ON A SEMI-DIRECT PRODUCT DECOMPOSITION OF AFFINE GROUPS OVER A FIELD OF CHARACTERISTIC 0

MITSUHIRO TAKEUCHI

(Received Feb. 8, 1972)

Let $G$ be an abstract group and $f_1: G \to GL_k(V_1)$ and $f_2: G \to GL_k(V_2)$ be two finite dimensional representations of $G$ over a field $k$ of characteristic 0. It is well known that if $f_1$ and $f_2$ are semi-simple then $f_1 \otimes f_2$ is also semi-simple [3, IV, §3, 3.5; 4, Th. 12.2]. In view of this fact we first show

**Proposition 0.** The coradical $R$ of a commutative Hopf algebra $A$ over a field $k$ of characteristic 0 is a sub-Hopf algebra.

**Proof.** The Jacobson radical of a ring is the intersection of its maximal left (or right) ideals. Dually the coradical of a coalgebra $C$ over a field is identical with the socle of $C$ as a right (or left) $C$-comodule. Since $k$ is perfect, the coradical of $\bar{k} \otimes_k A$ is $\bar{k} \otimes_k R$, where $\bar{k}$ is the algebraic closure of $k$. Hence we can assume that $k = \bar{k}$. Moreover $A$ can be assumed to be finitely generated as a $k$-algebra. Let $V_i, i = 1, 2$ be two finite dimensional right $A$-comodules. Because $G(A^\circ) = \text{Alg}_k(A, k)$ is dense in $A^\circ = \text{Hom}_k(A, k)$ [6, Lem. 3.6], $V_i$ is a semisimple $A$-comodule iff $V_i$ is a semisimple left $G(A^\circ)$-module. Hence by the remark above if $V_i$ are semisimple, then $V_1 \otimes V_2$ is also semi-simple. This means that $R \otimes R$ is a semisimple right $A$-comodule. Since the multiplication $\mu: A \otimes A \to A$ is a right $A$-comodule map, $R \cdot R$ is contained in $R$. Clearly $R$ is stable under the antipode of $A$. Hence $R$ is a sub-Hopf algebra of $A$.

The purpose of this paper is to prove

**Theorem 1.** Let $A$ be a commutative Hopf algebra over a field $k$ of characteristic 0 and $R$ its coradical. Then there exists a Hopf algebra map $\pi: A \to R$ such that $\pi = \text{identity}$ on $R$.

This follows from the following three propositions, where $A$ and $R$ are as in Theorem 1. We assume $k$ is of characteristic 0 throughout this paper.

**Proposition 2.** Let $B$ be a sub-Hopf algebra of $A$ which contains $R$. If $B \neq A$, then there exists a sub-Hopf algebra $C$ of $A$ which contains $B$ properly such that $C/(C \cdot B^+) \text{ is cocommutative}, \text{ where } B^+ = \text{Ker } (\varepsilon: B \to k)$. 
PROPOSITION 3. Let $B$ be a sub-Hopf algebra of $A$ which contains $R$. If $\pi : B \rightarrow R$ is a Hopf algebra map such that $\pi = \text{identity on } R$, then the ideal $I$ of $A$ generated by $\pi^{-1}(0)$ is a Hopf ideal of $A$ and we have $I \cap R = 0$.

PROPOSITION 4. Theorem 1 is valid when $A/(A \cdot R^+)$ is cocommutative.

First we show that Propositions 2, 3 and 4 imply Theorem 1. Indeed consider the set of pairs $(B, \pi)$, where $B$ is a sub-Hopf algebra of $A$ which contains $R$ and $\pi$ is a Hopf algebra map from $B$ to $R$ which is identity on $R$. Introducing the usual ordering on this set, take a maximal element $(B, \pi)$ by Zorn's lemma. Assume $B \neq A$. Then there exists a sub-Hopf algebra $C$ of $A$ as in Proposition 2. Let $I$ be the ideal of $C$ generated by $\pi^{-1}(0)$. Then by Proposition 3 we have $R \subseteq C/I$. $R$ can be identified with the coradical of $C/I$ [5, Exercise 4), p. 182]. Because we have $B^+ = \pi^{-1}(0) + R^+$, $(C/I)/(C/I \cdot R^+)$ is a quotient Hopf algebra of a cocommutative Hopf algebra $C/(C \cdot B^+)$ and therefore $R \subseteq C/I$ satisfies the condition in Proposition 4. Hence there exists a Hopf algebra map $\rho : C/I \rightarrow R$ such that $\rho = 1$ on $R$. Since we have $b - \pi(b) \in I$ for any $b \in B$, the composite $B \subseteq C \rightarrow C/I \xrightarrow{\rho} R$ is identical with $\pi$. Hence $(C, \rho)$ is properly larger than $(B, \pi)$. This is a contradiction. So $B$ is equal to $A$ and the proof of Theorem 1 is complete.

Let $B$ be a sub-Hopf algebra of $A$ which contains $R$. Then $H = A/(A \cdot B^+)$ is an irreducible Hopf algebra since its coradical is contained in the image of $R \rightarrow H$ [5, Exercise 4), p. 182]. We have shown in [6, Lemma 4.2] that $H$ is a quotient $A$-comodule of $A$ under the left $A$-comodule structure

$$\rho : A \rightarrow A \otimes A, \ a \mapsto \sum a_{(1)} S(a_{(2)}) \otimes a_{(2)}.$$ 

LEMMA 5. $P(H) = \{\text{the primitive elements of } H\}$ is a sub-$A$-comodule of $H$. If $H$ is cocommutative, then $H$ is a left $B$-comodule, i.e. $\rho(H) \subseteq B \otimes H$, where $\rho$ is the left $A$-comodule structure map of $H$.

PROOF. It is easy to see that we can assume $k$ is algebraically closed and $A$ is finitely generated. Then $G(A^e)$ is dense in $A^*$ since $k$ is of characteristic 0. Hence it suffices to show that $P(H)$ is $G(A^e)$-stable. But this is clear because the $G(A^e)$-action on $H$ is compatible with the Hopf algebra structure of $H$. Next we suppose that $H$ is cocommutative. Since $A$ is faithfully flat over $B$ [6, Th. 3.1], it suffices to show we have

$$\sum a_{(1)} S(a_{(2)}) \otimes 1 \otimes a_{(3)} = \sum 1 \otimes a_{(1)} S(a_{(2)}) \otimes a_{(3)}$$

in $A \otimes_B A \otimes H$ for any $a \in A$ in order to prove $\rho(H) \subseteq B \otimes H$. But we have an isomorphism [6, Lem. 3.9]
A \otimes_k A \rightarrow A \otimes H, x \otimes y \mapsto \sum xy_{(1)} \otimes y_{(3)}. \\
Through this isomorphism the equation above reduces to \\
\sum a_{(1)} S(a_{(3)}) \otimes 1 \otimes a_{(2)} = \sum a_{(1)} S(a_{(3)}) \otimes a_{(2)} S(a_{(4)}) \otimes a_{(5)}

in A \otimes H \otimes H, which is valid by the cocommutativity of H.

PROOF OF PROPOSITION 2. In the notation of Lemma 5, H = A/(A \cdot B^+) \neq k implies P(H) \neq 0 [5, Cor. 11.0.2]. On the other hand Lemma 5 means that the kernel J of A \rightarrow H \rightarrow H/H \cdot P(H) is a normal Hopf ideal of A [6, Def. 4.1]. Hence there exists a sub-Hopf algebra C of A such that J = A \cdot C^+ [6, Th. 4.3]. Because B \subset C \subset A, we have C/(C \cdot B^+) \subset A/(A \cdot B^+) [6, Proof of Th. 3.1]. In view of \\
H/(H \cdot (C/(C \cdot B^+))^+) = A/(A \cdot C^+) = H/(H \cdot P(H))

we have C/(C \cdot B^+) = k[P(H)] [6, Th. 3.10]. Because P(H) \neq 0, C contains B properly. Hence the proof of Proposition 2 is complete.

PROOF OF PROPOSITION 3. I is clearly a Hopf ideal of A. Because A is faithfully flat over B [6, Th. 3.1], we have I \cap B = \pi^{-1}(0) [2, I, §3, n° 5, Prop. 9, d)]. Hence I \cap R = 0.

PROOF OF PROPOSITION 4. First we show that H = A/(A \cdot R^+) can be assumed to be finitely generated. Indeed in the argument below Proposition 4, we can assume C/(C \cdot B^+) is finitely generated. Then (C/I)/(C/I \cdot R^+) is also finitely generated, and therefore \rho: C/I \rightarrow R exists. Now V = P(H) is a finite dimensional vector space over k and H is isomorphic as a Hopf algebra to U(V), the universal enveloping algebra of V, since H is irreducible cocommutative [5, Th. 13.0.1]. Hence \mathfrak{S}(H), the affine k-group represented by H, is isomorphic to \mathfrak{S}(V) = (V^*)^\times [3, II, §1.2.1]. Since [3, III, §4.6.6] can be extended to “torseurs durs” [3, III, §5.1.4], the following exact sequence

(*) 0 \rightarrow \mathfrak{S}(H) \rightarrow \mathfrak{S}(A) \rightarrow \mathfrak{S}(R) \rightarrow 1

in \text{Gr}_k [3, III, §3.7.2] is an “H-extension” [3, II, §3.2.1]. On the other hand by Lemma 5, P(H) is a sub-R-comodule of H. This means that the action of \mathfrak{S}(R) on \mathfrak{S}(H) = \mathfrak{S}(V), which is determined naturally by (*), is linear [3, II, §2.1.1]. Since R is co-semi-simple, the Hochschild cohomology H^2(\mathfrak{S}(R), \mathfrak{S}(H)) is zero [3, II, §3.3.7]. By [3, II, §3.2.3] the extension (*) splits and the proof of Theorem 1 is complete.

COMMENT 6. Theorem 1 is essentially proved in [4, Th. 14.2]. But the proof there is due to Levi’s theorem on the structure of Lie algebra [4, Th. 13.5]. Our proof is free from Lie algebra theory. Abe and Doi
announce the same results in [1, (3.5) and (3.6)]. But I think their proof has gaps. Now we have used the extension theory of affine algebraic $k$-groups in [3, III, §6] to prove Proposition 4. But it is easy to translate it into the Hopf algebra language and prove it again.

**COROLLARY 7.** Let $A$ be a commutative Hopf algebra over a field of characteristic 0 and $R$ its coradical. Let $\pi: A \to R$ be a Hopf algebra map such that $\pi = 1$ on $R$. Then we have an isomorphism of $k$-algebras

$$f: A \to (A/(A \cdot R^+)) \otimes R, a \mapsto \sum a_{(1)} \otimes \pi(a_{(2)}).$$

(If we introduce the concept of “semidirect product of Hopf algebras” of [1, §4], this becomes a Hopf algebra isomorphism. Compare our proof with that of [1, §5].)

**Proof.** Because $A$ is faithfully flat over $R$ [6, Th. 3.1] and $f$ is clearly $R$-linear, it suffices to construct the inverse of $f \otimes_R A: A \otimes_R A \to (A/(A \cdot R^+)) \otimes A, x \otimes y \mapsto \sum x_{(1)} \otimes \pi(x_{(2)})y$.

But the algebra map $h: A \otimes_R A \to A \otimes A, \sum x_{(1)} \otimes S(\pi(x_{(2)}))y \mapsto x \otimes y$ is 0 on $(A \cdot R^+) \otimes A$, and induces the inverse of $f \otimes_R A$.

**REFERENCES**