since $(k^2 \cdot 1 - (y_m)^2)^{-1}[g]$ is the largest projection right-annihilating $(1-g)(k^2 \cdot 1 - (y_m)^2)$, clearly $g \leqq (k^2 \cdot 1 - (y_m)^2)^{-1}[g]$. Put $f'_n = g_n \wedge ((k^2 \cdot 1 - (y_m)^2)^{-1}[g])$, so that $(1-g)(k^2 \cdot 1 - (y_m)^2) f'_n = 0$, $(k^2 \cdot 1 - (y_m)^2) f'_n = g(k^2 \cdot 1 - (y_m)^2) f'_n$, $f'_n(k^2 \cdot 1 - (y_m)^2) f'_n = f'_n g (k^2 \cdot 1 - (y_m)^2) f'_n$. Since $0 \leqq f'_n (g_n (k^2 \cdot 1 - (y_m)^2) g_n) f'_n = f'_n (k^2 \cdot 1 - (y_m)^2) f'_n \leqq \delta f'_n g f'_n \leqq 0$, necessary $\delta f'_n g f'_n = 0$, $g f'_n = 0$, $0 = g \wedge f'_n = g \wedge g_n$ for all n. By ([3], Theorem 5.4), $g = g - g \wedge g_n \sim g_n \vee g - g_n \leqq 1 - g_n \in \mathfrak{M}$. By the same argument used in the proof of Lemma 3.1, we have that $g = 0$, contradicting the above result $g \neq 0$. $k^2 \cdot 1 - (y_m)^2 \geqq 0$ follows, thus $\|y_n\| \leqq k$ for all n.

Let $y_n = w_n r_n$ be the polar decomposition of y_n where, w_n, $r_n \in \{u\}''$, $(w_n)^* w_n = w_n (w_n)^* = RP(y_n)$, $r_n = (y_n)^{1/2}$([11], Lemma 2.1). The uniqueness of this decomposition, together with the fact that $y_n e_m = y_m$ when $m<n$, shows that $w_n f_m = w_m$ and $r_n f_m = r_m$; thus $\{w_n, f_n\}$ and $\{r_n, f_n\}$ are EMO, and we have $[y_n, f_n] = [w_n, f_n]\, [r_n, f_n]$. Thus it is sufficient to show that $[w_n, f_n] = \bar{w}$ and $[r_n, f_n] = \bar{r}$ with $w, r \in M$. Modifying the proof of Lemma 4.3, we have that there exists a partial isometry $w \in \{u\}''$ such that $[w_n, f_n] = \bar{w}$. Finally since $r_n \uparrow$ and $r_n \leqq k1$, by [2], we can find $r = \sup\{r_n, n \geqq 1\}$ in the quasi complete lattice of self-adjoints of $\{u\}''$; since we may write $\{u\}''$ as the algebra $C(\Omega)$ of continuous complex-valued functions on a Stone space, and $r_n(\omega) \uparrow r(\omega)$ except on a set of first category, we have $r f_n = r_n$, $[r_n, f_n] = \bar{r}$ with $\|r\| \leqq k$. This completes the proof.

COROLLARY 5.1. *If* $\boldsymbol{x}=[x_n,e_n]$ *with* $\|e_n x_n e_n\| \leqq k$ *for all* n, *then* $\boldsymbol{x}=\bar{x}$ *for some* $x \in M$ *with* $\|x\| \leqq k$.

PROOF. Setting $y_n = e_n x_n e_n$, and $f_n = e_n \wedge ((x_n)^{-1}[e_n]) \wedge (((x_n)^*)^{-1}[e_n])$, $\{y_n, f_n\}$ is an EMO equivalent to $\{x_n, e_n\}$; hence $\boldsymbol{x} = [y_n, f_n]$ with $\|y_n\| \leqq k$ for all n. This completes the proof of Corollary 5.1.

Next we introduce the partial ordering of self-adjoints.

DEFINITION 5.1. An element $\boldsymbol{x} \in \mathcal{C}$ is positive ($\boldsymbol{x} \geqq 0$), if $\boldsymbol{x} = \boldsymbol{y}^*\boldsymbol{y}$ for some $\boldsymbol{y} \in \mathcal{C}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ are self-adjoint, write $\boldsymbol{x} \leqq \boldsymbol{y}$ in case $\boldsymbol{y}-\boldsymbol{x} \geqq 0$.

LEMMA 5.2. *If* $\boldsymbol{x}^*\boldsymbol{x} \leqq 1$, *then* $\boldsymbol{x} = \bar{x}$ *for some* $x \in M$ *and* $\|x\| \leqq 1$.

PROOF. By assumption, $\boldsymbol{x}^*\boldsymbol{x}+\boldsymbol{y}^*\boldsymbol{y}=1$ for some $\boldsymbol{y} \in \mathcal{C}$. Thus there exists an SDD $\{g_n\}$ such that $((x_n)^*x_n+(y_n)^*y_n)g_n = 1g_n$; $g_n(x_n)^*x_n g_n \leqq g_n(x_n)^*x_n g_n + g_n(y_n)^*y_n g_n = g_n \leqq 1$, $\|x_n g_n\| \leqq 1$, $\|g_n x_n g_n\| \leqq 1$. Since by remarks following Definition 3.3, we may suppose $\{x_n, g_n\}$ is an EMO, our assertion follows from Corollary 5.1.

An element $\boldsymbol{e} \in \mathcal{C}$ is a projection if $\boldsymbol{e}^*=\boldsymbol{e}=\boldsymbol{e}^2$; $\boldsymbol{w} \in \mathcal{C}$ is a partial isometry if $\boldsymbol{w}^*\boldsymbol{w}$ is a projection. The following theorem shows that $\mathcal{C}$ contains no new projections.

THEOREM 5.4. *In* $\mathcal{C}$, *every partial isometry has the form* $\boldsymbol{w}=\bar{w}$ *with* $w \in M_{pi}$. *In particular every projection* $\boldsymbol{e}$ *has the form* $\boldsymbol{e}=\bar{e}$ *with* $e \in M_p$. *Hence the projection of* $\mathcal{C}$ *form a complete lattice which is isomorphic to the projection lattice of* M *via the mapping* $\boldsymbol{e} \to \bar{e}$.

PROOF. Suppose $\boldsymbol{w} \in \mathcal{C}$, $\boldsymbol{w}^*\boldsymbol{w}=\boldsymbol{e}$, $\boldsymbol{e}$ a projection. Then $1-\boldsymbol{w}^*\boldsymbol{w}=1-\boldsymbol{e} =(1-\boldsymbol{e})^*(1-\boldsymbol{e})$, hence $\boldsymbol{w}^*\boldsymbol{w} \leqq 1$. The assertion is clear from Theorem 3.1 and Lemma 5.2.

In the numerical Cayley transform $\alpha=i(1+\lambda)(1-\lambda)^{-1}$, $\lambda=(\alpha-i)(\alpha+i)^{-1}$,

(1) $\qquad \alpha=0$ when $\lambda=-1$,

(2) $\qquad \alpha>0$ when $\lambda \in \{e^{i\theta}: -\pi<\theta<0\}$,

(3) $\qquad \alpha<0$ when $\lambda \in \{e^{i\theta}: 0<\theta<\pi\}$.

This is the basis of our theory of order in $\mathcal{C}$. If $\boldsymbol{x} \geqq 0$, and $\alpha \geqq 0$ is a real number, then $\alpha\boldsymbol{x} \geqq 0$. If $\boldsymbol{x} \geqq 0$ and $-\boldsymbol{x} \geqq 0$, $\boldsymbol{x} = 0$; for if $\boldsymbol{x} = \boldsymbol{y}^*\boldsymbol{y}$ and

$-\boldsymbol{x}=\boldsymbol{z}^*\boldsymbol{z}$, then $\boldsymbol{y}^*\boldsymbol{y}+\boldsymbol{z}^*\boldsymbol{z}=0$, by Lemma 4.4, $\boldsymbol{y}=0$, that is, $\boldsymbol{x}=0$. If $\boldsymbol{x}\geqq 0$ and $\boldsymbol{z}\in C$ is arbitrary, then $\boldsymbol{z}^*\boldsymbol{x}\boldsymbol{z}\geqq 0$. To show that the self-adjoint elements of C form a partially ordered real linear space with respect to the ordering defined in Definition 5.1, we have only to see: if $\boldsymbol{x}\geqq 0$ and $\boldsymbol{y}\geqq 0$, then $\boldsymbol{x}+\boldsymbol{y}\geqq 0$. This is clear from condition (2) of the following:

THEOREM 5.5. *Let $\boldsymbol{x}$ be a self-adjoint element of C, $\boldsymbol{u}=\bar{u}$ its Cayley transform. Then the following four conditions are equivalent*:

(1) $\boldsymbol{x}\geqq 0$;

(2) *we can write $\boldsymbol{x}=[y_n, f_n]$ with $y_n\geqq 0$*;

(3) *the spectrum of u is contained in $\{e^{i\theta}: -\pi\leqq\theta\leqq 0\}$*;

(4) *we may write $\boldsymbol{x}=[x_n, e_n]$ with $x_n, e_n\in\{u\}''$, $x_n\geqq 0$ and $x_ne_n=x_n$.*

PROOF. (1)→(2) is clear from Definition 5.1.

(2)→(3). Suppose $\lambda=e^{i\theta}$ with $0<\theta<\pi$; we must show that $u-\lambda 1$ has an inverse in M. Write $\lambda=(\alpha-i)(\alpha+i)^{-1}$, and $\alpha<0$, $\alpha=i(1+\lambda)(1-\lambda)^{-1}$. An easy calculation shows that $\boldsymbol{u}-\lambda 1=(1-\lambda)(\boldsymbol{x}-1)(\boldsymbol{x}+i1)^{-1}$, thus $\boldsymbol{u}-\lambda 1=(1-\lambda)[(y_n-\alpha 1)(y_n+i1)^{-1}, g_n]$ for a suitable SDD $\{g_n\}$. As $y_n\geqq 0$ for all n, each $y_n-\alpha 1$ is invertible in M; by Lemma 4.1 $\boldsymbol{u}-\lambda 1$ is invertible in C, and $(\boldsymbol{u}-\lambda 1)^{-1}=(1-\lambda)^{-1}[(y_n+i1)(y_n-\alpha 1)^{-1}, h_n]$ for a suitable SDD $\{h_n\}$. The numerical function $f(\eta)=(\eta^2+1)(\eta-\alpha)^{-2}$ defined for $\eta\geqq 0$ is bounded, say $f(\eta)\leqq k$; look at the functional representation for y_n, and we have that $\|(y_n+i1)(y_n-\alpha 1)^{-1}\|^2\leqq k$ for all n. By Theorem 5.3, $(\boldsymbol{u}-\lambda 1)^{-1}=\bar{x}$ for some $x\in M$, thus $u-\lambda 1$ is invertible in M.

(3)→(4). By assumption (3) and the proof of Theorem 5.2, the assertion is clear.

(4)→(1). Put $z_n=(x_n)^{1/2}$; if $m<n$ then $x_ne_n=x_m$, from the unicity of positive square roots we have $z_ne_m=z_m$. Hence $\{z_n,e_n\}$ is an EMO, and putting $\boldsymbol{y}=[z_n, e_n]$ we show that $\boldsymbol{y}^*=\boldsymbol{y}$, $\boldsymbol{x}=\boldsymbol{y}^2$; thus $\boldsymbol{x}\geqq 0$. This completes the proof.

COROLLARY 5.2. *If $\boldsymbol{x}\geqq 0$, then there is a unique $\boldsymbol{y}\geqq 0$ such that $\boldsymbol{x}=\boldsymbol{y}^2$; we have $\boldsymbol{y}\in\{\boldsymbol{x}\}''$.*

PROOF. From the above proof of (4)→(1), we have $\boldsymbol{x}=\boldsymbol{y}^2$ with $\boldsymbol{y}\geqq 0$, and $\boldsymbol{y}\in\{\boldsymbol{x}\}''$ follows from Theorem 5.2. Thus assuming $\boldsymbol{z}\geqq 0$, we must show that $\boldsymbol{y}=\boldsymbol{z}$. Clearly $\boldsymbol{xz}=\boldsymbol{zx}$, thus also $\boldsymbol{yz}=\boldsymbol{zy}$; then $(\boldsymbol{y}+\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z}) = \boldsymbol{y}^2-\boldsymbol{z}^2=0$, $(\boldsymbol{y}-\boldsymbol{z})(\boldsymbol{y}+\boldsymbol{z})(\boldsymbol{y}-\boldsymbol{z})=0$. Write $\boldsymbol{y}=\boldsymbol{r}^*\boldsymbol{r}$, $\boldsymbol{z}=\boldsymbol{s}^*\boldsymbol{s}$ for some $\boldsymbol{r},\boldsymbol{s}\in\mathcal{C}$, and we have $0=(\boldsymbol{y}-\boldsymbol{z})(\boldsymbol{r}^*\boldsymbol{r}+\boldsymbol{s}^*\boldsymbol{s})(\boldsymbol{y}-\boldsymbol{z})=\{\boldsymbol{r}(\boldsymbol{y}-\boldsymbol{z})\}^*\{\boldsymbol{r}(\boldsymbol{y}-\boldsymbol{z})\}+\{\boldsymbol{s}(\boldsymbol{y}-\boldsymbol{z})\}^*\{\boldsymbol{s}(\boldsymbol{y}-\boldsymbol{z})\}$. By Lemma 4.4, $\boldsymbol{r}(\boldsymbol{y}-\boldsymbol{z})=\boldsymbol{s}(\boldsymbol{y}-\boldsymbol{z})=0$, thus $\boldsymbol{r}^*\boldsymbol{r}(\boldsymbol{y}-\boldsymbol{z}) = \boldsymbol{s}^*\boldsymbol{s}(\boldsymbol{y}-\boldsymbol{z})=0$, $\boldsymbol{y}(\boldsymbol{y}-\boldsymbol{z})=\boldsymbol{z}(\boldsymbol{y}-\boldsymbol{z})=0$, $(\boldsymbol{y}-\boldsymbol{z})^*(\boldsymbol{y}-\boldsymbol{z})=0$.

DEFINITION 5.2. If $\boldsymbol{x}\geqq 0$ write $\boldsymbol{y}=\boldsymbol{x}^{1/2}$ for the unique $\boldsymbol{y}\geqq 0$ such that $\boldsymbol{x}=\boldsymbol{y}^2$. For $\boldsymbol{x}\in\mathcal{C}$, write $|\boldsymbol{x}|=(\boldsymbol{x}^*\boldsymbol{x})^{1/2}$.

REMARK. Let $\boldsymbol{x}$ be a positive element of $\mathcal{C}$, and, $\bar{u}$ the unique Cayley transform of $\boldsymbol{x}$. Then by Theorem 5.2, we can write $\boldsymbol{x}=[x_n, e_n]$, with $x_n, e_n\in\{u\}''$, and we have $\boldsymbol{x}=[x_ne_n, e_n]$. Looking at the functional representation of the elements of $\{u\}''$, $m<n$ implies $(x_ne_n)^pe_m=(x_me_m)^pe_m$ for an arbitrary non-negative real number p. Set $\boldsymbol{y}=[(x_ne_n)^p, e_n]$ and if $\boldsymbol{x}=[(x_n)', (e_n)']$ with $(x_n)', (e_n)'\in\{u\}''$, then $x_ne_n(e_n\wedge(e_n)')=(x_n)'(e_n)'(e_n\wedge(e_n)')$. By the same reason as above, we have that $(x_ne_n)^p(e_n\wedge(e_n)')=((x_n)'(e_n)')^p(e_n\wedge(e_n)')$, and hence $[(x_ne_n)^p, e_n]=[((x_n)'(e_n)')^p, (e_n)']$, that is, $\boldsymbol{y}$ is independent of the representation of $\boldsymbol{x}$ in $\{u\}''$ and is therefore unambiguously defined. We denote $\boldsymbol{y}$ by $\boldsymbol{x}^p$ (Note that $\boldsymbol{x}^p\in\{\boldsymbol{x}\}''$).

6. Algebraic structure of $\mathcal{C}$.

THEOREM 6.1. *Let $\boldsymbol{x}\in\mathcal{C}$, $\boldsymbol{x}\geqq 0$, and $\bar{u}$ be the Cayley transform of $\boldsymbol{x}$, writing $\{u\}''$, as the algebra $C(\Omega)$ of continuous complex-valued functions on a Stone space Ω ([2]), Ω_0^+ be the set $\{\omega;\ \omega\in\Omega,\ i(1+u(\omega))(1-u(\omega))^{-1}>0\} =\{\omega\in\Omega;\ u(\omega)=e^{i\theta},\ -\pi<\theta<0\}$. Then there is an element $\boldsymbol{y}\in\mathcal{C}$ and a projection $\bar{e}\in\mathcal{C}$, such that*

$$(1)\qquad \boldsymbol{xy}=\bar{e},\quad \bar{e}\boldsymbol{x}=\boldsymbol{x},\quad \bar{e}\boldsymbol{y}=\boldsymbol{y},$$

$$(2)\qquad \boldsymbol{y},\bar{e}\in\{\boldsymbol{x}\}'',\quad \boldsymbol{y}\geqq 0,$$

if and only if there exists a family of clopen sets $\{\Gamma_n\}$ such that $\bigcup_{n=1}^{\infty}\Gamma_n =\Omega_0^+$ and the characteristic function of $(\Omega_0^+)^- -\Gamma_n$ is an element of $\mathfrak{M}$ for all n (where E^- is the closure of a set E).

PROOF. Let $\boldsymbol{x}=[x_n, e_n]$, notation as in the proof of Theorem 5.2. If f_n (resp. f) is the characteristic function of Γ_n(resp. $\left(\bigcup_{n=1}^{\infty}\Gamma_n\right)^- =(\Omega_0^+)^-$), we have $f_n\uparrow f$ and $f-f_n\in\mathfrak{M}$. Put $g_n=f_n+(1-f)$, so that $g_n\uparrow 1$ and $1-g_n=f-f_n\in\mathfrak{M}$. Define $z_n=Ff_n$; since $F(\omega)=0$ for $\Omega_0\bigcap(\Omega-(\Omega_0^+)^-)$, we easily see that $\{z_n,$

$g_n\}$ is an EMO, and that the SDD $\{e_n g_n\}$ implements $\{x_n, e_n\} \equiv \{z_n, g_n\}$, thus $\boldsymbol{x} = [z_n, g_n]$. As $z_n(\omega) > 0$ for $\omega \in \Gamma_n$ (compact set), there exists a unique $y_n \in \{u\}''$ such that $z_n y_n = f_n$, $y_n f_n = y_n$. By the unicity we show that $y_n g_m = y_m$ when $m < n$, hence $\{y_n, g_n\}$ is an EMO. Then $\boldsymbol{y} = [y_n, g_n]$ and $e = f$ satisfy (1) and (2).

Conversely, suppose that there are $\boldsymbol{y}$ and $\bar{e}$ satisfying (1) and (2). Let $\bar{u}$ be the Cayley transform of $\boldsymbol{x}$, and setting $\boldsymbol{w} = ((\boldsymbol{x}+i1)/2)\boldsymbol{y}$, an easy calculation shows that

$$\bar{e}\boldsymbol{w} = \boldsymbol{w}\bar{e} = \boldsymbol{w}, \quad \boldsymbol{w} \in \{\boldsymbol{x}\}'',$$
$$\boldsymbol{w}(1+\bar{u}) = (1+\bar{u})\boldsymbol{w} = \bar{e},$$

and e is the characteristic function of $\{\omega\,;\ (1+u)(\omega) \neq 0\}^-$. Setting $\boldsymbol{w}_n = \boldsymbol{w}\bar{e}_n$, we have

$$\boldsymbol{w}_n \bar{e} = \bar{e}\boldsymbol{w}_n = \boldsymbol{w}_n$$

and

$$(*) \qquad \boldsymbol{w}_n(1+\bar{u}) = (1+\bar{u})\,\boldsymbol{w}_n = \bar{e}_n\bar{e}.$$

Let $\boldsymbol{w}_n = [w^n_m, g^n_m]$, and $\|w^n_m\| < r^n_m$ where r^n_m is a real number such that $r^n_m \uparrow \infty (m \uparrow \infty)$. Noting that

$$\{\omega\,;\ |(1+u)(\omega)| > 1/r^n_m\} \subset \{\omega\,;\ |(1+u)(\omega)| > 1/(r^n_{m+1})\},$$

the set $H^n_m = \{\omega;\ |(1+u)(\omega)| > 1/r^n_m\}^-$ is a clopen set ([2]) and putting $H^n_m \bigcap \Omega_n = \Omega^n_m$, an easy calculation shows that

$$\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\Omega^n_m = \{\omega\,;\ (1+u)(\omega) \neq 0\} \bigcap \{\omega\,;\ (1-u)(\omega) \neq 0\}$$
$$= \Omega_0^+.$$

Set h^n_m is the characteristic function of $(\Omega^n_m)^c$, and by the equation $(*)$, we can choose an SDD $\{g^n_m\}_{m=1}^\infty$ such that

$$w^n_m(1+u)g^n_m = (1+u)w^n_m g^n_m = e_n e g^n_m.$$

If $(e_n e) \wedge g^n_m \wedge h^n_m (\equiv f^n_m) \neq 0$, then

$$w^n_m(1+u)f^n_m = (1+u)w^n_m f^n_m = f^n_m,$$

and we have

$$1 = \|(1+u)w^n_m f^n_m\| = \|w^n_m(1+u)f^n_m\| \leqq \|w^n_m\|\,\|(1+u)f^n_m\|,$$

and since

$$\|(1+u)f_m^n\| = \|(1+u)h_m^n f_m^n\| \leqq 1/r_m^n,$$

we get $\|w_m^n\| \geqq r_m^n$. This is a contradiction and so $(e_n e h_m^n) \wedge g_m^n = 0$. Thus $e_n e h_m^n = e_n e h_m^n - (e_n e h_m^n) \wedge g_m^n \sim (e_n e h_m^n) \vee g_m^n - g_m^n \leqq 1 - g_m^n \in \mathfrak{M}$, and $e - e_n e(1-h_m^n) = e - ee_n + ee_n h_m^n \leqq 1 - e_n + ee_n h_m^n \in \mathfrak{M}$ ([3], Theorem 4.2). $\{\Omega_m^n\}_{n,m=1}^{\infty}$ meets all requirements.

THEOREM 6.2. *C is regular in the sense of* ([10], *Part II, Chap. II, Definition* 2.2) *if and only if M is finite.*

PROOF. Suppose M is finite, then by ([1], Corollary 7.1), C is regular. But for the sake of completeness, we sketch the proof. Since M is finite, for $|\boldsymbol{x}|$ $(\boldsymbol{x} \in C)$, the condition of Theorem 6.1 is always satisfied and hence there exist $\boldsymbol{s} \geqq 0$, and a projection $\boldsymbol{e}$, such that $|\boldsymbol{x}|\boldsymbol{s} = \boldsymbol{e}$, $\boldsymbol{e}|\boldsymbol{x}| = |\boldsymbol{x}|$, and $\boldsymbol{es} = \boldsymbol{s}$. Since $\boldsymbol{e} = \boldsymbol{s}^2|\boldsymbol{x}|^2 = (\boldsymbol{s}^2\boldsymbol{x}^*)\boldsymbol{x}$, we have $C\boldsymbol{e} \subset C\boldsymbol{x}$; conversely $|\boldsymbol{x}|\boldsymbol{e} = |\boldsymbol{x}|$, $|\boldsymbol{x}|^2(1-\boldsymbol{e}) = 0$, $\boldsymbol{x}^*\boldsymbol{x}(1-\boldsymbol{e}) = 0$, $(1-\boldsymbol{e})\boldsymbol{x}^*\boldsymbol{x}(1-\boldsymbol{e}) = 0$, $\boldsymbol{x}(1-\boldsymbol{e}) = 0$, $\boldsymbol{xe} = \boldsymbol{x}$, thus $C\boldsymbol{x} \subset C\boldsymbol{e}$.

Conversely suppose that C is regular. By ([3], Theorem 4.2), there exists a central projection e such that $M(1-e)$ is finite algebra, $e = 0$, or Me is a properly infinite algebra and $M = Me \oplus M(1-e)$. If $e \neq 0$, then Me is properly infinite and by ([3], Lemma 4.4), there is a family of increasing projections $\{e_i\}_{i=1}^{\infty} (\subset M_p)$ such that $1 - e_i \notin \mathfrak{M}$ and $e_i \uparrow 1$. Taking an increasing sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive real numbers such that $\lambda_i \uparrow \infty$ $(i \uparrow \infty)$, we define s_n by;

$$s_n = \sum_{i=2}^{n} (1/\lambda_i)(e_i - e_{i-1}) + (1/\lambda_1)e_1 \quad (\in M).$$

Then, as $\lambda_i \uparrow \infty (i \uparrow \infty)$, $s_n \leqq (1/\lambda_1)1$ for all n, and $\{s_n\}$ is the family of mutually commuting increasing positive elements majorized by $(1/\lambda_1)\cdot 1$. Considering a maximal commutative subalgebra $A\,(=C(\Delta)$, the algebra of all continuous complex-valued functions on a Stone space Δ [2]) generated by $\{e_n\}$, $\{s_n\}$ has the least upper bound s in A, and the right projection of s is 1; for if $se = 0$, $e \in M_p$, then e commutes with s and $e \in A$, and since $s_n(\delta) \uparrow s(\delta)$ without on a set of first category, we have that

$$e_n se = ee_n s = es_n = 0,$$

and

$$(1/\lambda_n)ee_n \leqq es_n = 0, \text{ that is, } ee_n = 0 \text{ for all } n.$$

By Lemma ([3], Lemma 2.2), $e = 0$. By the regularity of C, we can choose a projection $e\,(e \in M_p)$ such that $C\bar{s} = C\bar{e}$. An easy computation shows that

$e=1$ and $\bar{s}$ is invertible in C. Let $\boldsymbol{y}$ be the inverse of $\bar{s}$, we can write $\boldsymbol{y}=[x_n, f_n]$ with $(x_n)^*=x_n$. Then there exists an SDD $\{g_n\}$ such that

$$x_n s g_n = s x_n g_n = g_n \quad \text{for all } n.$$

Taking an increasing sequence $\{\mu_n\}_{n=1}^{\infty}$ of positive real numbers such that $\|x_n\|<\mu_n$ and $\mu_n \uparrow \infty$ $(n\uparrow\infty)$, let m_p be the largest integer k such that $\lambda_k \leqq \mu_p$. If $(1-e_{m_n})\wedge g_n \neq 0$, then by the same reason as above, we have

$$\begin{aligned} x_n s((1-e_{m_n})\wedge g_n) &= x_n[\sup\{\sum_{i=2}^{n}(1/\lambda_i)(e_i-e_{i-1})+(1/\lambda_1)e_1, n\geqq 1\}]((1-e_{m_n}\wedge g_n) \\ &= x_n[\sup\{\sum_{i=m_n+1}^{p}(1/\lambda_i)(e_i-e_{i-1}),\ p\geqq m_n+1\}]((1-e_{m_n})\wedge g_n) \\ &= (1-e_{m_n})\wedge g_n, \end{aligned}$$

and

$$\begin{aligned} &\|x_n\|\,\|\sup\{\sum_{i=m_n+1}^{p}(1/\lambda_i)(e_i-e_{i-1})\ \ p\geqq m_n+1\}\| \\ &\geqq \|x_n[\sup\{\sum_{i=m_n+1}^{p}(1/\lambda_i)(e_i-e_{i-1}),\quad p\geqq m_n+1\}]\| \\ &= 1. \end{aligned}$$

Noting that $0<(g_n\wedge(1-e_{m_n}))[\sup\{\sum_{i=m_n+1}^{p}(1/\lambda_i)(e_i-e_{i-1})\}]^2(g_n\wedge(1-e_{m_n}))$ $\leqq (1/\lambda_{m_n+1})^2(g_n\wedge(1-e_{m_n}))$, we have that

$$\begin{aligned} \|x_n\| &\geqq 1/\|[\sup\{\sum_{i=m_n+1}^{p}(1/\lambda_i)(e_i-e_{i-1}),\ \ p\geqq m_n+1\}]((1-e_{m_n})\wedge g_n)\| \\ &\geqq \lambda_{m_n+1} > \mu_n, \end{aligned}$$

and contradicting the inequality $\|x_n\|<\mu_n$. Thus $(1-e_{m_n})\wedge g_n=0$. $(1-e_{m_n})$ $=(1-e_{m_n})-(1-e_{m_n})\wedge g_n \sim g_n\vee(1-e_{m_n})-g_n \leqq 1-g_n \in \mathfrak{M}$, and contradicting the choice of $\{e_n\}$, that is, M is a finite algebra. This completes the proof of Theorem 6.2.

The polar decomposition of measurable operators is one of the important tools in the construction of non-commutative integration theory, and next we show that the decomposition is true in C.

THEOREM 6.3. *Let $\boldsymbol{x}\in C$, $\bar{u}$(resp. $\bar{v}$), the Cayley transform of $\boldsymbol{x}^*\boldsymbol{x}$(resp. $\boldsymbol{x}\boldsymbol{x}^*$), $e=LP(1+u)$ and $f=LP(1+v)$. Then we can write $\boldsymbol{x}=\boldsymbol{w}|\boldsymbol{x}|$ with*

w a partial isometry such that $w^*w=\bar{e}$, $ww^*=\bar{f}$. In particular $e\sim f$.

PROOF. The proof is a modification of the argument used in ([11], Lemma 2.1). By [2], we can write $\{u\}''$(resp. $\{v\}''$) as the algebra $C(\Omega)$(resp. $C(\Gamma)$) of continuous complex-valued functions on a Stone space Ω(resp. Γ). Then an easy calculation shows that e (resp. f) is the characteristic function of the set $\{\omega;\ u(\omega)\neq -1\}^-$(resp. $\{\gamma;\ v(\gamma)\neq -1\}^-$). By Theorem 5.2, we may write $x^*x=[y_n, e_n]$, $y_n, e_n\in C(\Omega)$, $\{e_n\}$ an SDD, $0\leqq y_n\leqq y_{n+1}$, and $y_ne_m=y_me_m=y_m$, when $m<n$. For $n, m=1, 2, \cdots$, there are positive elements c_m^n and projections e_m^n ($\in C(\Omega)$) with the following properties:

(1) $\qquad y_n(c_m^n)^2$ is a projection $\leqq ee_n$, $y_n(c_m^n)^2=e_m^n$.

(2) $\qquad y_n\geqq(1/m)e_m^n$, and $y_n\leqq(1/m)(e-e_m^n)$ in $(e-e_m^n)e_n$.

(3) $\qquad c_1^n\leqq c_2^n\leqq c_3^n\leqq\cdots$ for all n and $c_{m-1}^n(c_m^n-c_{m-1}^n)=0$

for $m=2, 3, \cdots$ for all n.

(4) $\qquad c_m^1\leqq c_m^2\leqq c_m^3\leqq\cdots$ for all m and $c_m^ke_k=c_m^ne_k(k<n)m=1, 2, \cdots$.

(5) $\qquad e_m^je_i=e_m^i$ if $j>i$ for all m.

Because, setting e_m^i is the characteristic function of the set $\{\omega;\ \omega\in\Omega_i,\ y_i(\omega)>(1/m)\}^-$ and $c_m^i(\omega)=(1/y_i(\omega))^{1/2}e_m^i(\omega)$, $\{e_m^i,\ c_m^i\}$ meets all requirements. By the Remark following Theorem 3.1, we have

$$\begin{aligned}(\overline{xc_m^n})^*(\overline{xc_m^n}) &= \overline{c_m^nx^*xc_m^n} = \overline{x^*x(c_m^n)^2} = [y_i, e_i][(c_m^n)^2, 1]\\ &= [y_i, e_i][e_n, 1][(c_m^n)^2, 1] \qquad \text{(by (5))}\\ &= [y_ne_n, 1][(c_m^n)^2, 1]\\ &= [y_n(c_m^n)^2, 1] = [e_m^n, 1]=\overline{e_m^n},\end{aligned}$$

and by Theorem 5.3, there is a partial isometry $w_m^n(\in M_{pi})$ such that $\overline{xc_m^n}=\overline{w_m^n}$ and $(w_m^n)^*w_m^n=e_m^n$. Since

$$xx^*(1-\bar{f})=i(1+\bar{v})(1-\bar{v})^{-1}(1-\bar{f})=i(1+\bar{v})(1-\bar{v})^{-1}=0,$$

we have $\bar{f}x=x$ and putting $w_m^n(w_m^n)^*=f_m^n(\in M_p)$,

$$\bar{f}\overline{f_m^n} = \bar{f}\overline{x(c_m^n)^2x^*} = \overline{x(c_m^n)^2x^*} = \overline{w_m^n(w_m^n)^*} = \overline{f_m^n},$$

and

$$\overline{f^n_{m-1}f^n_m} = \overline{x(c^n_{m-1})^2x^*x(c^n_m)^2x^*} = \overline{x(c^n_{m-1})^2e^n_m x^*}$$
$$= \overline{x(c^n_{m-1})^2x^*} = \overline{f^n_{m-1}},$$

thus we have $f^n_{m-1} \leqq f^n_m \leqq f$. Set $f_n = \sup\{f^n_m, m \geqq 1\}$ and noting that $(c^i_m)^2 \leqq (c^j_m)^2 (i < j)$, we see $f_n \uparrow$, and we write $f' = \sup\{f_n, n \geqq 1\} (\leqq f)$. Put $v^n_m = w^n_m(e^n_m - e^n_{m-1})$, where $f^n_0 = e_0 = v^n_0 = w^n_0 = 0$ for all n, and considering that $\overline{w^n_m e^n_{m-1}} = \overline{x c^n_m e^n_{m-1}} = \overline{x c^n_{m-1}} = \overline{w^n_{m-1}}$, we have

$$(v^n_m)^* v^n_m = e^n_m - e^n_{m-1},$$
$$v^n_m(v^n_m)^* = w^n_m(e^n_m - e^n_{m-1})(w^n_m)^*$$
$$= w^n_m e^n_m (w^n_m)^* - w^n_m e^n_{m-1}(w^n_m)^*$$
$$= f^n_m - f^n_{m-1}.$$

By ([4], Lemma 20), we can choose a partial isometry $w_n \in M_{pi}$ such that

$$(w_n)^* w_n = e_n e, \quad w_n(w_n)^* = f_n$$
$$w_n(e^n_m - e^n_{m-1}) = v^n_m,$$
$$(w_n)^*(f^n_m - f^n_{m-1}) = (v^n_m)^*,$$

and

$$w_n(e^n_m - e^n_{m-1}) = w^n_m(e^n_m - e^n_{m-1}).$$

Since $w^n_m e_{n-1} e = w^{n-1}_m$, we have

$$w_n(e^{n-1}_m - e^{n-1}_{m-1}) = w^n_m(e^{n-1}_m - e^{n-1}_{m-1}) \quad (e^n_m - e^n_{m-1} \geqq e^{n-1}_m - e^{n-1}_{m-1})$$
$$= w^{n-1}_m(e^{n-1}_m - e^{n-1}_{m-1})$$
$$= w_{n-1}(e^{n-1}_m - e^{n-1}_{m-1}).$$

By ([3], Lemma 2. 2) we have

$$w_n e_{n-1} e = w_{n-1} e_{n-1} e.$$

Set $v_n = w_n(e_n - e_{n-1})e$, it follows that

$$(v_n)^* v_n = (e_n - e_{n-1})e,$$
$$v_n(v_n)^* = w_n(e_n - e_{n-1})e(w_n)^* = w_n e_n e(w_n)^* - w_n e_{n-1} e(w_n)^*$$
$$= w_n e_n e(w_n)^* - w_{n-1} e_{n-1} e(w_{n-1})^* = f_n - f_{n-1}.$$

Again by ([4], Lemma 20), there is a partial isometry $w \in M_{pi}$ such that

$$w^*w = \sup\{e_n e,\ n \geqq 1\} = e,$$

$$ww^* = \sup\{f_n,\ n \geqq 1\} = f',$$

$$w(e_n - e_{n-1})e = w_n(e_n - e_{n-1})e \quad \text{where } e_0 = 0,$$

and

$$w^*(f_n - f_{n-1}) = (w_n)^*(f_n - f_{n-1}) \quad \text{where } f_0 = 0.$$

By mathematical induction we have $we_ne = w_ne_ne$.

Next we show $\boldsymbol{x} = \bar{w}|\boldsymbol{x}|$. By Lemma 4.5, it is sufficient to prove that $(\boldsymbol{x} - \bar{w}|\boldsymbol{x}|)\bar{e}_n = 0$ for all n. Since

$$\begin{aligned}(\boldsymbol{x} - \bar{w}|\boldsymbol{x}|)\overline{e_n} &= (\boldsymbol{x} - \overline{w^n_m}|\boldsymbol{x}| + \overline{w^n_m}|\boldsymbol{x}| - \bar{w}|\boldsymbol{x}|)\overline{e_n} \\ &= (\boldsymbol{x} - \overline{w^n_m}|\boldsymbol{x}|)\overline{e_n} + (\overline{w^n_m} - \bar{w})|\boldsymbol{x}|\overline{e_n} \\ &= \boldsymbol{x}(\bar{e} - \overline{c^n_m}|\boldsymbol{x}|)\overline{e_n} + (\overline{w^n_m} - \bar{w}_n)|\boldsymbol{x}|\overline{e_n} \\ &\qquad (\text{by } |\boldsymbol{x}|\overline{e_n} = \overline{e_n}|\boldsymbol{x}| \text{ and } \overline{w^n_m} = \boldsymbol{x}\overline{c^n_m}) \\ &= \boldsymbol{x}(\bar{e} - \overline{e^n_m})\overline{e_n} + (\overline{w^n_m} - \overline{w_n})|\boldsymbol{x}|\overline{e_n}.\end{aligned}$$

Then,

$$\begin{aligned}\{\boldsymbol{x}(\bar{e} - \overline{e^n_m})\overline{e_n}\}^*\{\boldsymbol{x}(\bar{e} - \overline{e^n_m})\overline{e_n}\} &= (\bar{e} - \overline{e^n_m})\overline{e_n}\boldsymbol{x}^*\boldsymbol{x}(\bar{e} - \overline{e^n_m}) \\ &= \boldsymbol{x}^*\boldsymbol{x}(\overline{e_n}\bar{e} - \overline{e^n_m}) = [y_n(e_ne - e^n_m),\ 1]\end{aligned}$$

and noting that $w_ne^n_m = w^n_me^n_m = w^n_m$, we have

$$\{(\overline{w^n_m} - \overline{w_n})|\boldsymbol{x}|\overline{e_n}\}^*\{(\overline{w^n_m} - \overline{w_n})|\boldsymbol{x}|\overline{e_n}\} = \overline{e_n}|\boldsymbol{x}|(\overline{e_n} - \overline{e^n_m})|\boldsymbol{x}| = \boldsymbol{x}^*\boldsymbol{x}(\overline{e_n}\bar{e} - \overline{e^n_m}).$$

By Theorem 5.3 and (2), we see that there exist elements $x_{(m)}$, $y_{(m)}(m = 1, 2, \cdots)$ such that $(\boldsymbol{x} - \bar{w}|\boldsymbol{x}|)\overline{e_n} = [x_{(m)} + y_{(m)},\ 1]$ and $\|x_{(m)}\| \leqq (1/m)^{1/2}$, $\|y_{(m)}\| \leqq (1/m)^{1/2}(m = 1, 2, \cdots)$. By Theorem 3.1, we can easily show that $(\boldsymbol{x} - \bar{w}|\boldsymbol{x}|)\overline{e_n} = 0$.

To see that $f' = f$, by the same way as in the case of $\boldsymbol{x}^*\boldsymbol{x}$, choosing for $\boldsymbol{x}\boldsymbol{x}^*$ families $\{(c^n_m)'\}\{(f^n_m)'\}$ satisfying the conditions (1)−(5), we have only to show that $f'(f^n_m)' = (f^n_m)'$ for all m, n. Considering that $\boldsymbol{x}\boldsymbol{x}^*(\overline{f'}) = \boldsymbol{x}\boldsymbol{x}^*$, the assertion is clear. Hence $f' \geqq f$, that is, $f' = f$.

Finally we shall prove the uniqueness. Let $\boldsymbol{x} = \boldsymbol{w}_1\boldsymbol{y}$ with $\boldsymbol{y} \geqq 0$, $(\boldsymbol{w}_1)^*\boldsymbol{w}_1 = \bar{e}$, $\bar{e}\boldsymbol{y} = \boldsymbol{y}$, then $\boldsymbol{x}^*\boldsymbol{x} = \boldsymbol{y}\bar{e}\boldsymbol{y} = \boldsymbol{y}^2$ and by Corollary 5.2, $\boldsymbol{y} = |\boldsymbol{x}|$, and $\boldsymbol{w}_1|\boldsymbol{x}| = \bar{w}|\boldsymbol{x}|$ implies $\boldsymbol{w}_1|\boldsymbol{x}|\overline{c^n_m} = \bar{w}|\boldsymbol{x}|\overline{c^n_m}$, $\boldsymbol{w}_1\overline{e^n_m} = \bar{w}\overline{e^n_m}$ for all m, n, $\boldsymbol{w}_1\bar{e} = \bar{w}\bar{e}$, that is, $\boldsymbol{w}_1 = \bar{w}$. This completes the proof of Theorem 6.3.

THEOREM 6.4. *C is a Baer*-ring in the sense of* ([6], *Definition* 2), *that is, if S is any subset of C, the right annihilator of S has the form eC, e a projection.*

PROOF. For $\boldsymbol{x} \in S$, using the same notation as in the proof of Theorem 6.3, $\boldsymbol{x}^*\boldsymbol{x}(1-\bar{e})=0$, that is, $\boldsymbol{x}=\boldsymbol{x}\bar{e}$. Thus the right annihilator of $\boldsymbol{x}$ includes $(1-\bar{e})C$. Conversely if $\boldsymbol{xy}=0$, then $\boldsymbol{x}^*\boldsymbol{xyy}^*=0$, $\overline{c_m^n}\boldsymbol{x}^*\boldsymbol{xyy}^*=0$, $\overline{e_m^n}\boldsymbol{yy}^*=0$ for all m, n. Choosing a family $\{d_m^n, g_m^n, g$ where $d_m^n \geqq 0$, g_m^n and g are projections$\}$ for $\boldsymbol{yy}^*$ satisfying the conditions (1)–(5), $e_m^n g_{m'}^{n'}=0$ for all n, m, n' and m', $e \leqq 1-g$, $(1-\bar{e})\boldsymbol{y}=(1-\bar{e})\bar{g}\boldsymbol{y}=\bar{g}\boldsymbol{y}=\boldsymbol{y}$, and $\boldsymbol{y} \in (1-\bar{e})C$. Since the right annihilator of S is the intersection of all the right annihilator of $\boldsymbol{x} \in S$, an easy calculation shows that the annihilator of $S=\bar{e}C$ for some projection e. This completes the proof of Theorem 6.4.

REMARK. By above Theorem 6.4, the projection e(resp. f) defined in Theorem 6.3 is the right (resp. left) projection of $\boldsymbol{x}$ in the sense of ([3], p.244), and $RP(\boldsymbol{x}) \sim LP(\boldsymbol{x})$ for all $\boldsymbol{x} \in C$.

REFERENCES

[1] S. K. BERBERIAN, The regular ring of a finite AW^*-algebra, Ann. of Math., 65(1957), 224–240.
[2] J. DIXMIER, Sur certains espaces considèrès par M. H. Stone, Summa Brasil. Math., 2(1951), 151–182.
[3] I. KAPLANSKY, Projections in Banach algebras Ann. of Math., 53(1951), 235–249.
[4] I. KAPLANSKY, Algebras of type I, Ann. of Math., 56(1952), 460–472.
[5] I. KAPLANSKY, Any orthocomplemented complete modular lattice is a continuous geometry, Ann. of Math., 61(1955), 524–541.
[6] I. KAPLANSKY, Rings of operators (Note prepared by S. K. Berberian with an appendix by R. Blattner), Univ. of Chicago Notes, 1955.
[7] T. OGASAWARA AND K. YOSHINAGA, A non-commutative theory of integration for operators, J. Sci. Hiroshima, 18(1955), 311–347.
[8] S. SAKAI, The theory of W^*-algebras, Mimeographed note, Yale Univ., 1962.
[9] I. E. SEGAL, A non-commutative extension of abstract integration, Ann. of Math., 57(1953), 401–457.
[10] J. VON. NEUMANN, Continuous geometry, Princeton, 1960.
[11] Ti Yen, Trace on finite AW^*-algebras, Duke Math. J., 22(1955), 207–222.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN