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KÄHLER SUBMANIFOLDS OF HOMOGENEOUS ALMOST HERMITIAN MANIFOLDS

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Let $M=(M, J, <\ ,\ >)$ be an almost Hermitian manifold, i.e., J is an almost complex structure on M, $<\ ,\ >$ is a Riemannian metric on M, and $<JX, JY> = <X, Y>$ for all vector fields X, Y on M. (Throughout this paper each almost Hermitian manifold M that we consider is always assumed to have a specific Riemannian metric and almost complex structure associated with it.) We define the *Kähler deficiency* of M to be the least integer $d(M)$ such that any Kähler submanifold of M must have (real) dimension $\leqq d(M)$. The purpose of this paper is to estimate the Kähler deficiency of certain homogeneous almost Hermitian manifolds of positive Euler characteristic.

The proofs of the following two propositions are easy.

PROPOSITION 1. *Let $M = M_1 \times \cdots \times M_r$ be a Riemannian product manifold. Suppose that $M, M_1, \cdots, M_r$ are all almost Hermitian with almost complex structures $J, J_1, \cdots, J_r$ such that $J = J_1 \oplus \cdots \oplus J_r$. Then*

$$d(M) = d(M_1) + \cdots + d(M_r).$$

PROPOSITION 2. *Let M be an almost Hermitian manifold and suppose $\widetilde{M}$ is a covering space of M. Then M is almost Hermitian and $d(\widetilde{M}) = d(M)$.*

Now let M be a compact homogeneous space of positive Euler characteristic. Because of Proposition 2, we may assume that M is simply connected and that $M=G/K$ where G is a compact semisimple Lie group acting effectively on M and K is the isotropy group at some point $p \in M$. Then rank G=rank K. We assume that the metric of M is determined by a biinvariant metric on G.

Denote by θ an automorphism of G of order 3 and assume that K is the

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fixed point set of θ. All such homogeneous spaces have been determined [5]. Since $\operatorname{rank} G = \operatorname{rank} K$, θ is an inner automorphism. Thus, on account of Proposition 1, we may assume that G is simple and M is irreducible. θ determines an almost complex structure J on M as follows. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Then we may write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \quad \text{(orthogonal direct sum)},$$

where $\mathfrak{m}$ may be identified with the tangent space to M at p. Let P denote the automorphism of $\mathfrak{g}$ determined by θ. Write

$$P|\mathfrak{m} = -\frac{1}{2} I + \frac{\sqrt{3}}{2} J,$$

where I denotes the identity. Then $J: \mathfrak{m} \to \mathfrak{m}$ is an almost complex structure. The action of G on M gives rise to an almost complex structure (also denoted by J) on all of M. We call this the *canonical almost complex structure* determined by θ. If the isotropy representation is irreducible (e.g., if K is a maximal subgroup) then up to sign the canonical almost complex structure is the only possible homogeneous almost complex structure on M [4].

The main result of this paper is the following theorem.

THEOREM 1. *Let $M=G/K$ be a compact irreducible simply connected homogeneous space of positive Euler characteristic. Assume that K is the fixed point set of an automorphism θ of G of order* 3 *and that J is the canonical almost complex structure on M determined by θ. Let M have a Riemannian metric determined by a biinvariant metric on G. If $M=G/K$ is not a Hermitian symmetric space, then*

$$d(M) \leqq \dim M - 4 .$$

PROOF. Let $X, Y \in \mathfrak{m}$. According to [5] we have

$$[X, JY]_{\mathfrak{m}} = -J[X, Y]_{\mathfrak{m}} .$$

Let $\overline{\nabla}$ be the Riemannian connection of M. Then

$$\begin{aligned} (1) \qquad \overline{\nabla}_X(J)(Y) &= \overline{\nabla}_X JY - J\overline{\nabla}_X Y \\ &= \frac{1}{2}[X, JY]_{\mathfrak{m}} - \frac{1}{2} J[X, Y]_{\mathfrak{m}} \\ &= -J[X, Y]_{\mathfrak{m}} . \end{aligned}$$

Thus if $M=G/K$ is not a Hermitian symmetric space, M is not Kählerian. We see, however, that $\overline{\nabla}_X(J)(X)=0$ for $X\in\mathfrak{m}$, and so M is *nearly Kählerian* in the sense of [3]. Now every nearly Kähler manifold of dimension 2 or 4 is Kählerian [2], and so we must have $\dim M\geqq 6$. Therefore it sufficess to prove that an almost complex submanifold N of dimension $\dim M-2$ cannot be Kählerian.

We may assume that $p\in N$. Then the tangent space of N at p can be identified with a subspace $\mathfrak{n}$ of $\mathfrak{m}$. Let T denote the configuration tensor field of N in M [2], and let ∇ be the Riemannian connection of N. Then

$$\overline{\nabla}_XY = \nabla_XY + T_XY$$

for $X,Y\in\mathfrak{n}$, and $T_XY\in\mathfrak{n}^\perp\subseteq\mathfrak{m}$. Hence by (1)

$$(2) \qquad -J[X,Y]_{\mathfrak{m}} = \nabla_X(J)(Y) + T_XJY - JT_XY\,.$$

Suppose that N is not totally geodesic in M at p. Then let $X\in\mathfrak{n}$ be the unit vector at which $Z\to\|T_ZZ\|^2$ achieves its maximum on the unit sphere of $\mathfrak{n}$. If $<X,Y> = <X,JY> = 0$, it is not hard to see that $<T_XX,T_XY> = 0$. Moreover, $<JT_XX,T_YY> = 0$ because $JT_XX=T_UU$ with $U=(X+JX)/\sqrt{2}$. Similarly T_XJY is perpendicular to both T_XX and JT_XX. Since T_XX and JT_XX span $\mathfrak{n}^\perp$ we must have $T_XY=T_XJY=0$.

Thus whether or not N is totally geodesic at p, there exists $X\in\mathfrak{n}$ with $\|X\|=1$ such that $T_XY=T_XJY=0$ whenever $<X,Y> = <X,JY> = 0$. Also, since M is nearly Kählerian, $T_XJX=JT_XX$. Now assume that N is Kählerian. Then (2) reduces to

$$(3) \qquad [X,Y]_{\mathfrak{m}}=0 \qquad \text{for all } Y\in\mathfrak{n}.$$

Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be the eigenspaces of J on $\mathfrak{m}\otimes\boldsymbol{C}$, and let $\mathfrak{n}_i=\mathfrak{m}_i\cap\mathfrak{n}$, $i=1,2$. Since N is an almost complex submanifold of M, we have $\mathfrak{n}\otimes\boldsymbol{C}=\mathfrak{n}_1\oplus\mathfrak{n}_2$. Then (3) implies the existence of $X\in\mathfrak{n}_1$, such that $X\neq 0$ and

$$(4) \qquad [X,Y]=0 \qquad \text{for all } Y\in\mathfrak{n}_1\,.$$

Next we decompose $\mathfrak{g}\otimes\boldsymbol{C}$ as

$$\mathfrak{g}\otimes\boldsymbol{C}=\mathfrak{h}\oplus\sum\{\mathfrak{g}^\alpha|\mathfrak{g}^\alpha\subseteq\mathfrak{k}\otimes\boldsymbol{C}\}\oplus\sum\{\mathfrak{g}^\alpha|\mathfrak{g}^\alpha\subseteq\mathfrak{m}_1\}\oplus\sum\{\mathfrak{g}^\alpha|\mathfrak{g}^\alpha\subseteq\mathfrak{m}_2\}$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}\otimes\boldsymbol{C}$ and $\mathfrak{g}\otimes\boldsymbol{C}$ and the $\mathfrak{g}^\alpha$ are the root spaces of $\mathfrak{g}\otimes\boldsymbol{C}$. For $i=1,2$ we set $\Delta_i=\{\alpha|\mathfrak{g}^\alpha\subseteq\mathfrak{m}_i\}$. We define an equivalence relation $\sim$ on Δ_1 as follows: $\alpha\sim\beta$ if and only if $\alpha=\beta$ or there exist roots $\alpha_0, \cdots, \alpha_p \in \Delta_1$ such that $\alpha = \alpha_0$, $\beta = \alpha_p$, and $\alpha_{i-1} + \alpha_i \in \Delta_2$ for $i = 1, \cdots, p$. From the classification of automorphisms of order 3 of a compact simple Lie algebra [5], it follows that all the roots in Δ_1 are equivalent to one another under $\sim$.

Let X and Y be as in (4). For each $\mathfrak{g}^\alpha$ let E_α be a basis vector of $\mathfrak{g}^\alpha$ such that $\langle E_\alpha, E_{-\alpha}\rangle = 1$ and $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ if $\alpha+\beta$ is a root. If $\alpha+\beta$ is not a root we set $N_{\alpha,\beta} = E_{\alpha+\beta} = 0$. Write

$$X = \sum \{x_\alpha E_\alpha \mid \alpha \in \Delta_1\} ,$$

$$Y = \sum \{y_\alpha E_\alpha \mid \alpha \in \Delta_1\} .$$

Then

$$0 = [X, Y] = \sum \{(x_\alpha y_\beta - x_\beta y_\alpha) N_{\alpha,\beta} E_{\alpha+\beta} \mid \alpha, \beta \in \Delta_1\} .$$

Since $\alpha \sim \beta$ for all $\alpha, \beta \in \Delta_1$, we must have

$$y_\alpha = \lambda x_\alpha \qquad \text{for all } \alpha \in \Delta_1 . \tag{5}$$

However, since $\dim N \geqq 4$, it is always possible to choose $Y \in \mathfrak{n}_1$ linearly independent from X, i.e., so that (5) is not satisfied.

Thus $\dim N = \dim M - 2$ is impossible. This completes the proof of Theorem 1.

COROLLARY. *In addition to the hypotheses of Theorem* 1 *suppose that* $\dim M = 6$. *Then M has no* 4*-dimensional almost complex submanifolds.*

The spaces to which this corollary applies consist of the following: $\dfrac{U(3)}{U(1)\times U(1)\times U(1)}$, $\dfrac{SO(5)}{U(1)\times SO(3)}$, $\dfrac{SO(5)}{U(2)}$, $\dfrac{SO(6)}{U(3)}$, $\dfrac{Sp(3)}{U(3)}$, and $\dfrac{G_2}{SU(3)} = S^6$. For the last space, the above corollary was proved in [3] by a different method.

In conclusion we prove a theorem about the curvature operator of a reductive homogeneous space G/K for which K is the fixed point set of an automorphism of order 3. Although $S^6 = G_2/SU(3)$ is locally symmetric, this is not true in general. However, it is possible to prove that a weak version of local symmetry holds.

THEOREM 2. *Let $M = G/K$ be a reductive homogeneous space for which K is the fixed point set of an automorphism θ of G of order* 3. *Let M have a pseudo-Riemannian metric determined by a biinvariant pseudo-Riemannian metric on G. Let σ be a geodesic in M with velocity vector* *field X. Then the holomorphic sectional curvature K_{XJX} is constant along σ, where J is the canonical almost complex structure defined by θ.*

PROOF. We may assume that $\sigma(0)=p$ where K is the isotropy subgroup of G at p. Then $X\in\mathfrak{m}$. Let $W, Y, Z\in\mathfrak{m}$. Then [1] we have

$$<\overline{\nabla}_W(\bar{R})_{YZ}Y, Z> = -<W, [[Y,Z]_{\mathfrak{k}}, [Y,Z]_{\mathfrak{m}}]>,$$

where $\bar{R}$ denotes the curvature operator of M. In particular

$$<\overline{\nabla}_W(\bar{R})_{XJX}X, JX> = -<W, [[X,JX]_{\mathfrak{k}}, [X,JX]_{\mathfrak{m}}]> = 0. \tag{6}$$

From (6) we obtain

$$X<\bar{R}_{XJX}X, JX> = 2<\bar{R}_{X\overline{\nabla}_X(J)(X)}X, JX> = 0. \tag{7}$$

Now Theorem 2 follows from (7).

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