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AUTOMORPHISM GROUPS OF HOPF SURFACES

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Introduction. Let $GL(2, \boldsymbol{C})$ be the group of non-singular (2×2)-matrices. An element $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL(2, \boldsymbol{C})$ operates on $\boldsymbol{C}^2$ as follows:

$$(z, w) \to (az + bw, cz + dw) .$$

Let M be a subset of $GL(2, \boldsymbol{C})$ defined by

$$M = \left\{ \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta, t \in \boldsymbol{C}, 0 < |\alpha| < 1, 0 < |\beta| < 1 \right\} .$$

Then M is a complex manifold. Let 0 be the origin of $\boldsymbol{C}^2$. We put $W = \boldsymbol{C}^2 - 0$. Let $u \in M$. Then u defines a properly discontinuous group

$$G_u = \{u^n \mid n \in \boldsymbol{Z}\}$$

of automorphisms (holomorphic isomorphisms) without fixed point of W. Hence we have a complex manifold

$$V_u = W/G_u .$$

V_u is easily seen to be compact. It is called a Hopf surface. It can be shown that the collection

$$\{V_u\}_{u \in M}$$

forms a complex analytic family (X, π, M). We denote by $\mathrm{Aut}\,(V_u)$ the group of automorphisms of V_u.

The purpose of this note is prove the following theorem.

THEOREM. *The disjoint union*

$$A = \coprod_{u \in M} \mathrm{Aut}\,(V_u)$$

admits a (reduced) analytic space structure such that

1) λ: $A \to M$ *is a surjective holomorphic map, where* λ *is the canonical projection,*

2) *the map*

$$A \times_M X \to X$$

defined by

$$(f, P) \to f(P) ,$$

is holomorphic, where

$$A \underset{M}{\times} X = \{(f, P) \in A \times X \mid \lambda(f) = \pi(P)\} ,$$

the fiber product of A and X over M,

3) *the map* $M \to A$ *defined by*

$$u \to 1_u$$

is holomorphic, where 1_u *is the identity map of* V_u,

4) *the map*

$$A \underset{M}{\times} A \to A$$

defined by

$$(f, g) \to g^{-1}f ,$$

is holomorphic, where

$$A \underset{M}{\times} A = \{(f, g) \in A \times A \mid \lambda(f) = \lambda(g)\} ,$$

the fiber product of A and A over M.

1. The complex analytic family of Hopf surfaces. By *a complex analytic family of compact complex manifolds,* we mean a triple (X, π, M) of complex manifolds X and M and a proper holomorphic map of X onto M which is of maximal rank at every point of X, i.e.,

$$\operatorname{rank} J(f)_P = \dim M$$

for all $P \in X$, where $J(f)_P$ is the Jacobian matrix of f at P. In this case, each fiber $\pi^{-1}(u)$, $u \in M$, is a compact complex manifold. M is called *the parameter space of the family* (X, π, M).

Now, let

$$M = \left\{\begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix} \in GL(2, \boldsymbol{C}) \mid \alpha, \beta, t \in \boldsymbol{C},\ 0 < |\alpha| < 1,\ 0 < |\beta| < 1\right\}$$

and

$$W = \boldsymbol{C}^2 - 0 .$$

We define a holomorphic map

$$\eta\colon M \times W \to M \times W$$

by

$$\eta(u, x) = (u, ux) .$$

Then η is an automorphism, for η^{-1} is given by

$$(u, x) \longrightarrow (u, u^{-1}x) .$$

We put $G = \{\eta^n \mid n \in \boldsymbol{Z}\}$.

LEMMA 1. *G is a properly discontinuous group of automorphisms without fixed point of $M \times W$.*

PROOF. We assume that $(u, u^n x) = (u, x)$ for an integer $n \neq 0$. Then $u^n x = x$. We write

$$u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}, \quad 0 < |\alpha| < 1 , \quad 0 < |\beta| < 1$$

and $x = (z, w)$. Then

$$\begin{aligned} u^n x &= \left(\alpha^n z + \frac{\alpha^n - \beta^n}{\alpha - \beta} tw, \beta^n w\right), && \text{if} \quad \alpha \neq \beta , \\ &= (\alpha^n z + n\alpha^{n-1} tw, \alpha^n w) , && \text{if} \quad \alpha = \beta . \end{aligned}$$

Since $0 < |\alpha| < 1$ and $0 < |\beta| < 1$, $u^n x = x$ implies that $w = 0$, so that $z = 0$, a contradiction. Hence G has no fixed point. In order to show that G is a properly discontinuous group, it is enough to show that, for a compact subset K_1 of M and a compact subset K_2 of W,

$$\{n \in \boldsymbol{Z} \mid \eta^n(K_1 \times K_2) \cap (K_1 \times K_2) \neq \emptyset\}$$

is a finite set. There are positive constants c and d such that

$$|\alpha|, |\beta| \leqq c < 1 \quad \text{and} \quad |t| \leqq d$$

for all $\begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix} \in K_1$. We define a norm $|\ \ |$ in $\boldsymbol{C}^2$ by

$$|(z, w)| = |z| + |w| .$$

Then there are positive constants a and b such that

$$a \leqq |x| \leqq b$$

for all $x \in K_2$. Now

$$|u^n x| = |\alpha^n z + \gamma_n tw| + |\beta^n w|$$

where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix} \in K_1$, $x = (z, w) \in K_2$ and

$$\begin{aligned} \gamma_n &= \frac{\alpha^n - \beta^n}{\alpha - \beta}, && \text{if} \quad \alpha \neq \beta , \\ &= n\alpha^{n-1} , && \text{if} \quad \alpha = \beta . \end{aligned}$$

Hence, for a positive integer n,

$$|u^n x| \leqq |\alpha|^n |z| + |\gamma_n||t||w| + |\beta|^n |w|$$
$$\leqq c^n b + nc^{n-1}db + c^n b \longrightarrow 0$$

as $n \longrightarrow +\infty$. Thus there is a positive integer N such that

$$|u^n x| < a$$

for all $n \geqq N$. Next, we show that there is a positive integer N' such that

$$|u^{-n} x| > b$$

for all $n \geqq N'$ and for all $(u, x) \in K_1 \times K_2$. We assume the converse. Then there are a sequence of points $\{(u_\nu, x_\nu)\}_{\nu=1,2,\cdots}$ of $K_1 \times K_2$ and a sequence of integers

$$n_1 < n_2 < \cdots$$

such that

$$|u_\nu^{-n_\nu} x_\nu| \leqq b\,, \qquad \nu = 1, 2, \cdots .$$

We put $y_\nu = u_\nu^{-n_\nu} x_\nu$, $\nu = 1, 2, \cdots$. Then $x_\nu = u_\nu^{n_\nu} y_\nu$, $\nu = 1, 2, \cdots$. We put

$$y_\nu = (z'_\nu, w'_\nu) \quad \text{and} \quad u_\nu = \begin{pmatrix} \alpha_\nu & t_\nu \\ 0 & \beta_\nu \end{pmatrix}, \qquad \nu = 1, 2, \cdots .$$

Then

$$x_\nu = u_\nu^{n_\nu} y_\nu = (\alpha_\nu^{n_\nu} z'_\nu + \gamma_\nu t_\nu w'_\nu, \beta_\nu^{n_\nu} w'_\nu)\,, \qquad \nu = 1, 2, \cdots ,$$

where

$$\gamma_\nu = \frac{\alpha_\nu^{n_\nu} - \beta_\nu^{n_\nu}}{\alpha_\nu - \beta_\nu}\,, \qquad \text{if} \quad \alpha_\nu \neq \beta_\nu\,,$$
$$= n_\nu \alpha_\nu^{n_\nu - 1}\,, \qquad \text{if} \quad \alpha_\nu = \beta_\nu\,.$$

Hence

$$|x_\nu| \leqq (c^{n_\nu} + n_\nu c^{n_\nu - 1} d + c^{n_\nu}) b \longrightarrow 0$$

as $\nu \longrightarrow +\infty$. This contradicts to

$$\{x_\nu\}_{\nu=1,2,\cdots} \subset K_2\,.$$

Now

$$\{n \in \boldsymbol{Z} \mid \eta^n(K_1 \times K_2) \cap (K_1 \times K_2) \neq \emptyset\}$$

is contained in

$$\{n \in \boldsymbol{Z} \mid -N' < n < N\}\,. \qquad \text{q.e.d.}$$

By Lemma 1, the quotient space

$$X = (M \times W)/G$$

is a complex manifold. Let $\tilde{\pi}: M \times W \to M$ be the canonical projection. Then $\tilde{\pi}\eta = \tilde{\pi}$. Hence there is a holomorphic map

$$\pi: X \to M$$

such that the diagram

$$\begin{array}{ccc} M \times W & \xrightarrow{p} & X \\ & {\scriptstyle \tilde{\pi}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

is commutative, where p is the canonical projection. Since p is a covering map, π is a surjective holomorphic map of maximal rank at every point of X.

LEMMA 2. *π is a proper map.*

PROOF. Let K be a compact subset of M. We show that $\pi^{-1}(K)$ is compact. Let $\{P_\nu\}_{\nu=1,2,\cdots}$ be a sequence of points in $\pi^{-1}(K)$. We want to choose a subsequence of $\{P_\nu\}_{\nu=1,2,\cdots}$ converging to a point of $\pi^{-1}(K)$. We may assume that $\{\pi(P_\nu)\}_{\nu=1,2,\cdots}$ converges to a point $u \in K$. We put $u_\nu = \pi(P_\nu)$, $\nu = 1, 2, \cdots$. We put

$$u_\nu = \begin{pmatrix} \alpha_\nu & t_\nu \\ 0 & \beta_\nu \end{pmatrix}, \qquad \nu = 1, 2, \cdots$$

and

$$u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}.$$

Then $\alpha_\nu \to \alpha$, $\beta_\nu \to \beta$ and $t_\nu \to t$ as $\nu \to +\infty$. We may assume that there are positive constants c_1, c_2 and d such that

$$c_1 \leqq |\alpha_\nu| \leqq c_2 < 1\,, \quad c_1 \leqq |\beta_\nu| \leqq c_2 < 1 \quad \text{and} \quad |t_\nu| \leqq d$$

for all ν. Let x_ν, $\nu = 1, 2, \cdots$, be points of W such that

$$p(u_\nu, x_\nu) = P_\nu\,, \qquad \nu = 1, 2, \cdots.$$

We put

$$x_\nu = (z_\nu, w_\nu)\,, \qquad \nu = 1, 2, \cdots.$$

We define a norm $|\ |$ in $\boldsymbol{C}^2$ by

$$|(z, w)| = |z| + |w|.$$

First, we assume that there is a subsequence

$$\nu_1 < \nu_2 < \cdots$$

such that

$$w_{\nu_k} = 0 , \qquad k = 1, 2, \cdots .$$

Then $z_{\nu_k} \neq 0$, $k = 1, 2, \cdots$. Thus there are integers n_k, $k = 1, 2, \cdots$, such that

$$c_1 \leqq |\alpha_{\nu_k}| \leqq |\alpha_{\nu_k}^{n_k} z_{\nu_k}| \leqq 1 .$$

We put $z'_{\nu_k} = \alpha_{\nu_k}^{n_k} z_{\nu_k}$, $k = 1, 2, \cdots$. We put $x'_{\nu_k} = (z'_{\nu_k}, 0)$, $k = 1, 2, \cdots$. Then $x'_{\nu_k} = u_{\nu_k}^{n_k} x_{\nu_k}$, $k = 1, 2, \cdots$. Hence

$$P_{\nu_k} = p(u_{\nu_k}, x_{\nu_k}) = p(u_{\nu_k}, x'_{\nu_k}) , \qquad k = 1, 2, \cdots .$$

Since $c_1 \leqq |x'_{\nu_k}| \leqq 1$, $k = 1, 2, \cdots$, we may assume that $\{x'_{\nu_k}\}_{k=1,2,\dots}$ converges to a point $x \in W$. Then $\{P_{\nu_k}\}_{k=1,2,\dots}$ converges to $p(u, x)$.

Now, we may assume that $w_\nu \neq 0, \nu = 1, 2, \cdots$. Since there are integers n_ν, $\nu = 1, 2, \cdots$, such that

$$c_1 \leqq |\beta_\nu| \leqq |\beta_\nu^{n_\nu} w_\nu| \leqq 1 , \qquad \nu = 1, 2, \cdots ,$$

we may assume that

$$c_1 \leqq |w_\nu| \leqq 1 , \qquad \nu = 1, 2, \cdots .$$

(We use $u_\nu^{n_\nu} x_\nu$ instead of x_ν.) Hence we may assume that $\{w_\nu\}_{\nu=1,2,\dots}$ converges to a point $w \in C$, $c_1 \leqq |w| \leqq 1$. Since the Riemann sphere $\hat{C}$ is compact, we may assume that $\{z_\nu\}_{\nu=1,2,\dots}$ converges to a point z of $\hat{C}$. If $z \neq \infty$, then $x = (z, w)$ is a point of W and $\{P_\nu\}_{\nu=1,2,\dots}$ converges to $p(u, x)$. If $z = \infty$, we may assume that

$$1 < |z_1| < |z_2| < \cdots \to +\infty .$$

Then there is a sequence of positive integers $\{n_\nu\}_{\nu=1,2,\dots}$ such that

$$c_1 \leqq |\alpha_\nu| \leqq |\alpha_\nu^{n_\nu} z_\nu| \leqq 1 , \qquad \nu = 1, 2, \cdots .$$

Let N be a positive integer such that

$$|n c_2^{n-1} d| \leqq \frac{c_1}{2}$$

for all $n \geqq N$. We may assume that $|z_1|$ is so large that

$$c_1^{-N} < |z_1| .$$

Then

$$|\alpha_\nu|^{-N} \leqq c_1^{-N} < |z_1| \leqq |z_\nu| , \qquad \nu = 1, 2, \cdots .$$

Hence

$$|\alpha_\nu^N z_\nu| > 1\,, \qquad \nu = 1, 2, \cdots .$$

This shows that

$$n_\nu > N\,, \qquad \nu = 1, 2, \cdots .$$

Hence

$$|n_\nu c_2^{n_\nu - 1} d| \leqq \frac{c_1}{2} \qquad \nu = 1, 2, \cdots .$$

We put

$$z'_\nu = \alpha_\nu^{n_\nu} z_\nu + \gamma_\nu t_\nu w_\nu\,, \qquad \nu = 1, 2, \cdots ,$$

where

$$\begin{aligned} \gamma_\nu &= \frac{\alpha_\nu^{n_\nu} - \beta_\nu^{n_\nu}}{\alpha_\nu - \beta_\nu}\,, && \text{if} \quad \alpha_\nu \neq \beta_\nu\,, \\ &= n_\nu \alpha_\nu^{n_\nu - 1}\,, && \text{if} \quad \alpha_\nu = \beta_\nu\,. \end{aligned}$$

Then

$$\begin{aligned} \frac{c_1}{2} = c_1 - \frac{c_1}{2} &\leqq |\alpha_\nu^{n_\nu} z_\nu| - |\gamma_\nu t_\nu w_\nu| \leqq |z'_\nu| \\ &\leqq |\alpha_\nu^{n_\nu} z_\nu| + |\gamma_\nu t_\nu w_\nu| \leqq 1 + \frac{c_1}{2}\,, \qquad \nu = 1, 2, \cdots . \end{aligned}$$

Hence we may assume that $\{z'_\nu\}_{\nu=1,2,\cdots}$ converges to a point $z' \in \boldsymbol{C}$, $c_1/2 \leqq |z'| \leqq 1 + c_1/2$. Since $|w_\nu| \leqq 1$, $\nu = 1, 2, \cdots$,

$$|\beta_\nu^{n_\nu} w_\nu| \leqq |\beta_\nu w_\nu| \leqq |\beta_\nu| \leqq c_2\,, \qquad \nu = 1, 2, \cdots .$$

We may assume that $\{\beta_\nu^{n_\nu} w_\nu\}_{\nu=1,2,\cdots}$ converges to $w' \in \boldsymbol{C}$. We put $x = (z', w') \in W$. Then $\{P_\nu\}_{\nu=1,2,\cdots}$ converges to $p(u, x)$. q.e.d.

Lemma 2 shows that (X, π, M) is a complex analytic family of compact complex manifolds. Each fiber $\pi^{-1}(u)$, $u \in M$, is called a Hopf surface. Each fiber can be written as

$$\pi^{-1}(u) = u \times V_u$$

where

$$V_u = W/G_u$$

and

$$G_u = \{u^n \mid n \in \boldsymbol{Z}\}\,.$$

A similar but simpler argument to the proof of Lemma 1 shows that G_u is a properly discontinuous group of automorphisms without fixed point of W. Henceforth, we identify $\pi^{-1}(u)$ with V_u.

2. **Automorphism groups of Hopf surfaces.** Let $u \in M$. Let V_u be the corresponding Hopf surface. Let $\mathrm{Aut}(V_u)$ be the group of automorphisms of V_u. Let

$$C_u = \{v \in GL(2, \boldsymbol{C}) \mid uv = vu\} .$$

Then C_u is a complex Lie subgroup of $GL(2, \boldsymbol{C})$. We define a homomorphism

$$h_u \colon C_u \longrightarrow \mathrm{Aut}(V_u)$$

by

$$v \longrightarrow \tilde{v}$$

where $\tilde{v}$ is an automorphism of V_u defined by

$$\tilde{v} \colon p(x) \longrightarrow p(vx)$$

for all $x \in W$, where $p \colon W \longrightarrow V_u$ is the canonical projection. Since $uv = vu$, $\tilde{v}$ is well defined.

LEMMA 3. $\ker(h_u) = G_u$.

PROOF. Let $u^n \in G_u$. Then

$$\tilde{u}_n \colon p(x) \longrightarrow p(u^n x) = p(x) .$$

Hence $G_u \subset \ker(h_u)$. Conversely, let $v \in \ker(h_u)$. Then

$$p(vx) = p(x)$$

for all $x \in W$. Hence, for each $x \in W$, there is an integer $k(x)$ such that

$$vx = u^{k(x)}x .$$

We show that

$$k(cx) = k(x)$$

if $c \in \boldsymbol{C}$ and $c \neq 0$. In fact

$$u^{k(cx)}cx = v(cx) = cvx = cu^{k(x)}x = u^{k(x)}cx$$

so that

$$u^{k(cx)-k(x)}cx = cx .$$

Since G_u operates on W without fixed point,

$$k(cx) = k(x) .$$

Thus we may consider k to be a $\boldsymbol{Z}$-valued function on $\boldsymbol{P}^1(\boldsymbol{C})$, the 1-dimensional projective space. Since the cardinal number of the set $\boldsymbol{P}^1(\boldsymbol{C})$ is greater than that of $\boldsymbol{Z}$, there are distinct points L_1 and L_2 in $\boldsymbol{P}^1(\boldsymbol{C})$ such that $k(L_1) = k(L_2)$. We put $k = k(L_1) = k(L_2)$. Let x_1 and x_2 be points in W such that $x_1 \in L_1$ and $x_2 \in L_2$. Then, for any point $x \in W$, there are complex numbers a and b such that

$$x = ax_1 + bx_2 .$$

Here we regard x, x_1 and x_2 as vectors $0x$, $0x_1$, and $0x_2$ respectively. Then

$$vx = v(ax_1 + bx_2) = avx_1 + bvx_2 = au^k x_1 + bu^k x_2 = u^k x .$$

Hence $v = u^k$. q.e.d.

Now, we determine $\mathrm{Aut}(V_u)$ following the argument in [1]. Let

$$f\colon V_u \to V_u$$

be an automorphism. Since W is the universal covering space of V_u, there is an automorphism

$$\tilde{f}\colon W \to W$$

such that the diagram

$$\begin{array}{ccc} W & \xrightarrow{\ \tilde{f}\ } & W \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ V_u & \xrightarrow{\ f\ } & V_u \end{array}$$

is commutative where p is the canonical projection. Moreover $\tilde{f}$ satisfies

$$\tilde{f}(ux) = u^g \tilde{f}(x)$$

for all $x \in W$, where u^g is a generator of G_u, ($u^g = u$ or u^{-1}). We show $u^g = u$. By Hartogs's theorem, $\tilde{f}$ is extended to an automorphism

$$\tilde{f}\colon \boldsymbol{C}^2 \to \boldsymbol{C}^2$$

which maps 0 to 0. If

$$\tilde{f}(ux) = u^{-1}\tilde{f}(x)$$

for all $x \in W$, then

$$u^n \tilde{f}(u^n x) = \tilde{f}(x)$$

for $n = 1, 2, \cdots$ and for all $x \in W$. We fix $x = (z, w) \in W$. We put $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$. Then

$$u^n x = (\alpha^n z + \gamma_n tw, \beta^n w)$$

where

$$\gamma_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \text{if} \quad \alpha \neq \beta,$$
$$= n\alpha^{n-1}, \qquad \text{if} \quad \alpha = \beta.$$

Hence

$$|u^n x| \leqq |\alpha|^n |z| + |\gamma_n||t||w| + |\beta|^n |w| \to 0$$

as $n \to +\infty$. Hence

$$\tilde{f}(u^n x) \to 0$$

as $n \to +\infty$, for the extended map $\tilde{f}$ maps 0 to 0. On the other hand, there is an integer N such that

$$|u^n \tilde{f}(u^n x)| < |\tilde{f}(u^n x)|$$

for all $n \geqq N$. In fact, it is enough to take N such that

$$|\beta^N| + |\gamma_N||t| < 1.$$

Hence

$$|u^n \tilde{f}(u^n x)| \to 0$$

as $n \to +\infty$. This contradicts to

$$u^n \tilde{f}(u^n x) = \tilde{f}(x), \qquad n = 1, 2, \cdots.$$

Hence

$$\tilde{f}(ux) = u\tilde{f}(x)$$

for all $x \in W$. We write the extended automorphism $\tilde{f}: \boldsymbol{C}^2 \to \boldsymbol{C}^2$ as

$$\tilde{f}(z, w) = (g(z, w), h(z, w)).$$

Then the above condition is written as

$$g(\alpha z + tw, \beta w) = \alpha g(z, w) + th(z, w),$$
$$h(\alpha z + tw, \beta w) = \beta h(z, w).$$

We expand g and h in the power series of z and w at the origin:

$$g(z, w) = \sum_{p+q>0} c_{pq} z^p w^q,$$
$$h(z, w) = \sum_{p+q>0} d_{pq} z^p w^q.$$

Case 1. $\beta = \alpha$ *and* $t = 0$; $u = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$.

In this case, above equations reduce to

$$\sum_{p+q>0} \alpha^{p+q} c_{pq} z^p w^q = \sum_{p+q>0} \alpha c_{pq} z^p w^q,$$

$$\sum_{p+q>0} \alpha^{p+q} d_{pq} z^p w^q = \sum_{p+q>0} \alpha d_{pq} z^p w^q .$$

Since $0 < |\alpha| < 1$, we get

$$c_{pq} = d_{pq} = 0 ,$$

if $p + q > 1$. Hence

$$g(z, w) = c_{10} z + c_{01} w ,$$
$$h(z, w) = d_{10} z + d_{01} w .$$

Since $\tilde{f}$ is an automorphism, the matrix $\begin{pmatrix} c_{10} & c_{01} \\ d_{10} & d_{01} \end{pmatrix}$ is non-singular. Thus

$$\mathrm{Aut}\,(V_u) \cong \frac{GL(2, C)}{G_u} = \frac{C_u}{G_u} , \qquad \dim \frac{C_u}{G_u} = 4 ,$$

for all $u = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$.

Case 2. $\beta = \alpha$ *and* $t \neq 0$; $u = \begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix}$.

In this case, above equations reduce to

$$\sum_{p+q>0} c_{pq}(\alpha z + tw)^p (\alpha w)^q = \sum_{p+q>0} (\alpha c_{pq} + t d_{pq}) z^p w^q ,$$
$$\sum_{p+q>0} d_{pq}(\alpha z + tw)^p (\alpha w)^q = \sum_{p+q>0} \alpha d_{pq} z^p w^q .$$

From the last equation, we have

$$d_{10} = 0 \quad \text{and} \quad d_{pq} = 0 , \quad \text{if} \quad p + q > 1 .$$

Hence

$$h(z, w) = d_{01} w .$$

Hence, from the first equation, we have

$$c_{10} = d_{01} \quad \text{and} \quad c_{pq} = 0 , \quad \text{if} \quad p + q > 1 .$$

Thus

$$g(z, w) = c_{10} z + c_{01} w ,$$
$$h(z, w) = c_{10} w .$$

Since $\begin{pmatrix} c_{10} & c_{01} \\ 0 & c_{10} \end{pmatrix} \in C_u$, we have

$$\mathrm{Aut}\,(V_u) \cong \frac{C_u}{G_u} , \qquad \dim \frac{C_u}{G_u} = 2 ,$$

for all $u = \begin{pmatrix} \alpha & t \\ 0 & \alpha \end{pmatrix}$, $t \neq 0$.

Case 3. $\beta \neq \alpha$ *and* $t=0$; $u=\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$.

In this case, above equations reduce to

$$\sum_{p+q>0} c_{pq}\alpha^p\beta^q z^p w^q = \sum_{p+q>0} \alpha c_{pq} z^p w^q ,$$

$$\sum_{p+q>0} d_{pq}\alpha^p\beta^q z^p w^q = \sum_{p+q>0} \beta d_{pq} z^p w^q .$$

Hence, if $p>0$ and $q>0$, then

$$c_{pq}=0 \quad \text{and} \quad d_{pq}=0 .$$

If $p=0$, then

$$c_{0q}(\beta^q-\alpha)=0 ,$$

$$d_{0q}(\beta^q-\beta)=0 .$$

Hence $c_{01}=0$ and $d_{0q}=0$, if $q>1$. If $q=0$, then

$$c_{p0}(\alpha^p-\alpha)=0 ,$$

$$d_{p0}(\alpha^p-\beta)=0 .$$

Hence $d_{10}=0$ and $c_{p0}=0$, if $p>1$. Case 3 is thus divided as follows.

Case 3-A. $\beta^q=\alpha$ *for some* $q \geqq 2$; $u=\begin{pmatrix}\beta^q & 0\\ 0 & \beta\end{pmatrix}$.

In this case, $\tilde{f}$ is generally written as

$$\tilde{f}\colon (z, w) \to (az+bw^q, dw)$$

where $ad \neq 0$ and b is arbitrary. We note that

$$C_u = \left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \middle| \, ad \neq 0\right\} .$$

We introduce a group operation in the set $C_u \times \boldsymbol{C}$ as follows:

$$(v', b')(v, b) = (v'v, a'b+b'd^q)$$

where $v=\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$ and $v'=\begin{pmatrix}a' & 0\\ 0 & d'\end{pmatrix}$. By this group operation, $C_u \times \boldsymbol{C}$ becomes a complex Lie group. C_u is then isomorphic to the complex Lie subgroup $C_u \times 0$ of $C_u \times \boldsymbol{C}$. The group $C_u \times \boldsymbol{C}$ is isomorphic to the group of automorphisms $\tilde{f}$ of W such that $\tilde{f}u=u\tilde{f}$. The isomorphism is given by

$$\left(\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}, b\right) \to \tilde{f}$$

where

$$\tilde{f}\colon (z, w) \to (az+bw^q, dw) .$$

Hence there is a surjective homomorphism

$$g_u: C_u \times \boldsymbol{C} \to \mathrm{Aut}(V_u) .$$

We show that $\ker(g_u)$ is equal to $G_u \times 0$. First, for an integer n, $u^n \times 0$ corresponds to the automorphism

$$\tilde{f} = u^n: (z, w) \to (\alpha^n z, \beta^n w)$$

of W which corresponds to the identity map of V_u. Next, let

$$\tilde{f}: (z, w) \to (az + bw^q, dw) , \qquad ad \neq 0 ,$$

be an automorphism of W which corresponds to the identity map of V_u. Then, for each $x = (z, w) \in W$, there is an integer $k(x)$ such that

$$az + bw^q = \alpha^{k(x)} z ,$$
$$dw = \beta^{k(x)} w .$$

In particular, let $x \in W'$ where

$$W' = \{(z, w) \in W \mid z \neq 0 \text{ and } w \neq 0\} .$$

Then, by the second equation, $d = \beta^{k(x)}$. Hence $k(x) = k$ is constant for $x \in W'$. By the first equation, $a = \alpha^k$ and $b = 0$. Hence $\ker(g_u)$ is equal to $G_u \times 0$. Thus

$$\mathrm{Aut}(V_u) \cong \frac{C_u \times \boldsymbol{C}}{G_u \times 0} , \qquad \dim\left(\frac{C_u \times \boldsymbol{C}}{G_u \times 0}\right) = 3 ,$$

for all $u = \begin{pmatrix} \beta^q & 0 \\ 0 & \beta \end{pmatrix}$, $q \geqq 2$. We note that the center of the group $C_u \times \boldsymbol{C}$ is

$$\left\{\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, b\right) \in C_u \times \boldsymbol{C} \mid a = d^q \text{ and } b = 0\right\} .$$

Hence $G_u \times 0$ is contained in the center.

Case 3-B. *$\alpha^p = \beta$ for some $p \geqq 2$; $u = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^p \end{pmatrix}$.*

In this case, $\tilde{f}$ is generally written as

$$\tilde{f}: (z, w) \to (az, dw + bz^p)$$

where $ad \neq 0$ and b is arbitrary. We note that

$$C_u = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \middle|\, ad \neq 0\right\} .$$

We introduce a group operation in the set $C_u \times \boldsymbol{C}$ as follows:

$$(v', b')(v, b) = (v'v, d'b + b'a^p)$$

where $v = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and $v' = \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}$. By this group operation, $C_u \times \boldsymbol{C}$ becomes a complex Lie group. C_u is then isomorphic to the complex Lie subgroup $C_u \times 0$ of $C_u \times \boldsymbol{C}$. By a similar argument to Case 3-A, we have

$$\mathrm{Aut}\,(V_u) \cong \frac{C_u \times \boldsymbol{C}}{G_u \times 0}\,, \qquad \dim\left(\frac{C_u \times \boldsymbol{C}}{G_u \times 0}\right) = 3\,,$$

for all $u = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^p \end{pmatrix}$, $p \geqq 2$. We note that the center of the group $C_u \times \boldsymbol{C}$ is

$$\left\{\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, b\right) \in C_u \times \boldsymbol{C} \,\middle|\, d = a^p \text{ and } b = 0\right\}.$$

Hence $G_u \times 0$ is contained in the center.

Case 3-C. $u = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, $\beta^q \neq \alpha$ *for any positive integer* q *and* $\alpha^p \neq \beta$ *for any positive integer* p.

In this case, $\tilde{f}$ is generally written as

$$\tilde{f}\colon (z, w) \longrightarrow (az, dw)$$

where $ad \neq 0$. We note that

$$C_u = \left\{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \,\middle|\, ad \neq 0\right\}.$$

Thus

$$\mathrm{Aut}\,(V_u) \cong \frac{C_u}{G_u}\,, \qquad \dim \frac{C_u}{G_u} = 2\,,$$

for all $u = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ such that $\beta^q \neq \alpha$ for any positive integer q and $\alpha^p \neq \beta$ for any positive integer p.

Case 4. $\alpha \neq \beta$ *and* $t \neq 0$, $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$.

Let $\tilde{u} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and $y = \begin{pmatrix} 1 & t/(\alpha - \beta) \\ 0 & 1 \end{pmatrix}$. Then $y^{-1} = \begin{pmatrix} 1 & -t/(\alpha - \beta) \\ 0 & 1 \end{pmatrix}$ and $\tilde{u} = yuy^{-1}$. Thus y induces a holomorphic isomorphism

$$\hat{y}\colon V_u \longrightarrow V_{\tilde{u}}$$

defined by

$$\hat{y}\colon p_u(z, w) \longrightarrow p_{\tilde{u}}(y(z, w)) = p_{\tilde{u}}\left(z + \frac{t}{\alpha - \beta} w, w\right)$$

where $p_u\colon W \longrightarrow V_u$ and $p_{\tilde{u}}\colon W \longrightarrow V_{\tilde{u}}$ are canonical projections. Hence

$$\mathrm{Aut}\,(V_u) \cong \mathrm{Aut}\,(V_{\tilde{u}})$$

by the correspondence

$$f \in \mathrm{Aut}(V_u) \to \hat{y} f \hat{y}^{-1} \in \mathrm{Aut}(V_{\tilde{u}}) .$$

Thus Case 4 reduces to Case 3. We note that, in Case 4,

$$C_u = \left\{ \begin{pmatrix} a & e \\ 0 & d \end{pmatrix} \middle| ad \neq 0,\ e = \frac{a-d}{\alpha - \beta} t \right\} .$$

Case 4-A. $\beta^q = \alpha$ *for some* $q \geqq 2$ *and* $t \neq 0$.

$$\mathrm{Aut}(V_u) \cong \frac{C_u \times \boldsymbol{C}}{G_u \times 0}, \qquad \dim\left(\frac{C_u \times \boldsymbol{C}}{G_u \times 0}\right) = 3 ,$$

where the group operation in $C_u \times \boldsymbol{C}$ is defined as in Case 3-A:

$$(v', b')(v, b) = (v'v, a'b + b'd^q)$$

where $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$ and $v' = \begin{pmatrix} a' & e' \\ 0 & d' \end{pmatrix}$, $e' = ((a'-d')/(\alpha-\beta))t$.

Case 4-*B*. $\alpha^p = \beta$ *for some* $p \geqq 2$ *and* $t \neq 0$.

$$\mathrm{Aut}(V_u) \cong \frac{C_u \times \boldsymbol{C}}{G_u \times 0}, \qquad \dim\left(\frac{C_u \times \boldsymbol{C}}{G_u \times 0}\right) = 3 ,$$

where the group operation in $C_u \times \boldsymbol{C}$ is defined as in Case 3-B:

$$(v', b')(v, b) = (v'v, d'b + b'a^p)$$

where $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$ and $v' = \begin{pmatrix} a' & e' \\ 0 & d' \end{pmatrix}$, $e' = ((a'-d')/(\alpha-\beta))t$.

Case 4-C. $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$, $t \neq 0$, $\beta^q \neq \alpha$ *for any positive integer* q *and* $\alpha^p \neq \beta$ *for any positive integer* p.

$$\mathrm{Aut}(V_u) \cong \frac{C_u}{G_u}, \qquad \dim\left(\frac{C_u}{G_u}\right) = 2 .$$

3. Proof of Theorem. In § 2, we have shown that $\mathrm{Aut}(V_u)$ is isomorphic to $C_u \times \boldsymbol{C}/G_u \times 0$ if u is in one of Case 3-A, Case 3-B, Case 4-A and Case 4-B, and is isomorphic to C_u/G_u if u is in one of other cases. We introduce an analytic space structure in the disjoint union of these quotient groups. If this is done, an analytic space structure in $\coprod_{u \in M} \mathrm{Aut}(V_u)$ is induced by it.

We consider closed subvarieties

$$Z_0, X_2, X_3, \cdots, Y_2, Y_3, \cdots$$

of $M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C}$ defined by

$$Z_0 = \{(u, v, b) \in M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C} \mid uv = vu, b = 0\} ,$$

$$X_k = \{(u, v, b) \in M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C} \mid uv = vu,\ \beta^k = \alpha\}$$

for $k = 2, 3, \cdots$, where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$, and

$$Y_k = \{(u, v, b) \in M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C} \mid uv = vu,\ \alpha^k = \beta\}$$

for $k = 2, 3, \cdots$, where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$. It is clear that $X_2, X_3, \cdots, Y_2, Y_3, \cdots$ are mutually disjoint, while each of them intersects Z_0. Let Z be the union of these subvarieties:

$$Z = Z_0 \cup \Big(\bigcup_{k \geqq 2} X_k\Big) \cup \Big(\bigcup_{k \geqq 2} Y_k\Big) .$$

LEMMA 4. *Z is a closed subvariety of $M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C}$.*

PROOF. First, we show that Z is closed in $M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C}$. Let $\{(u_\nu, v_\nu, b_\nu)\}_{\nu=1,2,\dots}$ be a sequence of points in Z converging to a point $(u, v, b) \in M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C}$. Since $u_\nu v_\nu = v_\nu u_\nu$, $\nu = 1, 2, \cdots$, we have $uv = vu$. We put $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$. We assume that

$$(u, v, b) \notin \Big(\bigcup_{k \geqq 2} X_k\Big) \cup \Big(\bigcup_{k \geqq 2} Y_k\Big) ,$$

i.e., $\alpha^k \neq \beta$, $\beta^k \neq \alpha$ for any $k \geqq 2$. Since α^k and β^k converge to 0 as $k \longrightarrow +\infty$, there is a positive number ε such that

(1) $$|\alpha^k - \beta| > \varepsilon \quad \text{and} \quad |\beta^k - \alpha| > \varepsilon$$

for all $k \geqq 2$. We may assume that

(2) $$\varepsilon < 3(1 - |\alpha|) \quad \text{and} \quad \varepsilon < 3(1 - |\beta|) .$$

We put $u_\nu = \begin{pmatrix} \alpha_\nu & t_\nu \\ 0 & \beta_\nu \end{pmatrix}$, $\nu = 1, 2, \cdots$. Then $\alpha_\nu \longrightarrow \alpha$, $\beta_\nu \longrightarrow \beta$ and $t_\nu \longrightarrow t$ as $\nu \longrightarrow +\infty$. Hence there is an integer N_0 such that

(3) $$|\alpha - \alpha_\nu| < \frac{\varepsilon}{3} \quad \text{and} \quad |\beta - \beta_\nu| < \frac{\varepsilon}{3}$$

for all $\nu \geqq N_0$. Now we show that there is an integer N, $N \geqq N_0$, such that

(4) $$|\alpha^k - \alpha_\nu^k| < \frac{\varepsilon}{3} \quad \text{and} \quad |\beta^k - \beta_\nu^k| < \frac{\varepsilon}{3}$$

for all $k \geqq 2$ and for all $\nu \geqq N$. We show the first half of (4). The second half is shown in a similar way. We assume the converse. Then there are a sequence $N_0 \leqq \nu_1 < \nu_2 < \cdots$ of integers and a sequence $k_1, k_2, \cdots$ of integers each of which is greater than 1 such that

$$|\alpha^{k_n} - \alpha_{\nu_n}^{k_n}| \geqq \frac{\varepsilon}{3}$$

for $n = 1, 2, \cdots$. If $\{k_1, k_2, \cdots\}$ is bounded, then there is a subsequence $k_{n_1}, k_{n_2}, \cdots$ such that

$$k_{n_1} = k_{n_2} = \cdots = k\ , \quad \text{a constant}\ .$$

Then

$$|\alpha^k - \alpha_{\nu_{n_m}}^k| \geqq \frac{\varepsilon}{3}$$

for $m = 1, 2, \cdots$. On the other hand, $\alpha_{\nu_{n_m}}^k \to \alpha^k$ as $m \to +\infty$, a contradiction. Hence we may assume that

$$k_1 < k_2 < \cdots\ .$$

Then

$$\frac{\varepsilon}{3} \leqq |\alpha^{k_n} - \alpha_{\nu_n}^{k_n}| \leqq |\alpha|^{k_n} + |\alpha_{\nu_n}|^{k_n} \leqq |\alpha|^{k_n} + \left(|\alpha| + \frac{\varepsilon}{3}\right)^{k_n}$$

(by (3)). The right hand side converges to 0 as $n \to +\infty$ by (2), a contradiction. This shows (4). By (1), (3) and (4),

$$|\beta_\nu^k - \alpha_\nu| > \frac{\varepsilon}{3} \quad \text{and} \quad |\alpha_\nu^k - \beta_\nu| > \frac{\varepsilon}{3}$$

for all $k \geqq 2$ and for all $\nu \geqq N$. This proves that

$$(u_\nu, v_\nu, b_\nu) \notin \left(\bigcup_{k \geqq 2} X_k\right) \cup \left(\bigcup_{k \geqq 2} Y_k\right)$$

for any $\nu \geqq N$. Hence $(u_\nu, v_\nu, b_\nu) \in Z_0$ for all $\nu \geqq N$. Hence $b_\nu = 0$ for all $\nu \geqq N$ so that $b = 0$, i.e., $(u, v, b) = (u, v, 0) \in Z_0$. Hence Z is closed.

Next, let $(u, v, b) \in X_k$. We put $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$. Then $\beta^k = \alpha$. We show that there is a positive number ε such that

$$(5) \qquad Z \cap \mu^{-1}(N(u, \varepsilon)) = (X_k \cup Z_0) \cap \mu^{-1}(N(u, \varepsilon))$$

where $\mu\colon M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C} \to M$ is the canonical projection and

$$N(u, \varepsilon) = \left\{\begin{pmatrix} \alpha' & t' \\ 0 & \beta' \end{pmatrix} \in M \,\middle|\, |\alpha - \alpha'| < \varepsilon \text{ and } |\beta - \beta'| < \varepsilon\right\}\ .$$

It is enough to claim that there is a positive number ε such that

$$(6) \qquad (\beta + \beta')^{k'} \neq \alpha + \alpha' \quad \text{and} \quad (\alpha + \alpha')^{k''} \neq \beta + \beta'$$

for any $k' \neq k$, $k' \geqq 1$, for any $k'' \geqq 1$ and for any β' and α' with $|\beta'| < \varepsilon$ and $|\alpha'| < \varepsilon$. (It is enough to prove (6) for any $k' \neq k$, $k' \geqq 2$ and for any $k'' \geqq 2$ for our present purpose. But we use the case $k' = k'' = 1$ afterwards.) We show the first half of (6). The second half is shown in a similar way. We assume the converse. Then there are sequences $\{\alpha'_\nu\}_{\nu=1,2,\cdots}$, $\{\beta'_\nu\}_{\nu=1,2,\cdots}$ such that

$$|\alpha'_\nu| < \frac{1}{\nu} \quad \text{and} \quad |\beta'_\nu| < \frac{1}{\nu}$$

for $\nu = 1, 2, \cdots$, and a sequence $k_1, k_2, \cdots$ of positive integers each of which is different from k such that

$$(7) \qquad (\beta + \beta'_\nu)^{k_\nu} = \alpha + \alpha'_\nu$$

for $\nu = 1, 2, \cdots$. If $\{k_1, k_2, \cdots\}$ is bounded, then there is a subsequence $k_{n_1}, k_{n_2}, \cdots$ such that

$$k_{n_1} = k_{n_2} = \cdots = k'(\neq k)\ , \quad \text{a constant}\ .$$

Then

$$(\beta + \beta'_{\nu_{n_m}})^{k'} = \alpha + \alpha'_{\nu_{n_m}}$$

for $m = 1, 2, \cdots$. The left hand side converges to $\beta^{k'}$ as $m \to +\infty$, while the right hand side converges to α. Hence $\beta^{k'} = \alpha$, a contradiction. Hence we may assume that

$$k_1 < k_2 < \cdots\ .$$

Then

$$|\beta + \beta'_\nu|^{k_\nu} \leqq (|\beta| + |\beta'_\nu|)^{k_\nu} \leqq \left(|\beta| + \frac{1}{\nu}\right)^{k_\nu}$$

$\to 0$ as $\nu \to +\infty$. Hence the left hand side of (7) converges to 0 as $\nu \to +\infty$, while the right hand side of (7) converges to α, a contradiction. Hence (5) is proved. Let $(u, v, b) \in Z_0 \cap X_k$. Then (5) shows that Z coincides with $Z_0 \cup X_k$ in a neighbourhood of (u, v, b). Let $(u, v, b) \in X_k - Z_0$. Then $b \neq 0$. The open subset

$$N = \{(u', v', b') \in \mu^{-1}(N(u, \varepsilon)) \mid b' \neq 0\}$$

of $\mu^{-1}(N(u, \varepsilon))$ does not intersect Z_0, and

$$(8) \qquad Z \cap N = X_k \cap N\ .$$

Thus Z coincides with X_k in a neighbourhood of (u, v, b). In a similar way to (5), we can show that, for every point $(u, v, b) \in Y_k$, there is a positive number ε such that

$$(9) \qquad Z \cap \mu^{-1}(N(u, \varepsilon)) = (Y_k \cup Z_0) \cap \mu^{-1}(N(u, \varepsilon))\ .$$

Let $(u, v, b) \in Z_0 \cap Y_k$. Then (9) shows that Z coincides with $Z_0 \cup Y_k$ in a neighbourhood of (u, v, b). Let $(u, v, b) \in Y_k - Z_0$. Then $b \neq 0$ and

$$Z \cap N = Y_k \cap N \tag{10}$$

where N is the open subset of $\mu^{-1}(N(u, \varepsilon))$ defined above. Hence Z coincides with Y_k in a neighbourhood of (u, v, b). Finally, let $(u, v, b) \in Z_0 - (\bigcup_{k\geq 2} X_k) \cup (\bigcup_{k\geq 2} Y_k)$. Then $b = 0$ and $uv = vu$. A similar proof to the proof of (5) shows that there is a positive number ε such that

$$Z \cap \mu^{-1}(N(u, \varepsilon)) = Z_0 \cap \mu^{-1}(N(u, \varepsilon)) . \tag{11}$$

This means that Z coincides with Z_0 in a neighbourhood of $(u, v, 0)$. This completes the proof of Lemma 4. q.e.d.

Let

$$\zeta: Z \to Z$$

be an automorphism defined by

$$(u, v, 0) \in Z_0 \to (u, uv, 0) \in Z_0 ,$$

$$(u, v, b) \in X_k \to (u, uv, \alpha b) \in X_k ,$$

$$(u, v, b) \in Y_k \to (u, uv, \beta b) \in Y_k ,$$

where uv is the product of matrices u and v and $u = \begin{pmatrix}\alpha & t \\ 0 & \beta\end{pmatrix}$. We note that $\zeta: Z_0 \to Z_0$ and $\zeta: X_k \to X_k$ (resp. $\zeta: Z_0 \to Z_0$ and $\zeta: Y_k \to Y_k$) coincide on $Z_0 \cap X_k$ (resp. $Z_0 \cap Y_k$). The inverse

$$\zeta^{-1}: Z \to Z$$

is given by

$$(u, v, 0) \in Z_0 \to (u, u^{-1}v, 0) \in Z_0 ,$$

$$(u, v, b) \in X_k \to \left(u, u^{-1}v, \frac{b}{\alpha}\right) \in X_k ,$$

$$(u, v, b) \in Y_k \to \left(u, u^{-1}v, \frac{b}{\beta}\right) \in Y_k .$$

We put

$$H = \{\zeta^n \mid n \in \mathbf{Z}\} .$$

LEMMA 5. *H is a properly discontinuous group of automorphisms without fixed point of Z.*

PROOF. Let $(u, v, b) \in Z$. We assume that $\zeta^n(u, v, b) = (u, v, b)$ for an integer n. Then $u^n v = v$. Hence $u^n = 1$ so that $n = 0$. Next, we show that, for any compact set K in Z,

$$\{n \in \boldsymbol{Z} \mid \zeta^n(K) \cap K \neq \emptyset\}$$

is a finite set. Let ρ and R be positive numbers such that

$$|\det u| \leqq \rho < 1 \quad \text{and} \quad \frac{1}{R} \leqq |\det v| \leqq R$$

for all $(u, v, b) \in K$, where $\det u$ is the determinant of u. Then there is a positive integer n_0 such that

$$\rho^{n_0} < \frac{1}{R^2} .$$

Then, for any positive integer $n \geqq n_0$,

$$|\det u^n v| = |\det u|^n |\det v| \leqq \rho^n R < \frac{1}{R}$$

and

$$|\det u^{-n} v| = |\det u|^{-n} |\det v| \geqq \rho^{-n} \frac{1}{R} > R .$$

Hence

$$\{n \in \boldsymbol{Z} \mid \zeta^n(K) \cap K \neq \emptyset\}$$

is contained in

$$\{n \in \boldsymbol{Z} \mid -n_0 < n < n_0\} .$$

q.e.d.

By Lemma 5, the quotient space

$$A = Z/H$$

is an analytic space such that the canonical projection

$$q: Z \to A$$

is a covering map. Let

$$\tilde{\lambda}: Z \to M$$

be the restriction to Z of the projection map

$$\mu: M \times GL(2, \boldsymbol{C}) \times \boldsymbol{C} \to M .$$

Then $\tilde{\lambda}\zeta = \tilde{\lambda}$. Hence there is a holomorphic map

$$\lambda: A \to M$$

such that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{q} & A \\ & \tilde{\lambda}\searrow \quad \swarrow\lambda & \\ & M & \end{array}$$

is commutative. Since $(u, 1, 0) \in Z_0 \subset Z$, where 1 is the identity matrix of $GL(2, C)$, $\tilde{\lambda}$ is surjective, so that λ is surjective. By the construction above, each fiber $\lambda^{-1}(u)$ is naturally isomorphic to

$$C_u \times C/G_u \times 0$$

if u is in one of Case 3-A, Case 3-B, Case 4-A and Case 4-B, and is isomorphic to

$$C_u/G_u$$

if u is in one of other cases.

Now, we prove 1)–4) of the theorem. 1) is already done. Next, we show 2). We define a holomorphic map

$$r\colon Z \underset{M}{\times} (M \times W) \to A \underset{M}{\times} X$$

by

$$((u, v, b), (u, x)) \to (q(u, v, b), p(u, x))$$

where $p\colon M \times W \to X$ is the canonical projection. Then r is a covering map. Let $((u, v, b), (u, x)) \in Z \underset{M}{\times} (M \times W)$. Let $\tilde{f}$ be the automorphism of W corresponding to (u, v, b), see § 2. Let f be the automorphism of V_u corresponding to $q(u, v, b)$. Since the diagram

$$\begin{array}{ccc} (\tilde{f}, (u, x)) \in Z \underset{M}{\times} (M \times W) & \xrightarrow{r} & (f, P) \in A \underset{M}{\times} X \\ \downarrow & & \downarrow \\ (u, \tilde{f}(x)) \in M \times W & \xrightarrow{p} & f(P) \in X\,, \end{array}$$

where $P = p(u, x)$, is commutative, and since r and p are covering maps, it is enough to show that $\tilde{f}(x)$ depends holomorphically on (u, v, b, x). Since the problem is local, it is enough to show that $\tilde{f}(x)$ depends holomorphically on (u, v, b, x) in a neighbourhood of any point (u_0, v_0, b_0, x_0).

Case A. $(u_0, v_0, b_0) \in Z_0 - (\bigcup_{k\geq 2} X_k) \cup (\bigcup_{k\geq 2} Y_k)$.

In this case, by (11) in the proof of Lemma 4, there is a positive number ε such that

$$Z \cap \mu^{-1}(N(u_0, \varepsilon)) = Z_0 \cap \mu^{-1}(N(u_0, \varepsilon))\,.$$

Let $(u, v, 0) \in Z \cap \mu^{-1}(N(u_0, \varepsilon)) = Z_0 \cap \mu^{-1}(N(u_0, \varepsilon))$. Let $\tilde{f}$ be the automorphism of W corresponding to $(u, v, 0)$. Then

$$\tilde{f}(x) = v(x)$$

for all $x \in W$, as the argument in § 2 shows. $v(x)$ depends holomorphically on (v, x).

Case B. $(u_0, v_0, b_0) \in X_k - Z_0$.

In this case, by (8) in the proof of Lemma 4,

$$Z \cap N = X_k \cap N ,$$

where $N = \{(u, v, b) \in \mu^{-1}(N(u_0, \varepsilon)) \mid b \neq 0\}$. Let $(u, v, b) \in Z \cap N = X_k \cap N$. Then $b \neq 0$ and $\beta^k = \alpha$ where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$. Let $\tilde{f}$ be the automorphism of W corresponding to (u, v, b). Let $x = (z, w) \in W$. Then $\tilde{f}(x)$ is written as

$$\tilde{f}(x) = \left(az + \frac{a-d}{\alpha-\beta}tw + bw^k, dw\right)$$

where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$ and $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$. In fact, $\tilde{f} = y^{-1}\tilde{g}y$ where $y = \begin{pmatrix} 1 & t/(\alpha-\beta) \\ 0 & 1 \end{pmatrix}$ and $\tilde{g}(z, w) = (az + bw^k, dw)$, (see Case 4-A in § 2). Hence $\tilde{f}(x)$ depends holomorphically on $(u, v, b, x) \in (Z \cap N) \times W$.

Case C. $(u_0, v_0, b_0) \in X_k \cap Z_0$.

In this case, by (5) in the proof of Lemma 4,

$$Z \cap \mu^{-1}(N(u_0, \varepsilon)) = (X_k \cup Z_0) \cap \mu^{-1}(N(u_0, \varepsilon)) .$$

Let

$$(u, v, b) \in Z \cap \mu^{-1}(N(u_0, \varepsilon)) = (X_k \cup Z_0) \cap \mu^{-1}(N(u_0, \varepsilon)) .$$

Let $\tilde{f}$ be the automorphism of W corresponding to (u, v, b). Let $x = (z, w) \in W$. Then it is easy to see that $\tilde{f}(x)$ is written as

$$\tilde{f}(x) = \left(az + \frac{a-d}{\alpha-\beta}tw + bw^k, dw\right)$$

for all $(u, v, b, x) \in (Z \cap \mu^{-1}(N(u_0, \varepsilon))) \times W$, where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$ and $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$. (We note that $\alpha \neq \beta$ in $Z \cap \mu^{-1}(N(u_0, \varepsilon)) = (X_k \cup Z_0) \cap \mu^{-1}(N(u_0, \varepsilon))$ by (6) of the proof of Lemma 4.) This shows that $\tilde{f}(x)$ depends holomorphically on (u, v, b, x).

Case D. $(u_0, v_0, b_0) \in Y_k - Z_0$.

In this case, by (10) in the proof of Lemma 4,

$$Z \cap N = Y_k \cap N ,$$

where $N = \{(u, v, b) \in \mu^{-1}(N(u_0, \varepsilon)) \mid b \neq 0\}$. Let $(u, v, b) \in Z \cap N = Y_k \cap N$. Then $b \neq 0$ and $\alpha^k = \beta$ where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$. Let $\tilde{f}$ be the automorphism of W corresponding to (u, v, b). Let $x = (z, w) \in W$. Then $\tilde{f}(x)$ is written as

$$\tilde{f}(x) = \Big(az + \frac{a-d}{\alpha-\beta}tw - \frac{bt}{\alpha-\beta}\Big(z + \frac{t}{\alpha-\beta}w\Big)^k ,$$
$$b\Big(z + \frac{t}{\alpha-\beta}w\Big)^k + dw\Big)$$

where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$ and $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$. In fact, $\tilde{f} = y^{-1}\tilde{g}y$ where $y = \begin{pmatrix} 1 & t/(\alpha-\beta) \\ 0 & 1 \end{pmatrix}$ and $\tilde{g}(z, w) = (az, dw + bz^k)$, (see Case 4-B in § 2). Hence $\tilde{f}(x)$ depends holomorphically on $(u, v, b, x) \in (Z \cap N) \times W$.

Case E. $(u_0, v_0, b_0) \in Y_k \cap Z_0$.

In this case, by (9) in the proof of Lemma 4,

$$Z \cap \mu^{-1}(N(u_0, \varepsilon)) = (Y_k \cup Z_0) \cap \mu^{-1}(N(u_0, \varepsilon)) .$$

Let $\tilde{f}$ be the automorphism of W corresponding to (u, v, b). Let $x = (z, w) \in W$. Then

$$\tilde{f}(x) = \Big(az + \frac{a-d}{\alpha-\beta}tw - \frac{bt}{\alpha-\beta}\Big(z + \frac{t}{\alpha-\beta}w\Big)^k ,$$
$$b\Big(z + \frac{t}{\alpha-\beta}w\Big)^k + dw\Big)$$

for all $(u, v, b, x) \in (Z \cap \mu^{-1}(N(u_0, \varepsilon))) \times W$, where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$ and $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$. Hence $\tilde{f}(x)$ depends holomorphically on (u, v, b, x).
This completes the proof of 2) of the theorem.

Next, we prove 3) of the theorem. Let 1 be the (2×2)-identity matrix. Then the map

$$u \in M \longrightarrow (u, 1, 0) \in Z_0 \subset Z$$

is holomorphic. Hence the map

$$u \in M \longrightarrow q(u, 1, 0) \in A$$

is holomorphic. It is clear that $q(u, 1, 0)$ corresponds to the identity map of V_u.

Finally we show 4) of the theorem. We define a holomorphic map

$$s\colon Z \underset{M}{\times} Z \longrightarrow A \underset{M}{\times} A$$

by

$$((u, v, b), (u, v', b')) \to (q(u, v, b), q(u, v', b'))\ .$$

Then s is a covering map. Let $((u, v, b), (u, v', b')) \in Z \underset{M}{\times} Z$. We define a product

$$(u, v', b')(u, v, b) \in Z$$

by

$$(1) \qquad (u, v', b')(u, v, b) = (u, v'v, a'b + b'd^q)\ ,$$

if u is in Case 3-A or 4-A of §2, where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$, $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$ and $v' = \begin{pmatrix} a' & e' \\ 0 & d' \end{pmatrix}$, $e' = ((a'-d')/(\alpha-\beta))t$,

$$(2) \qquad (u, v', b')(u, v, b) = (u, v'v, d'b + b'a^p)\ ,$$

if u is in Case 3-B or 4-B of §2, where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$, $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$ and $v' = \begin{pmatrix} a' & e' \\ 0 & d' \end{pmatrix}$, $e' = ((a'-d')/(\alpha-\beta))t$,

$$(3) \qquad (u, v', 0)(u, v, 0) = (u, v'v, 0)\ ,$$

if u is in one of other cases. Then, as in the proof of 2) of the theorem, by diving in various cases, we can easily see that the map

$$((u, v, b), (u, v', b')) \in Z \underset{M}{\times} Z \to (u, v', b')(u, v, b) \in Z$$

is holomorphic. We define a product

$$q(u, v', b')q(u, v, b) \in A$$

by

$$q(u, v', b')q(u, v, b) = q((u, v', b')(u, v, b))\ .$$

This is well defined, as is easily shown by dividing in various cases. Since the map s defined above and the map q are covering maps, the map

$$(q(u, v, b), q(u, v', b')) \in A \underset{M}{\times} A \to q(u, v', b')q(u, v, b) \in A$$

is holomorphic. It is clear that $q(u, v', b')q(u, v, b)$ corresponds to the composition gf of automorphisms g and f of V_u corresponding to $q(u, v', b')$ and $q(u, v, b)$ respectively.

Now, we define a holomorphic map

$$\tilde{\theta}: Z \to Z$$

by

$$\tilde{\theta}\colon (u, v, 0) \in Z_0 \longrightarrow (u, v^{-1}, 0) \in Z_0 ,$$

$$\tilde{\theta}\colon (u, v, b) \in X_k \longrightarrow \left(u, v^{-1}, -\frac{b}{ad^k}\right) \in X_k ,$$

$$\tilde{\theta}\colon (u, v, b) \in Y_k \longrightarrow \left(u, v^{-1}, -\frac{b}{a^k d}\right) \in Y_k ,$$

where $u = \begin{pmatrix} \alpha & t \\ 0 & \beta \end{pmatrix}$ and $v = \begin{pmatrix} a & e \\ 0 & d \end{pmatrix}$, $e = ((a-d)/(\alpha-\beta))t$. We note that $\tilde{\theta}\colon Z_0 \longrightarrow Z_0$ and $\tilde{\theta}\colon X_k \longrightarrow X_k$ (resp. $\tilde{\theta}\colon Z_0 \longrightarrow Z_0$ and $\tilde{\theta}\colon Y_k \longrightarrow Y_k$) coincide on $Z_0 \cap X_k$ (resp. $Z_0 \cap Y_k$). It is easy to see that $\tilde{\theta}\zeta = \zeta\tilde{\theta}$. Hence we can define a map

$$\theta\colon A \longrightarrow A$$

by

$$\theta(q(u, v, b)) = q(\tilde{\theta}(u, v, b)) .$$

Since q is a covering map, θ is holomorphic. It is clear that $\theta(q(u, v, b))$ corresponds to the inverse f^{-1} of the automorphism f of V_u corresponding to $q(u, v, b)$. This completes the proof of 4) of the theorem.

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