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LEFT INVERTIBILITY OF CLOSED OPERATORS MODULO AN α-COMPACT OPERATOR

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In a recent work, Peter Fillmore, Joseph Stampfli and James Williams listed seven conditions which, for a closed linear operator with dense domain, are equivalent to left invertibility modulo a compact operator. Cf. Theorem 1.1 of [3]. (The equivalence of some of these conditions was first shown by F. Wolf [8].) In an independent work, written around the same time, the author, together with Gerald Edgar and Sa Ge Lee, obtained a theorem (Theorem 2.6 of [2]) listing conditions, for a bounded operator, which are equivalent to left invertibility modulo an α-compact operator. (Let $\mathscr{H}$ be a Hilbert space of dimension h and let α denote a cardinal, $\aleph_0 \leqq \alpha \leqq h$. An α-compact operator is any bounded operator which is contained in $\bar{\mathscr{I}}_\alpha$, the closed two-sided ideal which is the norm closure of the ideal $\mathscr{I}_\alpha$ of bounded operators of rank less than α. Cf. Theorem 0 of [2]. The $\aleph_0$-compact operators are precisely the compact operators.)

The purpose of this paper is to present one comprehensive theorem, listing many conditions equivalent to left invertibility modulo an α-compact operator, which integrates the two theorems mentioned above, and generalizes both. The Fillmore, Stampfli and Williams result is generalized to obtain appropriately modified conditions which are equivalent to left invertibility modulo each of the closed two-sided ideals of the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators. The Edgar, Ernest, Lee result is generalized to unbounded operators (specifically closed operators with dense domain).

Such a theory, formulated so as to describe the phenomena uniformly for all the closed two-sided ideals of $\mathscr{B}(\mathscr{H})$, without preferential treatment of the smallest such ideal (the compact operators) appears to us to be a necessary step preliminary to any investigation of similar questions in an arbitrary von Neumann algebra, where the collection of closed two-sided ideals is still more complicated.

The first section is preliminary. We generalize the notion of approximate nullity introduced in [2], to closed operators with dense domain. (This is also a generalization of the notion of approximate nullity as used by T. Kato in [7].) In our next section we present our main theorem.

We conclude with two propositions describing two conditions closely related to left invertibility modulo an α-compact operator, but which we were not able to prove equivalent at this level of generality.

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1. The notion of approximate nullity for unbounded operators. We begin with two basic lemmas, which are unbounded versions of lemmas 1.1 and 1.2 of [2]. Since we omitted the proofs of these lemmas in [2], we shall give here the proof for this unbounded case.

LEMMA 1.1. *Let T be a closed linear transformation with domain $\mathscr{D}_T$ dense in a Hilbert space $\mathscr{H}$ and let $\varepsilon > 0$. Then there exists a closed subspace $\mathscr{K}$ of $\mathscr{H}$ such that*

$$\ker T \subset \mathscr{K} \subset \mathscr{D}_T$$

and

$$\| Tf \| < \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K},\ f \neq 0 ,$$

and

$$\| Tf \| \geqq \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K}^{\perp} \cap \mathscr{D}_T .$$

If T is self-adjoint, there exists such a closed subspace $\mathscr{K}$ which reduces T in the sense that $T\mathscr{K} \subset \mathscr{K}$ and $T(\mathscr{K}^{\perp} \cap \mathscr{D}_T) \subset \mathscr{K}^{\perp}$.

PROOF. Consider first the case where T is self-adjoint. In this way we may use the spectral representation of T given by Theorem 5, page 1209 of [1]. We may therefore consider, without loss of generality, that T is acting on a space of the form

$$\sum_{\gamma} L_2(\mu_\gamma)$$

where $\{\mu_\gamma\}$ is a family of finite positive measures defined on the Borel sets of the plane and vanishing on the complement of the spectrum $\Lambda(T)$ of T, and where

$$\mathscr{D}_T = \left\{ f \colon f \in \sum_{\gamma} L_2(\mu_\gamma) \quad \text{and} \quad \sum_{\gamma} \int_{\Lambda(T)} | \lambda f_\gamma(\lambda) |^2 d\mu_\gamma(\lambda) < \infty \right\}$$

and, for $f \in \mathscr{D}_T$,

$$(Tf)_\gamma(\lambda) = \lambda f_\gamma(\lambda) \quad \text{for} \quad \mu_\gamma\text{-almost all } \lambda .$$

Let $\mathscr{K}$ be the closed subspace of $\sum_\gamma L_2(\mu_\gamma)$ of those f which, for every γ, f_γ vanishes μ_γ almost everywhere off the interval $(-\varepsilon, +\varepsilon)$. Then $\mathscr{K}^{\perp}$ is the closed subspace of $\sum_\gamma L_2(\mu_\gamma)$ consisting of those f which, for every γ, f_γ vanishes almost everywhere on $(-\varepsilon, +\varepsilon)$. Clearly the conditions of the lemma hold for this space $\mathscr{K}$.

If T is not self-adjoint we may use the canonical factorization of T given by Theorem 7, page 1249 of [1], and write T uniquely in the form $T = UP$ where U is a partial isometry whose initial domain is the closure of the range $\mathscr{R}(T^*)$ of T^* and P is a positive transformation such that $\overline{\mathscr{R}(P)} = \overline{\mathscr{R}(T^*)}$.

We may therefore apply our prior considerations to the self-adjoint operator P to obtain a closed subspace $\mathscr{K}$ such that

$$\ker P \subset \mathscr{K} \subset \mathscr{D}_P = \mathscr{D}_T$$

and

$$\| Tf \| = \| UPf \| = \| Pf \| < \varepsilon \| f \| \quad \text{for all } f \text{ in } \mathscr{K}, f \neq 0 .$$

Since the range of P is contained in the initial space of U we always have $\| UPf \| = \| Pf \|$. Hence

$$\| Tf \| = \| UPf \| = \| Pf \| \geqq \varepsilon \| f \| \quad \text{for all } f \in \mathscr{K}^\perp \cap \mathscr{D}_P = \mathscr{K}^\perp \cap \mathscr{D}_T .$$

LEMMA 1.2. *Let T be a closed linear operator with domain $\mathscr{D}_T$ dense in $\mathscr{H}$ and let $\varepsilon > 0$. Suppose $\mathscr{K}$ is a closed subspace of $\mathscr{H}$, $\mathscr{K} \subset \mathscr{D}_T$ such that*

$$\| Tf \| < \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K}, f \neq 0$$

and suppose $\mathscr{L}$ is a closed subspace such that

$$\| Tf \| \geqq \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{L}^\perp \cap \mathscr{D}_T .$$

Then

$$\dim \mathscr{K} \leqq \dim \mathscr{L} .$$

PROOF. Let E denote the projection onto $\mathscr{L}$. If $f \in \mathscr{K}$, $f \neq 0$, then $f \in \mathscr{D}_T$ and $\| Tf \| < \varepsilon \| f \|$. Hence $f \notin \mathscr{L}^\perp$ and thus $Ef \neq 0$. Thus E, restricted to $\mathscr{K}$ is a one-to-one bounded linear map of $\mathscr{K}$ into $\mathscr{L}$. Thus $\dim \mathscr{K} \leqq \dim \mathscr{L}$. (Cf. problem 42, page 27 of [6].)

The subspace $\mathscr{K}$ given in Lemma 1.1 is not unique. However if we have two spaces satisfying Lemma 1.1, then Lemma 1.2 implies they have the same dimension. We are therefore justified in defining $\delta_\varepsilon(T)$ to be the common dimension of all subspaces satisfying the conditions of Lemma 1.1.

DEFINITION 1.3. We define the *approximate nullity* $\delta(T)$ of an operator T (a closed linear operator with dense domain) to be

$$\delta(T) = \min_{\varepsilon > 0} \delta_\varepsilon(T) .$$

DEFINITION 1.4. If T is a closed linear operator with dense domain, we define the *approximate point spectrum of weight* α, of T, denoted $\Pi_\alpha(T)$ to be

$$\Pi_\alpha(T) = \{\lambda\colon \lambda \in \mathrm{C} \quad \text{and} \quad \delta(T - \lambda I) \geqq \alpha\} .$$

In the case where T is bounded, the approximate point spectrum of weight α was shown to be a non-empty compact subset of the spectrum of T. (Cf. Definition 3.1 and Theorems 3.2 and 3.7 of [2].) In the case where $\alpha = \aleph_0$, this concept has been studied independently in [3], under the term "left essential spectrum."

2. Conditions for left invertibility modulo the ideal $\mathscr{J}_\alpha$ of α-compact operators. Let T denote a closed linear operator with dense domain. We say T is left invertible modulo $\mathscr{J}_\alpha$ if there is a bounded linear operator X on the same Hilbert space $\mathscr{H}$ such that $I - XT$ has a unique extension to a bounded operator defined on all of $\mathscr{H}$, which is α-compact, i.e., $I - XT \in \mathscr{J}_\alpha$.

THEOREM 2.1. *Let T be a closed linear transformation with domain $\mathscr{D}_T$ dense in $\mathscr{H}$. Then the following conditions are equivalent.*

(1) *T is left invertible modulo $\mathscr{J}_\alpha$.*

(2) *$|T| = (T^*T)^{1/2}$ is invertible modulo $\mathscr{J}_\alpha$.*

(3) *T is left invertible modulo $\mathscr{I}_\alpha$, where $\mathscr{I}_\alpha$ is the ideal of bounded operators of rank less than α.*

(4) $\delta(T) < \alpha$.

(5) $0 \notin \Pi_\alpha(T)$.

(6) *There exists a closed subspace $\mathscr{K}$ such that $\dim \mathscr{K}^\perp < \alpha$ and T is bounded below on $\mathscr{K} \cap \mathscr{D}_T$.*

(7) *There exists $\varepsilon > 0$ such that, whenever $\mathscr{K}$ is a closed subspace of $\mathscr{H}$ for which $\mathscr{K} \subset \mathscr{D}_T$ and $\|Tf\| \leqq \varepsilon \|f\|$ for all f in $\mathscr{K}$, then* $\dim \mathscr{K} < \alpha$.

(8) *If $\{f_\lambda\}$ is a net of unit vectors in $\mathscr{D}_T$, spanning a space of dimension at least α and converging to zero weakly, then $\{Tf_\lambda\}$ does not converge to 0 strongly.*

(9) *If $\{e_\lambda\}$ is an orthonormal net of cardinality at least α, contained in $\mathscr{D}_T$, then $\{Te_\lambda\}$ does not converge to 0 strongly.*

(10) *There exists a $\gamma > 0$ such that $E(0, \gamma)\mathscr{H}$ has dimension at least α, where E is the spectral resolution of $(T^*T)^{1/2} = |T|$.*

REMARKS. The easily verified equivalence of (1) and (2) means that there are in fact another seven conditions equivalent to left invertibility modulo $\mathscr{J}_\alpha$, namely those obtained by applying conditions 3 through 9 to the positive operator $|T|$ rather than to T.

The last three conditions are generalizations of conditions of F. Wolf [8]. In [3], Fillmore, Stampfli and Williams state conditions equivalent to *non-invertibility* of an operator modulo the ideal of compact operators. Thus conditions 1, 5, 7, 8, 9, 10 of the above theorem are generalizations of the negations of conditions 7, 5, 8, 1, 2, 3 respectively, of Theorem 1.1 of [8]. To see that our condition 5 generalizes the Fillmore, Stampfli, Williams condition 5 one need only note that when $\alpha = \aleph_0$, $\Pi_{\aleph_0}(|T|)$ is precisely the Weyl essential spectrum of the Hermitian operator $|T|$, and the Fillmore, Stampfli, Williams condition 5 simply states that 0 is contained in the essential spectrum of $|T|$. Conditions 3, 1, 6, 4 of the above theorem generalize (to the case of unbounded operators) conditions i), ii), iii), iv) of Theorem 2.6 of [2].

PROOF OF THE THEOREM. We first note the equivalence of conditions (1) and (2). Let $T = U|T|$ denote the polar decomposition of T. (Cf. Theorem XII. 7.7, page 1249 of [1].) If X is a left inverse for T mod $\mathscr{J}_\alpha$, then XU is a left inverse for $|T|$ mod $\mathscr{J}_\alpha$. Similarly if X is a left inverse for $|T|$ mod $\mathscr{J}_\alpha$, then XU^* is a left inverse for T mod $\mathscr{J}_\alpha$. Also the equivalence of 4 and 5 is obvious from Definition 1.4.

We next embark on the proof cycle, $6 \Rightarrow 7 \Rightarrow 4 \Rightarrow 3 \Rightarrow 1 \Rightarrow 6$.

$(6 \Rightarrow 7)$. By condition 6 there exists a closed subspace $\mathscr{L}$ and an $\varepsilon' > 0$ such that

$$\| Tf \| \geqq \varepsilon' \| f \|$$

for all f in $\mathscr{L}^\perp \cap \mathscr{D}_T$ and $\dim \mathscr{L} < \alpha$. Then condition 7 holds with $\varepsilon = \varepsilon'/2$. Indeed if $\mathscr{K}$ is any closed subspace of $\mathscr{H}$ such that $\mathscr{K} \subset \mathscr{D}_T$ and $\| Tf \| \leqq \varepsilon'/2 \| f \| < \varepsilon' \| f \|$ for all $f \in \mathscr{K}$, $f \neq 0$, then Lemma 1.2 implies

$$\dim \mathscr{K} \leqq \dim \mathscr{L} < \alpha .$$

$(7 \Rightarrow 4)$. Let ε be the positive quantity given by condition 7. By Lemma 1.1 there exists a closed subspace $\mathscr{K}$ such that

$$\| Tf \| < \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K},\ f \neq 0$$

and

$$\| Tf \| \geqq \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K}^\perp \cap \mathscr{D}_T .$$

By condition 7, $\dim \mathscr{K} < \alpha$. Hence, by Definition 1.3, $\delta_\varepsilon(T) < \alpha$ and thus $\delta(T) < \alpha$.

$(4 \Rightarrow 3)$. Since $\delta(T) < \alpha$, there is an $\varepsilon > 0$ such that $\delta_\varepsilon(T) < \alpha$. Hence there exists a closed subspace $\mathscr{K}$ such that

$$\ker T \subset \mathscr{K} \subset \mathscr{D}_T$$

and

$$\| Tf \| < \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K},\ f \neq 0$$

and

$$\| Tf \| \geqq \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K}^{\perp} \cap \mathscr{D}_T ,$$

and $\dim \mathscr{K} < \alpha$.

The restriction T_0 of T to $\mathscr{K}^{\perp}$ is a closed operator from $\mathscr{K}^{\perp}$ to $\mathscr{H}$, with domain $\mathscr{K}^{\perp} \cap \mathscr{D}_T$. (Since $\mathscr{K} \subset \mathscr{D}_T$ and $\mathscr{D}_T$ is dense in $\mathscr{H}$ it follows that $\mathscr{K}^{\perp} \cap \mathscr{D}_T$ is dense in $\mathscr{K}^{\perp}$.) Further the minimum modulus $\gamma(T_0)$ (cf. Definition IV. 1.3, page 96 of [4]) is greater than or equal ε. Hence (cf. Theorem IV. 1.6, page 98 of [4]) ran $T_0 = T(\mathscr{K}^{\perp} \cap \mathscr{D}_T)$ is a closed subspace of $\mathscr{H}$, which we shall denote $\mathscr{L}$. Define X on $\mathscr{L}$ to be the closed linear operator which is the inverse of T_0 and define $X = 0$ on $\mathscr{L}^{\perp}$. Since X is a closed linear operator defined on all of $\mathscr{H}$, the closed graph theorem (cf. Theorem II. 1.9, page 45 of [4]) implies X is a bounded operator on $\mathscr{H}$.

We now examine the operator $(I - XT)$. This operator annihilates $\mathscr{K}^{\perp} \cap \mathscr{D}_T$, and since $\mathscr{K}^{\perp} \cap \mathscr{D}_T$ is dense in $\mathscr{K}^{\perp}$, it has a unique extension to an operator (namely the zero operator) on all of $\mathscr{K}^{\perp}$.

Since T is bounded on $\mathscr{K}$, $I - XT$ is a bounded operator on $\mathscr{K}$. Let $T = UP$ denote the polar decomposition of T. It follows from the proof of Lemma 1.1 that $\mathscr{K}$ can be chosen such that it reduces P. We next show T maps $\mathscr{K}$ into $\mathscr{L}^{\perp}$. Indeed if $f \in \mathscr{K}$ and $h \in \mathscr{L}$ then there exists $g \in \mathscr{K}^{\perp} \cap \mathscr{D}_T$ such that $Tg = h$. Further by Theorem XII. 7.7, page 1249 of [1], U^*U is the projection onto $\overline{\mathscr{R}(T^*)} = \overline{\mathscr{R}(P)}$. Hence

$$\begin{aligned} (Tf, h) &= (Tf, Tg) = (UPf, UPg) \\ &= (Pf, U^*UPg) = (Pf, Pg) = 0 \end{aligned}$$

since $\mathscr{K}$ reduces P and $f \in \mathscr{K}$, $g \in \mathscr{K}^{\perp} \cap \mathscr{D}_T$. Thus $T\mathscr{K} \subset \mathscr{L}^{\perp}$ and since X annihilates $\mathscr{L}^{\perp}$, $I - XT$ is just the identity on $\mathscr{K}$. Thus $I - XT$ is the projection onto $\mathscr{K}$. Since $\dim \mathscr{K} < \alpha$ we have $I - XT \in \mathscr{I}_{\alpha}$.

Since $3 \Rightarrow 1$ is obvious, this proof cycle will be completed if we show $1 \Rightarrow 6$. However the proof of this assertion is essentially the same as the proof that condition ii) implies condition iii) in Theorem 2.6 of [2], and will not be given here.

We next consider the proof cycle $8 \Rightarrow 9 \Rightarrow 10 \Rightarrow 8$. The arguments here are slight modifications of the arguments used by Fillmore, Stampfli and Williams to show the equivalences of conditions (1), (2) and (3) in Theorem 1.1 of [3]. The implication $8 \Rightarrow 9$ is trivial. We show $9 \Rightarrow 10$ by way of contradiction. For each n, let $\{e_{n\lambda}\}$ denote an orthonormal basis for $E(1/n+1, 1/n]\mathscr{H} \subset \mathscr{D}_T$, where $\lambda \in \Lambda_n$, a well-ordered index set. Then $\{e_{n\lambda}\}$ is a well-ordered net in the lexographic ordering and, for every n,

$$\operatorname{card}\{e_{m\lambda}\colon \lambda \in \Lambda_m,\ m \geqq n\} \geqq \alpha$$

since $\dim E(0, 1/n)\mathscr{H} \geqq \alpha$ for every n. However $\| Te_{n\lambda} \| \leqq 1/n$ and hence $Te_{n\lambda}$ converges to zero strongly, contradicting condition 9. The implication $10 \Rightarrow 8$ follows from the inequality

$$\| f_\lambda - E(0, \gamma) f_\lambda \| \leqq 1/\gamma \, \| Tf_\lambda \|$$

whose verification is given in [3].

We complete the proof by connecting the two proof cycles with the verification that 10 is equivalent to 4. If 10 holds, note that $\mathscr{K} = E(0, \gamma)\mathscr{H}$ satisfies the conditions of Lemma 1.1 and hence $\delta(|T|) = \min_{\varepsilon > 0} \delta_\varepsilon(|T|) \leqq \delta_\gamma(|T|) < \alpha$. This implies $\delta(T) < \alpha$ since we have already seen that conditions 1, 2 and 4 are equivalent. Conversely if $\delta(T) < \alpha$ then $\delta(|T|) < \alpha$. Then there exists $\gamma > 0$ such that $\delta_\gamma(|T|) < \alpha$. Since $\mathscr{K} = E(0, \gamma)\mathscr{H}$ satisfies the condition of Lemma 1.1, it follows from the remark following Lemma 1.2 that

$$\dim E(0, \gamma)\mathscr{H} = \delta_\gamma(|T|) < \alpha .$$

3. Two more conditions. In this section we consider two more conditions closely related to invertibility modulo $\mathscr{J}_\alpha$, but which we have not been able to show equivalent at this level of generality. We first recall a definition from [2]. A subspace $\mathscr{K}$ of a Hilbert space $\mathscr{H}$ is called *α-closed* if there is a closed subspace $\mathscr{L}$ of $\mathscr{H}$ such that $\mathscr{L} \subset \mathscr{K}$ and such that $\dim(\mathscr{K} \cap \mathscr{L}^\perp) < \alpha$. Consider the following two conditions on a closed linear operator T, with dense domain.

(11) If P is a projection such that $TP \in \mathscr{J}_\alpha$, then $\dim P < \alpha$.

(12) Range T is α-closed and the nullity of T, $\nu(T)$, is less than α.

Condition 11 is the obvious generalization of the negation of condition 6 of Theorem 1.1 of [3]. Condition 12 is the same as condition v (for bounded operators) of Theorem 2.6 of [2], as well as a generalization of the negation of condition 4 of Theorem 1.1 of [3]. Thus condition 12 is known to be equivalent to left invertibility modulo $\mathscr{J}_\alpha$ either if T is bounded or if $\alpha = \aleph_0$.

Following Bernhard Gramsch (cf. Definition 1.3 of [5]) we define a cardinal α to be $\aleph_0$-irregular if it is the sum of countably many cardinals strictly smaller than α.

PROPOSITION 3.1. *Let T be a closed linear operator with dense domain. If T is left invertible modulo $\mathscr{J}_\alpha$, then condition* 11 *holds. Furthermore if α is an $\aleph_0$-irregular cardinal, then condition* 11 *is equivalent to left invertibility of T modulo $\mathscr{J}_\alpha$.*

PROOF. We suppose T is left invertible modulo $\mathscr{J}_\alpha$ and also that P is a projection such that TP is α-compact. Thus there exists a bounded operator X such that $I - XT$ is α-compact. Hence both XTP and $P-XTP$ are α-compact and thus P itself is α-compact. Hence, by any one of the conditions characterizing α-compact in Theorem 0 of [2], we have that $\dim P < \alpha$.

We next suppose α is an $\aleph_0$-irregular cardinal and we wish to show that condition 11 implies left invertibility modulo $\mathscr{J}_\alpha$. First note that if condition 11 holds for T, then it holds for the positive part $|T|$ of T, where $T = U|T|$ is the polar decomposition of T. By the equivalence of conditions 1 and 2 of Theorem 2.1 we may assume, without loss of generality, that T is positive.

Again by Theorem 2.1, it is sufficient to prove that condition 10 of that theorem holds for T. Proceeding by way of contradiction, suppose it does not hold. Then

$$\dim E[0, 1/n]\mathscr{H} \geqq \alpha$$

for $n = 1, 2, \cdots$, where E is the spectral resolution of T. Since α is $\aleph_0$-irregular we may express $\alpha = \sum_{n=1}^{\infty} \alpha_n$ where $\alpha_n < \alpha$ and $\alpha_{n+1} \geqq \alpha_n$ for each n. For each n, choose an orthonormal set S_n of cardinality α_n in $E[0, 1/n]\mathscr{H}$. Let P denote the projection onto the closed subspace generated by $\bigcup_{m=1}^{n-1} S_m$. Then $\| TP - TP_n \| = \| T(P - P_n) \| \leqq 1/n$ and thus $TP_n \rightarrow TP$, while $TP_n \in \mathscr{J}_\alpha$ (the ideal of operators of rank less than α) since $\dim P_n < \alpha$. Hence $TP \in \mathscr{J}_\alpha$. But dimension $P = \alpha$, contradicting condition 11.

PROPOSITION 3.2. *Let T be a closed linear operator with dense domain. If T is left invertible modulo $\mathscr{J}_\alpha$, then condition* 12 *holds.*

PROOF. By the equivalence of conditions (1) and (4) of Theorem 2.1 we have $\nu(T) \leqq \delta(T) < \alpha$.

Since $\delta(T) < \alpha$, there exists a closed subspace $\mathscr{K}$, of dimension less than α, which reduces $|T|$ and such that

$$\| Tf \| < \varepsilon \| f \| \qquad \text{for all } f \in \mathscr{K},\ f \neq 0$$

and

$$\| Tf \| \geqq \varepsilon \| f \| \qquad \text{for all } f \text{ in } \mathscr{K}^{\perp} \cap \mathscr{D}_T .$$

Now it is a fairly easy exercise to show that $\mathscr{L} = T(\mathscr{K}^{\perp} \cap \mathscr{D}_T)$ is a closed subspace and $\mathscr{L}^{\perp} \cap \operatorname{ran} T = T(\mathscr{K})$. Since T is bounded on $\mathscr{K}$ we have

$$\dim (\mathscr{L}^{\perp} \cap \operatorname{ran} T) = \dim (T(\mathscr{K})) \leqq \dim \mathscr{K} < \alpha .$$

Hence ran T is α-closed.

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