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SOME REMARKS ON LOCAL MARTINGALES

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Let (Ω, F, P) be a basic probability space where F is complete with respect to P and let $\{F(t)\}_{0\leqq t<\infty}$ be an increasing family of Borel subfields of F. In what follows, we suppose that the family $\{F(t)\}_{0\leqq t<\infty}$ is right continuous and has no time of discontinuity.

We call a family $T=\{F(t), \tau_t\}_{0\leqq t<\infty}$ a *time change function* with respect to the family $\{F(t)\}_{0\leqq t<\infty}$, if

(1) for each $u\in[0, \infty)$, τ_u is a stopping time with respect to the family $\{F(t)\}_{0\leqq t<\infty}$ and $\tau_u<\infty$,

(2) for almost all ω, $[0, \infty)\ni u\rightarrow\tau_u(\omega)$ is a continuous and strictly increasing function with $\tau_0(\omega)=0$.

For a right continuous stochastic process $X=\{x_t, F(t)\}_{0\leqq t<\infty}$ and a time change function $T=\{F(t), \tau_t\}_{0\leqq t<\infty}$, we can define a new stochastic process $TX=\{x_{\tau_t}, F(\tau_t)\}_{0\leqq t<\infty}$ and we call it the stochastic process obtained from X by a time change with respect to T. In particular, if $X^a=\{x_t^a, F(t)\}_{0\leqq t<\infty}$, $a\in A$, where A is an arbitrary set, is a collection of continuous stochastic processes such that

$$\sup\{|x_s^a-x_0^a|;\ 0\leqq s\leqq t,\ a\in A\}$$

is continuous, then we call the time change function

$$\Theta=\{F(t), \theta_t\}_{0\leqq t<\infty},$$

the *stopping process* or the ***brake*** of the processes X^a, where $\theta_t=\inf\{u;\lambda_u>t\}$ and $\lambda_t=t+\sup\{|x_s^a-x_0^a|;\ 0\leqq s\leqq t,\ a\in A\}$.

In the followings we assume that $x_0=0$ and X is quasi-continuous from the left. We call a martingale $X=\{x_t, F(t)\}_{0\leqq t<\infty}$ an L_∞^**-martingale* if for each t

$$P\{\sup_{0\leqq u\leqq t}|x_u|\leqq c_t\}=1$$

where c_t is some constant with $c_0=c_{0+}=0$. Let M designate the set of all right continuous local martingales which can be transformed into L_∞^*-martingales by means of time changes.

K. E. Dambis [1, 1965] has proved that any q-martingale (that is to say, any continuous local martingale [2, 1968]) belongs to M. The purpose of this note is to give a necessary and sufficient condition for the assertion that a right continuous local martingale belongs to M.

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THEOREM 1. *In order that a right continuous local martingale* $X=\{x_t, F(t)\}_{0\leq t<\infty}$ *belongs to* M, *it is necessary and sufficient that there exists a continuous stochastic process* $Y=\{y_t, F(t)\}_{0\leq t<\infty}$ *with* $y_0=0$ *such that we have*

$$P\{\sup_{0\leqq u\leqq t}|x_u|\leqq\sup_{0\leqq u\leqq t}|y_u|\}=1$$

for each t. *Here we need not assume that* y_t *is integrable.*

PROOF. Necessity. If $X=\{x_t,F(t)\}_{0\leqq t<\infty}$ belongs to M, by the definition of M there exist non negative constant c_t and some time change function $T=\{F(t),\tau_t\}_{0\leqq t<\infty}$ satisfying $c_0=c_{0+}=0$ and

$$P\{\sup_{0\leqq u\leqq t}|x_{\tau_u}|\leqq c^+\}=1$$

for each t. We put $c_t'=\inf_{t\leqq u}c_u$, $c_t''=\lim_{h\downarrow 0}c_{t+h}'$ and $c_t^*=\dfrac{1}{t}\displaystyle\int_t^{2t}c_u''\,du$. Then c_t^* is continuous in t and clearly we have

$$P\{\sup_{0\leqq u\leqq t}|x_{\tau_u}|\leqq c_t^*\}=1$$

for each t. Therefore we have $P\{\sup_{0\leqq u\leqq t}|x_u|\leqq c_{\phi_t}^*\}=1$, where $\phi_t=\inf\{u\,;\tau_u>t\}$, and $\{c_{\phi_t}^*, F(t)\}_{0\leqq t<\infty}$ is a continuous stochastic process satisfying $c_{\phi_0}^*=c_0^*=0$. Put $y_t=c_{\phi_t}^*$ and we have the desired.

Sufficiency. Let $Y=\{y_t, F(t)\}_{0\leqq t<\infty}$ be a continuous stochastic process satisfying $y_0=0$ and $P\{\sup_{0\leqq u\leqq t}|x_u|\leqq\sup_{0\leqq u\leqq t}|y_u|\}=1$ for each t. Then the brake of Y, $\Theta=\{F(t),\theta_t\}_{0\leqq t<\infty}$, is a time change function and it is easy to see that

$$P\{\sup_{0\leqq u\leqq t}|x_{\theta_u}|\leqq\sup_{0\leqq u\leqq t}|y_{\theta_u}|\leqq t\}=1$$

for each t and $\Theta X=\{x_{\theta_t}, F(\theta_t)\}_{0\leq t<\infty}$ is an L^*_∞-martingale. This implies that $X=\{x_t, F(t)\}_{0\leq t<\infty}$ is an element of M.

This completes the proof.

It is easy to show that $\alpha X^{(1)}+\beta X^{(2)}$ belongs to M for any $X^{(k)}=\{x_t^{(k)}, F(t)\}_{0\leq t<\infty}\in M$, $k=1,2$ and any real numbers α, β. Let $F^{(n)}$ be a sequence of σ-fields. By $\lim_{n\to\infty} F^{(n)}$ we mean the σ-field of all sets A for which there exists an $A^{(n)}\in F^{(n)}$ for each n such that $\lim_{n\to\infty} P(A\triangle A^{(n)})=0$.

Let $X^{(n)}=\{x_t^{(n)}, F^{(n)}(t)\}_{0\leq t<\infty}$, $n=1,2,\cdots$ and $X=\{x_t, F(t)\}_{0\leq t<\infty}$ be stochastic processes. We say that the sequence $X^{(n)}$ converges uniformly almost surely to X if $F(t)=F'(t+0)$ where $F'(t)=\lim_{n\to\infty} F^{(n)}(t)$ and there exist continuous processes

$$Y^{(n)}=\{y_t^{(n)}, F^{(n)}(t)\}_{0\leq t<\infty} \quad\text{and}\quad Y=\{y_t, F(t)\}_{0\leq t<\infty}$$

such that for each t

$$\lim_{n\to\infty}\sup_{0\leq u\leq t}\max\{|x_u^{(n)}-x_u|,\ |y_u^{(n)}-y_u|\}=0,$$

$$y_0^{(n)}=0,\quad \sup_{0\leq u\leq t}|x_u^{(n)}|\leqq\sup_{0\leq u\leq t}|y_u^{(n)}|,\quad n=1,2,\cdots$$

with probability 1.

THEOREM 2. *In order that the stochastic process* $X=\{x_t, F(t)\}_{0\leq t<\infty}$ *belongs to* M, *it is necessary and sufficient that there exists a sequence* $X^{(n)}=\{x_t^{(n)}, F(t)\}_{0\leq t<\infty}$ *of right continuous martingales converging uniformly almost surely to* X.

PROOF. Sufficiency. We shall divide the proof into three portions.

Case(1). First we shall consider the simple case that $x_t^{(n)}$ is uniformly integrable with respect to n, for fixed t.

Let $A\in F(s)$. Then for each n

$$\int_A x_s^{(n)}dP=\int_A x_t^{(n)}dP,\qquad s\leqq t.$$

Passing to the limit, we see, in view of the uniform integrability of $x_s^{(n)}$ and $x_t^{(n)}$, that $X=\{x_t, F(t)\}_{0\leq t<\infty}$ is a martingale. It is easy to see that we have

$$\sup_{0\leq u\leq t}|x_u|\leqq\sup_{0\leq u\leq t}|y_u|,\quad y_0=0\quad\text{a.s.}$$

In view of Theorem 1, $X=\{x_t, F(t)\}_{0\leqq t<\infty}$ belongs to M.

Case (2). Assume that we can define the brake $\Theta=\{F(t), \theta_t\}_{0\leqq t<\infty}$ of $\{Y^{(n)}, Y; n=1,2,\cdots\}$. As $\lambda_t = t + \sup_{\substack{0\leqq u<t \\ n=1,2,\cdots}} \max(|y_u^{(n)}|, |y_u|)$, clearly we have

$$P\{\sup_{0\leqq u<t} \max[|y_{\theta_u}^{(n)}|, |y_{\theta_u}|]\leqq t\} = 1 .$$

That is to say, $x_{\theta_t}^{(n)}$ is uniformly integrable with respect to n, for fixed t. As $\Theta Y^{(n)}$, $n=1,2,\cdots$ and ΘY are continuous stochastic processes and $\Theta X^{(n)}$ converges uniformly almost surely to ΘX, we have $\Theta X \in M$ in view of case (1). Hence X is an element of M.

Case (3). Let us go over to the general case. Let $\Theta^{(n)}=\{F(t), \theta_t^{(n)}\}_{0\leqq t<\infty}$ and $\Theta=\{F(t), \theta_t\}_{0\leqq t<\infty}$ be the brakes of $\{Y^{(n)}, Y\}$ and Y respectively. It is easy to show that we have with probability 1

$$\theta_t^{(n)} \leqq \theta_t, \quad \sup_{0\leqq u\leqq t}|x_{\theta_u}| \leqq t ,$$

$$\lambda_t^{(n)} \equiv t + \sup_{0\leqq u\leqq t} \mathrm{Max}(|y_u^{(n)}|, |y_u|)$$

tends to $\lambda_t \equiv t + \sup_{0\leqq u\leqq t}|y_u|$ by the convergence property of $X^{(n)}$ and then

$$\lim_{n\to\infty} \theta_t^{(n)} = \theta_t$$

for each t. We put $\theta_N^{*(n)} = \inf\{\theta_N^{(k)};\ k\geqq n\}$. Then each $\theta_N^{*(n)}$ is a stopping time with respect to the family $\{F(t)\}_{0\leqq t<\infty}$, and

$$P\{\theta_N^{*(n)} \uparrow \theta_N(n\uparrow\infty)\} = 1$$

for each $N=1,2,\cdots$.

By the triangle inequality we have

$$|x^{(n)}_{\theta_N^{*(n)}\wedge t} - x_{\theta_N\wedge t}| \leqq |x^{(n)}_{\theta_N^{*(n)}\wedge t} - x_{\theta_N^{*(n)}\wedge t}| + |x_{\theta_N^{*(n)}\wedge t} - x_{\theta_N\wedge t}|$$

$$\leqq \sup_{0\leqq u\leqq t} |x_u^{(n)} - x_u| + |x_{\theta_N^{*(n)}\wedge t} - x_{\theta_N\wedge t}| .$$

and we have

$$P\{\lim_{n\to\infty} x^{(n)}_{\theta_N^{*(n)}\wedge t} = x_{\theta_N\wedge t}\} = 1$$

for each N, because $X^{(n)}$ converges uniformly almost surely to X and the process X is quasi-continuous from the left. For each n and N, $\{x^{(n)}_{\theta_N^{*(n)}\wedge t}, F(\theta_N^{*(n)}\wedge t)\}_{0\leqq t\leqq\infty}$ is a martingale and from the fact $|x^{(n)}_{\theta_N^{*(n)}\wedge t}|\leqq N$, $x^{(n)}_{\theta_N^{*(n)}\wedge t}$ is uniformly integrable with respect to n, for fixed t. On the other hand, by the assumption that the family $\{F(t)\}_{0\leqq t<\infty}$ has no time of discontinuity, for any $A \in F_{\theta_N \wedge s}$ there exists $A^{(n)} \in F_{\theta^*_N{}^{(n)}\wedge s}$ such that $P(A \triangle A^{(n)})$ converges to 0. Therefore for each n we have

$$\int_{A^{(n)}} x^{(n)}_{\theta^*_N{}^{(n)}\wedge s}\, dP = \int_{A^{(n)}} x^{(n)}_{\theta^*_N{}^{(n)}\wedge t}\, dP, \quad s \leqq t\,.$$

In view of the Lebesgue bounded convergence theorem we have

$$\int_A x_{\theta_N \wedge s}\, dP = \int_A x_{\theta_N \wedge t}\, dP, \quad s \leqq t\,.$$

Thus for fixed N, $\{x_{\theta_N\wedge t}, F(\theta_N \wedge t)\}_{0\leqq t<\infty}$ is a martingale. On the other hand, as $A[\theta_N > s] \in F(\theta_N \wedge s)$ for $A \in F(s)$, we have for $s<t$

$$\begin{aligned}\int_A x_{\theta_N\wedge s}\, dP &= \int_{A[\theta_N \leqq s]} x_{\theta_N \wedge s}\, dP + \int_{A[\theta_N > s]} x_{\theta_N\wedge s}\, dP \\ &= \int_{A[\theta_N \leqq s]} x_{\theta_N \wedge t}\, dP + \int_{A[\theta_N > s]} x_{\theta_N\wedge t}\, dP \\ &= \int_A x_{\theta_N\wedge t}\, dP\,.\end{aligned}$$

Therefore $\{x_{\theta_N\wedge t}, F(t)\}_{0\leqq t<\infty}$ is a martingale.

Now we put $X^{(n)} = \{x_{\theta_N\wedge t}, F(t)\}_{0\leqq t<\infty}$ and $Y^{(N)} = \{y_{\theta_N\wedge t}, F(t)\}_{0\leqq t<\infty}$.

Then the brake of $\{Y^{(n)}, Y\,; N = 1, 2, \cdots\}$ exists and coincides with the brake Θ of Y. As $X^{(N)}$ converges uniformly almost surely to X, from the case (2) we see that X is an element of M. This completes the proof of the sufficiency.

Necessity. Let $X = \{x_t, F(t)\}_{0\leqq t<\infty}$ be an element of M, that is to say, we assume that there exists a continuous stochastic process $Y^* = \{y^*_t, F(t)\}_{0\leqq t<\infty}$ with $y_0 = 0$ satisfying

$$\sup_{0\leqq u\leqq t} |x_u| \leqq \sup_{0\leqq u\leqq t} |y^*_u| \qquad \text{a. s}\,.$$

for each t. If $\Theta = \{F(t), \theta_t\}_{0\leqq t<\infty}$ is the brake of Y^*, then $X^{(n)} = \{x_{t\wedge\theta_n}, F(t)\}_{0\leqq t<\infty}$ satisfies that

$$\sup_{0\leqq t<\infty} |x_{t\wedge\tau_n}| \leqq n \qquad \text{a. s.}$$

for each $n=1,2,\cdots$ From this fact we may deduce that $X^{(n)}$ is a martingale for each n.

We put $x_t^{(n)}=x_{t\wedge\theta_n}$. Then clearly we have for each t

$$\sup_{0\leqq u\leqq t} |x_u^{(n)}| \leqq \sup_{0\leqq u\leqq t}|y^*_{u\wedge\theta_n}| = \sup_{0\leqq u\leqq t}|y^*_{\theta_{\lambda_u}\wedge u}| \leqq n\wedge\lambda_t \quad \text{a. s.}$$

where $\lambda_t=\inf\{u\,;\,\theta_u>t\}$. If we put $y_t^{(n)}=n\wedge\lambda_t$ and $y_t=\lambda_t$, then $Y^{(n)}=\{y_t^{(n)}, F(t)\}_{0\leqq t<\infty}$ and $Y=\{y_t, F(t)\}_{0\leqq t<\infty}$ are continuous stochastic processes and

$$\lim_{n\to\infty}\sup_{0\leqq u\leqq t}|y_u^{(n)}-y_u| = 0, \quad y_0^{(n)}=y_0=0 \qquad \text{a. s.}$$

As $\lim_{n\to\infty}\sup_{0\leqq u\leqq t}|x_u^{(n)}-x_u|=0$ a. s., $X^{(n)}=\{x_t^{(n)}, F(t)\}_{0\leqq t<\infty}$ converges uniformly almost surely to X. This completes the proof.

THEOREM 3. *If a sequence $X^{(n)}=\{x_t^{(n)}, F(t)\}_{0\leqq t<\infty}$ of M satisfies that for each t*

$$\lim_{m,n\to\infty}\sup_{0\leqq u\leqq t}\max\{|x_u^{(n)}-x_u^{(m)}|, |y_u^{(n)}-y_u^{(m)}|\} = 0$$

almost surely, then $X^{(n)}$ converges uniformly almost surely to some element $X=\{x_t, F(t)\}_{0\leqq t<\infty}$ of M.

PROOF. It is easy to see that $X^{(n)}$ converges uniformly almost surely to some process $X=\{x_t, F(t)\}_{0\leqq t<\infty}$, where $x_t=\lim_{n\to\infty} x_t^{(n)}$ a. s., and there exists a brake $\Theta=\{F(t), \theta_t\}_{0\leqq t<\infty}$ of $\{Y^{(n)}; n=1,2,\cdots\}$. Then a sequence $\Theta X^{(n)}=\{x_{\theta_t}^{(n)}, F(\theta_t)\}_{0\leqq t<\infty}$ of right continuous martingales converges uniformly almost surely to $\Theta X=\{x_{\theta_t}, F(\theta_t)\}_{0\leqq t<\infty}$. In view of Theorem 2, ΘX belongs to M and therefore X belongs to M. This completes the proof.

We can generalize Theorem 1 as the follows:

THEOREM 4. *Let $X=\{x_t, F(t)\}_{0\leqq t<\infty}$ be a right continuous stochastic process (not necessary $x_0=0$).*

In order that for some time change function $T=\{F(t), \tau_t\}_{0\leqq t<\infty}$,

$$\sup_{0\leqq u\leqq t}|x_{\tau_u}| \leqq c_t+\xi_0 \qquad a.\,s.$$

where $\{c_t\}_{0\leqq t<\infty}$ is a constant process and ξ_0 is a F_0-measurable random *variable, it is necessary and sufficient that there exists a continuous process* $Y=\{y_t, F(t)\}_{0\leq t<\infty}$ *(not necessary* $y_0=0$*) satisfying the inequality*

$$\sup_{0\leqq u\leqq t}|x_u| \leqq \sup_{0\leqq u\leqq t}|y_u| \qquad a.s.$$

for each t.

PROOF. Sufficiency. Let $Y=\{y_t, F(t)\}_{0\leq t<\infty}$ be a continuous stochastic process satisfying for each t

$$\sup_{0\leqq u\leqq t}|x_u| \leqq \sup_{0\leqq u\leqq t}|y_u|$$

with probability 1. Then the brake $\Theta=\{F(t), \theta_t\}_{0\leq t<\infty}$ of the process $\{y_t-y_0, F(t)\}_{0\leq t<\infty}$ is a time change function and it is easy to check

$$\sup_{0\leqq u\leqq t}|x_{\theta_u}| \leqq \sup_{0\leqq u\leqq t}|y_{\theta_u}-y_0|+|y_0| \leqq t+|y_0| \text{ a. s.}$$

Necessity. We put $c'_t=\inf_{t\leqq u} c_u$ and $c''_t=c'_{t+0}-c'_{0+}$. Then $\{c^*_t\}_{0\leq t<\infty}$ where $c^*_t=\dfrac{1}{t}\displaystyle\int_t^{t} c''_u du$, is a continuous process with $c^*_0=0$ and we have

$$\sup_{0\leqq u\leqq t}|x_{\tau_u}| \leqq c^*_t+\xi^*_0 \qquad \text{a. s.}$$

where $\xi^*_0=\xi_0+c'_{0+}$ (this is clearly F_0-measurable).

Therefore we have

$$\sup_{0\leqq u\leqq t}|x_u| \leqq c^*_{\phi_t}+\xi^*_0 \qquad \text{a. s.}$$

where $\phi_t=\inf\{u\,;\,\tau_u>t\}$, and $\{c^*_{\phi_t}+\xi^*_0, F(t)\}_{0\leq t<\infty}$ is a continuous stochastic process. This completes the proof.

REMARK. We may assume that c_t is continuous. Then $\xi_0=0$ if and only if $y_0=0$. Moreover for each p, ξ_0 is L_p-integrable if and only if y_0 is so.

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