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A NOTE ON QF-1 ALGEBRAS

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Let A be a finite-dimensional associative algebra with an identity element over a commutative field K, and every A-module be finite-dimensional as a vector space over K. R. M. Thrall [5] gave the following definition.

DEFINITION. An algebra A is said to be a $QF-1$ algebra if every faithful right A-module has the double centralizer property.

In this paper we shall establish a necessary and sufficient condition for an algebra to be a $QF-1$ algebra.

THEOREM. *The following statements are equivalent*:
(1) *An algebra A is a $QF-1$ algebra*,

(2) *For each exact sequence*

$$0 \longrightarrow A_A \longrightarrow M_A$$

such that $D(1) = M_A$, *where* $D = \mathrm{Hom}_A(M_A, M_A)$, *there exists the exact sequence*

$$0 \longrightarrow A_A \longrightarrow M_A \longrightarrow \prod M_A.$$

Preliminaries. Throughout this paper, rings will have an identity element and modules will be unital. M_R will denote, as usual, the fact that M_R is a right R-module. If M_R is a right R-module, $E(M_R)$ will denote the injective envelope of M_R. We adopt the notation that homomorphisms of modules will be written on the side opposite the scalars.

For the right R-module M_R, we define

$$D = D(M_R) = \mathrm{Hom}_R(M_R, M_R) \quad \text{and} \quad Q = Q(M_R) = \mathrm{Hom}_D({}_DM, {}_DM).$$

The ring $Q(M_R)$ is said to be a double centralizer of M_R. We say that M_R has the double centralizer property if every element of $Q(M_R)$ is obtained by the right multiplication of an element of R.

If there exists an R-monomorphism : $M_R \longrightarrow N_R$, then we express it also as $M_R \hookrightarrow N_R$ and identify M_R with the isomorphic image of M_R. A right R-module M_R is U_R-torsionless in case $M_R \hookrightarrow \prod U_R$, where $\prod U_R$ is a direct product of copies of U_R. It is easy to see that M_R is U_R-torsionless if and only if, for each $0 \neq m \in M_R$, there exists $f \in \mathrm{Hom}_R(M_R, U_R)$ such that $f(m) \neq 0$.

LEMMA 1. *If M_R has the double centralizer property and N_R satisfies one of the following conditions :*

(1) *N_R is M_R-torsionless,*

(2) *R-homomorphic images of M_R into N_R generate N_R,*

then $U_R = M_R \oplus N_R$ (direct sum) has the double centralizer property.

PROOF (see [3]). Let $q \in Q(U_R)$. It is clear that $(M_R)q \subset M_R$ and $(N_R)q \subset N_R$. Then there exists an element $r \in R$ such that $(m)q = mr$ for each $m \in M_R$. We set

$$(u)\bar{q} = (u)q - ur \text{ for each } u \in U_R.$$

Since $q \in Q(U_R)$ and $(M_R)\bar{q} = 0$, it is sufficient to prove $(n)\bar{q} = 0$ for each $n \in N_R$. Suppose that there thxsts $n \in N_R$ such that $(n)\bar{q} \neq 0$. If N_R satisfies the condition (1), then there exists $f \in \mathrm{Hom}_R(N_R, M_R)$ such that $f(nq) \neq 0$. On the other hand $f(n\bar{q}) = (fn)\bar{q} \in (M_R)\bar{q} = 0$. This is a contradiction.

In case N_R satisfies the condition (2), for each $n \in N_R$, there exist $f_i \in \mathrm{Hom}_R(M_R, N_R)$ and $m_i \in M_R$ such that $n = \sum_{finite} f_i(m_i)$. Therefore $(n)\bar{q} = \left(\sum f_i(m_i)\right)\bar{q} = \sum f_i(m_i\bar{q}) \in \sum f_i(M\bar{q}) = 0$. This proves the lemma.

The next lemma is a special case of Lemma 1.

LEMMA 2. *If M_R has the double centralizer property, then $\oplus M_R$ and $\prod M_R$ also have the double centralizer property.*

LEMMA 3. *If $\oplus M_R$ has the double centralizer property, then M_R also has the double centralizer property.*

PROOF (see [1]). Let $q \in Q(M_R) = \mathrm{Hom}_D({}_DM, {}_DM)$, where $D = \mathrm{Hom}_R(M_R, M_R)$. We define a mapping $\bar{q}$ of $\oplus M_R$ into $\oplus M_R$ by setting

$$(m_\alpha)\bar{q} = (m_\alpha q) \text{ for each } (m_\alpha) \in \oplus M_R.$$

It is sufficient to prove $\bar{q} \in \bar{Q} = \mathrm{Hom}_{\bar{D}}(\oplus M_R, \oplus M_R)$, where $\bar{D} = \mathrm{Hom}_R(\oplus M_R, \oplus M_R)$.

Let p be a projection : $\oplus M_R \longrightarrow M_R$, $d \in \overline{D}$ and $(m_\alpha) \in \oplus M_R$. We shall show $p[d\{(m_\alpha)\overline{q}\}] = p[\{d(m_\alpha)\}\overline{q}]$. Since $pdi_\alpha \in D$, where i_α denotes an injection : $M_R \longrightarrow \oplus M_R$, we have that

$$
\begin{aligned}
p[d\{(m_\alpha)\overline{q}\}] &= p[d(m_\alpha q)] \\
&= p\left[d\left\{\sum_{finite} i_\alpha(m_\alpha q)\right\}\right] \\
&= \sum pdi_\alpha(m_\alpha q) \\
&= \sum \{pdi_\alpha(m_\alpha)\} q \\
&= \left[pd\left\{\sum i_\alpha(m_\alpha)\right\}\right] q \\
&= \{pd(m_\alpha)\} q \\
&= p[\{d(m_\alpha)\}\overline{q}],
\end{aligned}
$$

and this completes the proof.

For the fixed right R-module M_R, we define

$$N_R^* = \mathrm{Hom}_R(N_R, M_R) \quad \text{and} \quad N_R^{**} = \mathrm{Hom}_D(N_R^*, {}_DM).$$

It is easy to see that N_R is M_R-torsionless if and only if the natural R-homomorphism $\sigma_{N_R} : N_R \longrightarrow N_R^{**}$ is R-monomorphism. A right R-module N_R is said to be M_R-reflexive if σ_{N_R} is R-isomorphism. In case M_R to be a faithful right R-module, M_R has the double centralizer property if and only if R_R is M_R-reflexive.

THEOREM 4. *There exists the exact sequence*

$$0 \longrightarrow R_R \longrightarrow M_R$$

such that $D(1) = M_R$, *where* $D = \mathrm{Hom}_R(M_R, M_R)$. *Then the following statements are equivalent* :

(1) *M_R has the double centralizer property,*

(2) $M_R/R_R \subsetneq \prod M_R$.

PROOF. Since there exists the R-exact sequence

$$0 \longrightarrow R_R \longrightarrow M_R \longrightarrow M_R/R_R \longrightarrow 0,$$

we have the D-exact sequence

$$0 \longrightarrow (M_R/R_R)^* \longrightarrow M_R{}^* \longrightarrow R_R{}^*.$$

It follows that the right hand D-homomorphism is D-epimorphism by $D(1)=M_R$. Then the D-exact sequence

$$0 \longrightarrow (M_R/R_R)^* \longrightarrow M_R{}^* \longrightarrow R_R{}^* \longrightarrow 0$$

induces the R-exact sequence

$$0 \longrightarrow R_R{}^{**} \longrightarrow M_R{}^{**} \longrightarrow (M_R/R_R)^{**}.$$

Furthermore we can easily show that the following diagram is commutative with all rows and columns R-exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & R_R & \longrightarrow & M_R & \longrightarrow & M_R/R_R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R_R{}^{**} & \longrightarrow & M_R{}^{**} & \longrightarrow & (M_R/R_R)^{**} & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

In this commutative diagram, we can prove that $\sigma_{R_R} : R_R \longrightarrow R_R{}^{**}$ is R-epimorphism if and only if $\sigma_{M_R/R_R} : M_R/R_R \longrightarrow (M_R/R_R)^{**}$ is R-monomorphism. This shows the equivalence of the theorem.

By Lemma 2, Lemma 3 and Theorem 4, we can prove the next result.

COROLLARY (H. Tachikawa [4], T. Kato [2]). *The following conditions on a ring R are equivalent* :

(1) *The injective envelope* $E(R_R)$ *of* R_R *has the double centralizer property,*

(2) $E(R_R)/R_R \hookrightarrow \prod E(R_R)$,

(3) *Each finitely-faithful, injective right R-module has the double centralizer property.*

Here a right R-module M_R is said to be a finitely-faithful right R-module if there exists the following exact sequence

$$0 \longrightarrow R_R \longrightarrow \bigoplus_{finite} M_R.$$

QF−1 Algebra. In this section, we shall prove our main theorem.

THEOREM. *The following statements are equivalent* :
(1) *An algebra* A *is a* $QF-1$ *algebra,*
(2) *For each exact sequence*

$$0 \longrightarrow A_A \longrightarrow M_A$$

such that $D(1)=M_R$ *where* $D=\mathrm{Hom}_A(M_A, M_A)$, *there exists the exact sequence*

$$0 \longrightarrow A_A \longrightarrow M_A \longrightarrow \prod M_A.$$

PROOF. It is clear that (1) implies (2). We shall only show that (2) implies (1). Let $\{m_1, \cdots, m_n\}$ be a K-basis of a faithful right A-module M_A. We define a D-homomorphism of $\overset{n}{\oplus} D$ (where $D=\mathrm{Hom}_A(M_A, M_A)$) into ${}_DM$ by setting

$$d=(d_1, \cdots, d_n) \longrightarrow d_1m_1 + \cdots + d_nm_n,$$
$$\text{for each } d=(d_1, \cdots, d_n) \in \overset{n}{\oplus} D.$$

Since a mapping $k_R(k \in K): m \longrightarrow mk$ for each $m \in M_A$ is an element of D, we have the D-exact sequence

$$\overset{n}{\oplus} D \longrightarrow {}_DM \longrightarrow 0$$

It follows that there exists the Q-exact sequence

$$0 \longrightarrow Q = \mathrm{Hom}_D({}_DM, {}_DM) \longrightarrow \mathrm{Hom}_D(\overset{n}{\oplus} D, {}_DM).$$

A right regular A-module A_A is naturally imbedded in Q by a faithfulness of M_A. Thus there exists the A-exact sequence

$$0 \longrightarrow A_A \xrightarrow{\ f\ } \overset{n}{\oplus} M_A,$$

where $f(a)=(m_1a, \cdots, m_na)$ for each $a \in A_A$. If we pay attention to the fact that

$$m_1D + \cdots + m_nD = M_A \quad \text{and} \quad \overline{D} = \mathrm{Hom}_A(\overset{n}{\oplus} M_A, \overset{n}{\oplus} M_A) \approx \overset{n}{\oplus}\overset{n}{\oplus} D,$$

we can easily show that $\overline{D}(1)=\overset{n}{\oplus} M_A$. Therefore (2) implies that the right A-module $\overset{n}{\oplus} M_A$ has the double centralizer property, by Theorem 4. This shows that the right A-module M_A has the double centralizer property, by Lemma 3. This completes the proof.

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