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TOPOLOGIES ON GROUPS AND A CERTAIN L-IDEAL OF MEASURE ALGEBRAS

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1. Introduction. Let G_{τ_0} be a non-discrete locally compact abelian group with a topology τ_0. Let $M(G_{\tau_0})$ be the commutative semisimple Banach algebra consists of bounded regular Borel measures on G_{τ_0}. We write $\mathfrak{M}$ the maximal ideal space of $M(G_{\tau_0})$. For $\mu \in M(G_{\tau_0})$, we put $\mu^*(E) = \overline{\mu(-E)}$ for all Borel subset E of G_{τ_0}. Then we have $\mu^* \in M(G_{\tau_0})$ and $M(G_{\tau_0})$ is considered a Banach $*$-algebra. Let Δ be the set of all symmetric multiplicative linear functionals on $M(G_{\tau_0})$, that is $\Delta = \{f \in \mathfrak{M}: f(\mu^*) = \overline{f(\mu)}$ for all $\mu \in M(G_{\tau_0})\}$. A closed subspace (subalgebra, ideal) N of $M(G_{\tau_0})$ will be called an *L-subspace* (*L-subalgebra*, *L-ideal*) if N satisfies the condition; $\mu \in M(G_{\tau_0})$, $\nu \in N$ and μ is absolutely continuous with respect to ν, then $\mu \in N$. For a subspace N of $M(G_{\tau_0})$, we put $N^{\perp} = \{\mu \in M(G_{\tau_0}): \mu$ is mutually singular with $\nu \in N\}$.

In this note, we consider the following subspace of $M(G_{\tau_0})$; $M(\Delta) = \{\mu \in M(G_{\tau_0}): \hat{\mu}(f) = 0$ for all $f \notin \Delta\}$. J. H. Williamson ([9]) showed that for every $\mu \in M(\Delta)$, $|\hat{\mu}_d(f)| < |\hat{\mu}_c(f)|$ for all $f \in \mathfrak{M}$, where μ_d and μ_c are the discrete part and the continuous part of μ, respectively. And he conjectured that $\mu_d = 0$ for every $\mu \in M(\Delta)$ ([9]). Using the results of J. L. Taylor ([7]), T. Shimizu ([6]) showed that $M(\Delta)$ is a proper L-ideal of $M(G_{\tau_0})$ and Williamson's conjecture is true. For a locally compact group topology τ on G which is strictly stronger than τ_0, we may consider $M(G_{\tau})$ a prime L-subalgebra of $M(G_{\tau_0})$ with natural injection ([3]). It is clear that $M_c(G_{\tau_0})^{\perp} = M(G_{\tau_d})$, where τ_d is the discrete topology on G and $M_c(G_{\tau_0}) = \{\mu \in M(G_{\tau_0}): \mu$ is continuous$\}$. From the above fact, we have the following conjecture:

Conjecture I. $M(\Delta)$ is contained in $M(G_{\tau})^{\perp}$.

For $M(G_{\tau})$, there is a Raikov system $\mathfrak{F}$ such that $M(G_{\tau}) = M(\mathfrak{F})$, where $M(\mathfrak{F}) = \{\mu \in M(G_{\tau_0})$: there is $A \in \mathfrak{F}$ such that μ is concentrated on $A\}$. Thus we have a more generally conjecture as follows:

Conjecture II. For a proper Raikov system $\mathfrak{F}$, we have $M(\Delta) \subset M(\mathfrak{F})^{\perp}$.

In § 1, we show that our conjecture II is true, if G_{τ_0} is metrizable. In § 2, we show that our conjecture I is true. In § 3, we show a property of the

Gelfand transforms of $M(\Delta)$, using Taylor's structure semigroup of $M(G_{\tau_0})$.

2. **Metrizable group.** Throughout this section, let G be a non-discrete locally compact abelian group. A subset of G is called type F_σ if it is a countable union of compact subsets of G. A collection of subsets of G of type F_σ is called a *Raikov system* if the following properties hold:

(1) If $A_1 \in \mathfrak{F}$ and A_2 is a subset of A_1 of type F_σ, then $A_2 \in \mathfrak{F}$.

(2) The union of a countable collection of sets in $\mathfrak{F}$ also in $\mathfrak{F}$.

(3) If $A \in \mathfrak{F}$ and $t \in G$, then $A - t \in \mathfrak{F}$.

(4) If $A \in \mathfrak{F}$, then $A + A \in \mathfrak{F}$.

Let m be a Haar measure on G. A Raikov system $\mathfrak{F}$ such that $m(A) = 0$ for every $A \in \mathfrak{F}$, will be called proper. For a σ-compact subset A, there exists a minimal Raikov system containing A. Such a Raikov system will be called a single generated Raikov system. For a Raikov system $\mathfrak{F}$, we put $M(\mathfrak{F}) = \{\mu \in M(G)$: there exists $A \in \mathfrak{F}$ such that μ is concentrated on $A\}$. For a Raikov system $\mathfrak{F}$, if $A \in \mathfrak{F}$ implies $-A \in \mathfrak{F}$, then $\mathfrak{F}$ is a symmetric Raikov system. For a single generated symmetric Raikov $\mathfrak{F}$, there is a group which generates $\mathfrak{F}$.

J. L. Taylor ([7]) showed that there exists a compact topological semigroup S and an isometric isomorphism θ from $M(G)$ into $M(S)$ such that the image of θ is weak* dense in $M(S)$ and the maximal ideal space of $M(G)$ is identified with the set $\hat{S}$ of all continuous semicharacters on S. For $\mu \in M(G)$, the Gelfand transform $\hat{\mu}$ of μ is given by $\hat{\mu}(f) = \int_S f d\theta\mu$ for every $f \in \hat{S}$.

For a given subset E of G which contains 0, we shall say a subset F of G is $(E, 1)$-independent if the following relation holds:

$\sum_{r=1}^N n_r x_r \in E$ if and only if $n_r = 0$ for $1 \leqq r \leqq N$, where $x_1, \cdots, x_N$ are distinct elements of F and $n_1, \cdots, n_N$, are integers with $|n_r| \leqq 1$.

THEOREM 1. *Let $\mathfrak{F}$ be a proper symmetric Raikov system with a single generator. Let H be a group which generates $\mathfrak{F}$. If there exists a perfect compact $(H, 1)$-independent set P, then we have $M(\Delta) \subset M(\mathfrak{F})^\perp$.*

PROOF. Let μ_0 be a positive continuous measure concentrated on P, with $\|\mu_0\| = 1$. We put $\mu = (1/2)(\mu_0 + \mu_0^*)$, then $\mu = \mu^*$ and μ is concentrated on $Q = P \cup (-P)$. For a non-negative measure $\omega_0 \in M(\mathfrak{F})$ with $\|\omega_0\| = 1$, we put $\omega = (1/2)(\omega_0 + \omega_0^*)$ and $\sigma = \omega^2 - \mu^2$. As the proof of Proposition 2 of [10] we obtain that $\mu^{n_1}\omega^{m_1} \perp \mu^{n_2}\omega^{m_2}$ for $(n_1, m_1) \neq (n_2, m_2)$ where n_i, m_i $(i = 1, 2)$ are positive integers. So we have

$$\|\sigma^n\| = \left\|\sum_{k=0}^n \binom{n}{k}(-1)^k \mu^{2k}\omega^{2(n-k)}\right\| = \sum_{k=0}^n \binom{n}{k}\|\mu^{2k}\omega^{2(n-k)}\| = \sum_{k=0}^n \binom{n}{k} = 2\,,$$

and the spectral norm of σ is 2. Hence there is a complex homomorphism h of $M(G)$ such that $|h(\sigma)|=2$. Since $||\mu||=1$, we have that $|h(\mu^2)|\leqq 1$, and $|h(\omega^2)-h(\mu^2)|=|h(\sigma)|=2$ if and only if $h(\omega^2)=-h(\mu^2)$ and $|h(\omega)|=|h(\mu)|=1$. This shows that h is non-symmetric. Let f be a continuous semicharacters on S such that $h(\lambda)=\int_S f d\theta\lambda$ for every $\lambda\in M(G)$. Since $||\omega||=1$, $|f|\leqq 1$ and $|h(\omega)|=1$, we obtain $\operatorname{supp}\theta\omega\subset\{x\in S: |f(x)|=1\}$. By Shimizu [6], we have $\omega\in M(\varDelta)^{\perp}$. Then $M(\varDelta)\subset M(\mathfrak{F})^{\perp}$. q.e.d.

COROLLARY 2. *If G is metrizable, then we have $M(\varDelta)\subset M(\mathfrak{F})^{\perp}$ for a proper Raikov system $\mathfrak{F}$.*

PROOF. For any $\mu\in M(\mathfrak{F})$, there exists a single generated Raikov system $\mathfrak{F}_0$ such that $\mu\in M(\mathfrak{F}_0)$ and $M(\mathfrak{F}_0)\subset M(\mathfrak{F})$. If $\mathfrak{F}_0$ is a non-symmetric Raikov system, we can easily see $M(\varDelta)\subset M(\mathfrak{F}_0)^{\perp}$. If $\mathfrak{F}_0$ is a symmetric Raikov system, there is a group H that generates $\mathfrak{F}_0$. Then there is a perfect compact $(H, 1)$-independent set P as in the proof of Proposition 1 of [10]. By Theorem 1, we have $M(\varDelta)\subset M(\mathfrak{F}_0)^{\perp}$. Thus we have $M(\varDelta)\subset M(\mathfrak{F})^{\perp}$. q.e.d.

3. Topologies on groups and $M(\varDelta)$. Let G be a non-discrete locally compact abelian group and $\hat{G}$ be the dual group of G. Let H be a closed subgroup of G and φ the canonical continuous homomorphism from G onto G/H.

PROPOSITION 3. *We put $\Phi\mu(E)=\mu(\varphi^{-1}(E))$ for every Borel set E of G/H. Then we have the followings:*

(a) *Φ is a norm decreasing positive homomorphism from $M(G)$ onto $M(G/H)$.*

(b) *For every non-negative measure $\nu\in M(G/H)$, there exists a non-negative measure $\mu\in M(G)$ such that $\Phi\mu=\nu$.*

(c) *$\Phi(\mu^*)=(\Phi\mu)^*$ for every $\mu\in M(G)$.*

PROOF. At first, we shall show (a). For every Borel subset E of G/H, we have

$$\int_{G/H}\chi_E(y)d\Phi\mu(y)=\int_G\chi_E(\varphi(x))d\mu(x)$$

for every $\mu\in M(G)$, where χ_E is a characteristic function of E. Then for every Borel function f on G/H, we have

$$\int_{G/H}f(y)d\Phi\mu(y)=\int_G f(\varphi(x))d\mu(x) \tag{5}$$

for every $\mu\in M(G)$. Let $\varLambda$ be the annihilator of H, then we may consider $\varLambda$ as the dual group of G/H. By (5), we have $(\widehat{\Phi\mu})(\gamma)=\hat{\mu}(\gamma)$ for every

$\gamma \in \Lambda$. Then we get $(\Phi(\mu * \nu))^{\wedge}(\gamma) = (\Phi\mu * \Phi\nu)^{\wedge}(\gamma)$ for every $\gamma \in \Lambda$. From the uniqueness theorem, we obtain $\Phi(\mu * \nu) = \Phi\mu * \Phi\nu$. By the definition of Φ, Φ is a positive norm decreasing linear mapping. Moreover, Φ is an onto mapping by ([5]; p. 54). Thus (a) is proved. (c) is clear by the definition. Finally, we shall show (b). For every non-negative measure $\nu \in M(G/H)$, there exists $\mu_0 \in M(G)$ such that $\Phi\mu_0 = \nu$. Let $\mu_0 = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ be the Jordan decomposition of μ_0, where $\mu_i \geqq 0$ $(n = 1, 2, 3, 4)$. Since $\Phi\mu_0 \geqq 0$, we have $\Phi\mu_0 = \Phi\mu_1 - \Phi\mu_2$. Since $\Phi\mu_2 \geqq 0$, we get $\Phi\mu_1 \geqq \Phi\mu_0 \geqq 0$. Then from Radon-Nikodym's theorem, there exists a non-negative Borel measurable function $f \in L^1(\Phi\mu_1)$ such that $\|f\|_\infty \leqq 1$ and $\Phi\mu_0 = f\Phi\mu_1$. We put

$$\mu(E) = \int_E f(\varphi(x))d\mu_1(x) \tag{6}$$

for every Borel subset E of G. Then μ is a non-negative measure on G. By (5) and (6), we get

$$\begin{aligned}\Phi\mu(A) &= \int_G \chi_A(\varphi(x))f(\varphi(x))d\mu_1(x) = \int_{G/H} \chi_A \cdot f d\Phi\mu_1 \\ &= \int_{G/H} \chi_A d\Phi\mu_0 = \nu(A)\end{aligned}$$

for every Borel subset A of G/H. Thus we have $\Phi\mu = \nu$. q.e.d.

PROPOSITION 4. *Let H be a σ-compact closed subgroup of G. If E is a σ-compact subset of G/H, then $\varphi^{-1}(E)$ is a σ-compact subset of G.*

PROOF. Without loss of generality, we may assume that E is compact. Let H be a σ-compact open subgroup of G. Since $\varphi(H_0)$ is open, there exists a finite set $\{x_1, \cdots, x_n\} \subset G$ such that $E \subset \bigcup_{k=1}^n (\varphi(x_k) + \varphi(H_0))$. Then we have $\varphi^{-1}(E) \subset \bigcup_{k=1}^n (x_k + H_0 + H)$. Since H_0 and H are σ-compact, $H_0 + H$ is σ-compact. Then $\bigcup_{k=1}^n (x_k + H_0 + H)$ is σ-compact. Since $\varphi^{-1}(E)$ is a closed set, $\varphi^{-1}(E)$ is σ-compact. q.e.d.

Let G_{τ_0} be an abelian group G with a non-discrete locally compact abelian group topology τ_0. Let τ be a locally compact abelian group topology on G strictly stronger than τ_0. Now we consider that G_{τ_0} and τ are fixed. Let η be the continuous identity mapping from G_τ to G_{τ_0}. For $\mu \in M(G_\tau)$, we put $\Psi\mu$ the restriction of μ to the Borel field of G_{τ_0}. Then Ψ is an isometric isomorphism from $M(G_\tau)$ into $M(G_{\tau_0})$ and we may consider that $M(G_\tau)$ is a prime L-subalgebra of $M(G_{\tau_0})$. The following proposition is important for our purpose.

PROPOSITION 5 (J. Inoue [3]). *For $\mu \in M(G_{\tau_0})$, $\mu \in M(G_\tau)$ if and only if there exists a Borel set C of G_{τ_0} such that $\eta^{-1}(C)$ is a σ-compact subset* *of G_τ and μ is concentrated on C.*

COROLLARY 6. *For $\mu \in M(G_{\tau_0})$, $\mu \in M(G_\tau)^\perp$ if and only if $\mu(C) = 0$ for every Borel set C of G_{τ_0} such that $\eta^{-1}(C)$ is σ-compact.*

Let H be a closed subgroup of G_{τ_0}, and φ_1, φ_2 be the canonical homomorphisms from G_{τ_0} onto G_{τ_0}/H, from G_τ onto G_τ/H, respectively. Let ψ be a continuous identity mapping from G_τ/H to G_{τ_0}/H. Then we have the following commutative diagram.

$$\begin{array}{ccc} G_\tau & \xrightarrow{\eta} & G_{\tau_0} \\ {\scriptstyle\varphi_2}\downarrow & & \downarrow{\scriptstyle\varphi_1} \\ G_\tau/H & \xrightarrow{\psi} & G_{\tau_0}/H\,. \end{array}$$

Let Φ be a canonical homomorphism from $M(G_{\tau_0})$ onto $M(G_{\tau_0}/H)$ induced by φ_1. The following proposition is followed by Lebesgue's decomposition theorem.

PROPOSITION 7. *Let N be an L-subspace of $M(G_{\tau_0})$, then $N^\perp$ is an L-subspace and $M(G_{\tau_0}) = N \oplus N^\perp$.*

PROPOSITION 8. *Suppose H is a closed subgroup of G_{τ_0} and a σ-compact subset of G_τ. Then we have*

(7) $\Phi(M(G_\tau)) = M(G_\tau/H)$ *and*

(8) $\Phi(M(G_\tau)^\perp) = M(G_\tau/H)^\perp$.

PROOF. At first, we shall show (7). Let $\mu \in M(G_\tau)$, then by Proposition 5 there exists a Borel set C of G_{τ_0} such that $\eta^{-1}(C)$ is σ-compact and μ is concentrated on C. Then $\Phi\mu$ is concentrated on $\varphi_1(C)$. Since $\psi^{-1}(\varphi_1(C)) = \varphi_2(\eta^{-1}(C))$ and $\eta^{-1}(C)$ is a σ-compact subset of G_τ, $\psi^{-1}(\varphi_1(C))$ is a σ-compact subset of G_τ/H. Then we have $\Phi\mu \in M(G_\tau/H)$ by Proposition 5. Let $\nu \in M(G_\tau/H)$, then there exists a Borel set C_1 of G_{τ_0}/H such that $\psi^{-1}(C_1)$ is σ-compact and ν is concentrated on C_1. There exists $\lambda \in M(G_{\tau_0})$ such that $\Phi\lambda = \nu$ by Proposition 3. We put $\lambda_0(E) = \lambda(E \cap \varphi_1^{-1}(C_1))$ for every Borel set E of G_{τ_0}. Then we have $\Phi\lambda_0 = \nu$. Since $\eta^{-1}(\varphi_1^{-1}(C_1)) = \varphi_2^{-1}(\psi^{-1}(C_1))$, $\eta^{-1}(\varphi_1^{-1}(C_1))$ is a σ-compact subset of G_τ by Proposition 4. By Proposition 5, we have $\lambda_0 \in M(G_\tau)$. Then $\Phi(M(G_\tau)) = M(G_\tau/H)$. Next, we shall show (8). For every Borel set C_2 of G_{τ_0}/H such that $\psi^{-1}(C_2)$ is a σ-compact subset of G_τ/H, $\eta^{-1}(\varphi_1^{-1}(C_2))$ is σ-compact. Then $\Phi\mu(C_2) = \mu(\varphi^{-1}(C_2)) = 0$ for every $\mu \in M(G_\tau)^\perp$. By Corollary 6, we have $\Phi\mu \in M(G_\tau/H)^\perp$. Conversely, for $\nu \in M(G_\tau/H)^\perp$, there exists $\mu \in M(G_{\tau_0})$ such that $\Phi\mu = \nu$. By Proposition 6, we have $\mu = \mu_1 + \mu_2$ where $\mu_1 \in M(G_\tau)$ and $\mu_2 \in M(G_\tau)^\perp$. Since $\Phi\mu_1 \in M(G_\tau/H)$ and $\Phi\mu_2 \in M(G_\tau/H)^\perp$, we have $\Phi\mu_2 = \nu$. Then $\Phi(M(G_\tau)^\perp) = M(G_\tau/H)^\perp$.

q.e.d.

The following lemma is essential to show our main theorem.

LEMMA 9. *Let K be a σ-compact open subgroup of G_τ. Then there exists a compact subgroup H of G_τ such that G_{τ_0}/H contains a perfect compact $(\varphi(K), 1)$-independent subset, where φ is the canonical map from G_{τ_0} onto G_{τ_0}/H.*

PROOF. Let $K = \bigcup_{m=1}^{\infty} K_m$, such that $K_1 \subset K_2 \subset, \cdots, K_m \subset \cdots$ $(m = 1, 2, \cdots)$ are compact subsets of G_τ. There exists a countable family $\{U_n\}$ $(n = 1, 2, \cdots)$, where U_n is a compact neighborhood of $0 \in G_\tau$ such that

(9) $U_n = -U_n$ $(n = 1, 2, \cdots)$,

(10) $U_n \supset U_{n+1} + U_{n+1}$ $(n = 1, 2, \cdots)$.

Let $K_0 = \bigcap_{n=1}^{\infty} U_n$, then K_0 is a compact subgroup of G_τ. By Proposition 3 of [1], there exists a countable family $\{W_{m,n}\}$ $(m, n = 1, 2, \cdots)$, where $W_{m,n}$ is a compact neighborhood of $0 \in G_{\tau_0}$ such that

(11) $W_{m,n} = -W_{m,n}$,

(12) $W_{m,n} \supset W_{m,n+1} + W_{m,n+1}$, and

(13) $W_{m,n} \cap K_m \subset U_m$.

Let $V_n = \bigcap_{j=1}^{n} \bigcap_{k=1}^{n} W_{j,k}$, then $\{V_n\}$ $(n = 1, 2, \cdots)$ has the following properties:

(14) $V_n = -V_n$,

(15) $V_n \supset V_{n+1} + V_{n+1}$, and

(16) $V_n \cap K_n \subset U_n$ $(n = 1, 2, \cdots)$.

Let $H_0 = \bigcap_{n=1}^{\infty} V_n$, then H_0 is a compact subgroup of G_{τ_0}. For $x \in H_0 \cap K$, there exists a positive integer n_0 such that $x \in K_n \cap V_n$ for every $n \geqq n_0$. Since $U_1 \supset U_2 \supset \cdots$, we have $x \in K_0$. Thus we get that $H_0 \cap K \subset K_0$, and $H_0 \cap K$ is a compact subgroup of G_τ. Let φ_0 be the canonical map from G_{τ_0} onto $G_{\tau_0}/H_0 \cap K$, then

(17) $\varphi_0(H_0) \cap \varphi_0(K) = \{0\}$.

We consider the following two cases.

Case I. Suppose $\varphi_0(H_0)$ is an infinite compact subgroup of $G_{\tau_0}/H_0 \cap K$. Then there exists a perfect independent set of $\varphi_0(H_0)$. By (17), it is a $(\varphi_0(K), 1)$-independent set. Thus $H = H_0 \cap K$ and $\varphi = \varphi_0$ satisfy this lemma.

Case II. Suppose $\varphi_0(H_0)$ is a finite compact subgroup of $G_{\tau_0}/H_0 \cap K$. Let φ_1 be the canonical map from G_{τ_0} to G_{τ_0}/H_0. Since $\varphi_0(H_0)$ is finite, H_0 is a compact subgroup of G_τ. Now, we show $\varphi_1(K)$ is a set of the first category in G_{τ_0}/H_0. Otherwise there exists a positive integer n such that $\varphi_1(K_n)$ contains an interior point. Since $\varphi_1^{-1}(\varphi_1(K_n)) = K_n + H_0$, $K_n + H_0$ contains an interior point in G_{τ_0}. Then we have $m_{\tau_0}(K_n + H_0) > 0$, where m_{τ_0} is a Haar measure on G_{τ_0}. Since $K_n + H_0$ is a compact subset of G_τ, by Proposition 5 we have $m_{\tau_0}(K_n + H_0) = 0$, a contradiction. Since G_{τ_0}/H_0 is metrizable ([2]), there is a $(\varphi_1(K), 1)$-independent compact perfect subset of G_{τ_0}/H_0 ([10]). We put $H = H_0$ and $\varphi = \varphi_1$, then the proof is complete. q.e.d.

THEOREM 10. *Let τ be a locally compact abelian group topology on G strictly stronger than τ_0, then we have $M(\Delta) \subset M(G_\tau)^\perp$.*

PROOF. Since $M(\Delta)$ and $M(G_\tau)$ are L-ideals, it is sufficient to show that for any non-negative $\mu \in M(G_\tau)$, we have $\mu \notin M(\Delta)$. Let K be a σ-compact open subgroup of G_τ. We take H and φ satisfying Lemma 9. Let Φ be the homomorphism from $M(G_{\tau_0})$ to $M(G_{\tau_0}/H)$ induced by φ. Let $\mathfrak{F}$ be the Raikov system generated by $\varphi(K)$, then $M(\mathfrak{F}) = M(G_\tau/H)$. By Proposition 8 and Theorem 1, for any nonzero $\mu \in M(G_\tau)$, there exists a non-symmetric complex homomorphism f on $M(G_{\tau_0}/H)$ such that $f\circ\Phi(\mu) = f(\Phi\mu) \neq 0$. From (c) of Proposition 3, $f\circ\Phi$ is a nonsymmetric homomorphism f on $M(G_{\tau_0})$. Thus we have $\mu \notin M(\Delta)$. q.e.d.

3. Gelfand transforms of $M(\Delta)$. Let G be a nondiscrete locally abelian group, and S be Taylor's structure semigroup of $M(G)$. The maximal ideal space of $M(G)$ is identified with $\hat{S}$, with the weak$*$-topology of $M(G)$, the set of all nonzero continuous semicharacters on S. We may consider $\hat{S}$, a compact separately continuous abelian semigroup. Let $H = \{f \in \hat{S}: |f|^2 = |f|\}$, then $\hat{S}\backslash H \neq \emptyset$ (c.f. [7]).

B. E. Johnson [4] showed that $(\hat{S}\backslash H) \cap \Delta \neq \emptyset$. In this section, we give a topological characterization of $(\hat{S}\backslash H) \cap \Delta$. For $f \in \hat{S}\backslash H$, we put $J(f) = \{x \in S: f(x) = 0\}$ and $\mathfrak{M}(J(f)) = \{\mu \in M(G): \operatorname{supp}\theta\mu \subset J(f)\}$. Let C be the complex field and $C^+ = \{z \in C: \operatorname{Re} z > 0\}$.

THEOREM 11. *$\hat{S}\backslash H$ is contained in the weak*-closure of $\hat{S}\backslash\Delta$ in $\hat{S}$, that is $\overline{\hat{S}\backslash\Delta} \supset \hat{S}\backslash H$.*

PROOF. Let $f \in \hat{S}\backslash H$ and $f \in \Delta$. Then there exists $h_f \in H$ such that $f = h_f|f|$ by the polar decomposition theorem ([7]). We put $f_z = h_f|f|^z$ for $z \in C^+$, then $f_z \in \hat{S}$. Let V be any neighborhood of f. We may assume that

$$V = \{g \in \hat{S}: |\hat{\mu}_i(f) - \hat{\mu}_i(g)| < \varepsilon,\ \mu_i \in M(G),\ i = 1, 2, \cdots, n\}\ .$$

Since $f_z \to f (z \to 1)$ is uniformly convergent, there exists $\delta > 0$ such that $f_z \in V$ for $z \in \{x \in C^+: |1 - x| < \delta\}$. Since $f \in \hat{S}\backslash H$, there exists $x_0 \in S$ such that $0 < |f(x_0)| < 1$. We take a neighborhood $U(x_0)$ of x_0 such that $0 < |f(x)| < 1$ on $U(x_0)$. The image of $M(G)$ is weak*-dense in $M(S)$, then there exists $\mu \in M(G)$ such that the support of $\theta\mu$ is contained in $U(x_0)$

and $\hat{\mu}(f) \neq 0$. We put $F(z) = \hat{\mu}(f_z)$ for $z \in C^+$. Then $F(z)$ is a nonconstant analytic function on C^+. Suppose that $\{f_z: |1-z| < \delta\} \subset \Delta$. Then we have $\overline{F(z)} = \overline{\hat{\mu}(f_z)} = \widehat{\mu^*}(f_z)$ for $z \in \{x \in C^+: |1-x| < \delta\}$, and $F(z)$ is an analytic function on $\{x \in C^+: |1-x| < \delta\}$. Thus $\overline{F(z)}$ is a constant on $\{x \in C^+: |1-x| < \delta\}$. By identity theorem, $F(z)$ is a constant function on C^+. This is a contradiction. Then there exists $g \in \{f_z: |1-z| < \delta\}$ such that $g \notin \Delta$. q.e.d.

COROLLARY 12. *If $\mu \in M(\Delta)$, then $\hat{\mu}(f) = 0$ for all $f \in \hat{S} \backslash H$.*

Let $\hat{G}$ be the dual group of G. T. Shimizu ([6]) showed that

$$(\hat{S} \backslash \Delta) \cdot \hat{G} \subset (\hat{S} \backslash \Delta) .$$

Since $\hat{S}$ is a separetely continuous topological semigroup, we have

$$\overline{(\hat{S} \backslash \Delta)} \cdot \hat{G} \subset \overline{(\hat{S} \backslash \Delta)} .$$

COROLLARY 13. *If $f \in \hat{S} \backslash H$, then $M(\Delta) \subset \mathfrak{M}(J(f))$.*

PROOF. Since $f \in \overline{\hat{S} \backslash \Delta}$, we have $f \cdot \hat{G} \subset \overline{\hat{S} \backslash \Delta}$. Let $\mu \in M(\Delta)$, then we have $\hat{\mu}(g) = 0$ for $g \in f \cdot \hat{G}$. This shows that $\mu \in \mathfrak{M}(J(f))$ by Shimizu ([6]). Thus we have $M(\Delta) \subset \mathfrak{M}(J(f))$. q.e.d.

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