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CUT LOCI IN RIEMANNIAN MANIFOLDS

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1. Introduction. In this paper we study the structure of the cut locus $C(p)$ of a point p in a Riemannian manifold M, in the complement of the set $Q_0(p)$ of points conjugate to p along a minimizing geodesic. The principal tool is a vector field constructed as follows: Let $V \subset T_pM$ be an open subset on which $\exp_p$ is a diffeomorphism onto an open set in M. For each $X \in V$ let $\gamma_X(t) = \exp_p tX$ be the geodesic from p in the direction of X; and for each t such that $tX \in V$, let $Y_{\exp_p tX}$ be the tangent to γ_X at t having length equal to the length of the segment of γ_X going from p to $\exp_p tX$. Using the vector fields Y obtained in this way we are able to apply differential methods to the study of the cut locus away from $Q_0(p)$.

The main results are: (i) A description of $C(p) - Q_0(p)$ locally as an intersection of a finite number of smooth $(n-1)$-dimensional manifolds and finitely many open sets given by smooth inequalities ($n = \dim M$). (ii) For real analytic manifolds, a triangulation theorem for open subsets of $C(p) - Q_0(p)$ whose closure is disjoint from $Q_0(p)$. This second result is a consequence of a theorem of Łojasiewicz [3] on the triangulability of semi-analytic sets.

Also, we give a description of what happens to the cut locus when taking quotients by groups Γ of isometries which act properly discontinuously (and freely). We show that if Δ_p is the fundamental domain of Γ centered at p, $E_p = M - C(p)$, and if $\pi: M \to M/\Gamma$ is the projection, then $C(\pi p) = \pi(\partial(\Delta_p \cap E_p))$.

2. Local structure of the cut locus. Let M be a complete C^∞ Riemannian manifold of dimension n, $\langle \cdots, \cdots \rangle$ the Riemannian inner product, and $\|\cdots\|$ the associated norm. For each $p \in M$, $\exp_p: T_pM \to M$ denotes the exponential map, $C(p)$ the cut locus of p in M, and $Q(p)$ the first conjugate locus of p in M. Let $\tilde{C}(p)$ and $\tilde{Q}(p)$ be the corresponding loci in T_pM, and let $\tilde{Q}_0(p) = \tilde{C}(p) \cap \tilde{Q}(p)$. Then define $Q_0(p) = \exp_p \tilde{Q}_0(p)$; so $Q_0(p)$ is the set of all points which are conjugate to p along some minimizing geodesic. It is easy to see that all of these sets are closed.

For each $X \in T_pM$, we will identify $T_X(T_pM)$ with T_pM in the usual way. If $X \in T_pM$ and $\exp_p$ is non-singular at X, then let $V \subset T_pM$ be an open set about X on which $\exp_p$ is a diffeomorphism onto an open set in M; and define a vector field Y on $\exp_p V$ by: $Y_q = (\exp_p)_*(Z)$ for all $q \in \exp_p V$, where $q = \exp_p Z$, $Z \in V$. Then Y is a C^∞ vector field on $\exp_p V$.

DEFINITION 2.1. Y is the *distance vector field* determined by p and V.

Alternatively, Y can be described as the field of tangents to the geodesics $\gamma_Z\colon t \mapsto \exp_p tZ$, as Z ranges over V, which have length at each $\exp_p tZ$ equal to the length of the segment of γ_Z from p to $\exp_p tZ$.

LEMMA 2.2. *For each $q \in \exp_p V$ and each $X_q \in T_qM$,*

$$X_q\|Y\| = \langle X_q, Y_q\rangle/\langle Y_q, Y_q\rangle^{1/2} \quad \text{if} \quad \|Y_q\| \neq 0 .$$

PROOF. Let $q = \exp_p Z$, $Z \in V$; and let $X_0 = (\exp_{p*})^{-1}X_q$. Define a variation of the geodesic $\gamma_Z\colon t \mapsto \exp_p tZ$ by: $Q(t, s) = \exp_p [t((Z/\|Z\|) + sX_0)]$. This is a one-parameter family of geodesics emanating from p which are all parameterized proportionally to arc-length, so the longitudinal curves have parallel tangent vector fields. For $s = 0$, t is arc-length. Let $L(s) = \int_0^a \langle T, T\rangle^{1/2} dt$ where $Q_*(\partial/\partial t) = T$ and $a = \|Y_q\|$ (= constant). By the formula for first variation of arc-length and the fact that $Q_*(\partial/\partial s) = 0$ when $t = 0$, $s = 0$, we get $L'(0) = \langle X_q, T_q\rangle = \langle X_q, Y_q\rangle\langle Y_q, Y_q\rangle^{-1/2}$. Since $L'(0) = X_q\|Y\|$, the result follows. q.e.d.

Suppose $r \in C(p) - Q_0(p)$, and $\lambda = \rho(p, r)$ where $\rho(p, r)$ is the Riemannian distance. Then there are at least two minimizing geodesics from p to r, so $\exp_p^{-1}(r) \cap S_\lambda(0)$ has at least two elements. (Here, $S_\lambda(0) = \{X \in T_pM \mid \|X\| = \lambda\}$). If $\{X_i\}$ is a sequence of infinitely many vectors in $\exp_p^{-1}(r) \cap S_\lambda(0)$, then there is a convergent subsequence which we again denote by $\{X_i\}$. If $X_i \to X_0$ as $i \to \infty$, then $X_0 \in \exp_p^{-1}(r) \cap S_\lambda(0)$ is a conjugate point of p contrary to hypothesis. Therefore $\exp_p^{-1}(r) \cap S_\lambda(0)$ is a finite set which we denote by $\{X_i \mid 1 \leqq i \leqq k_p(r)\}$. $k_p(r)$ is the number of distinct minimizing geodesics from p to r. For each $i = 1, \cdots, k_p(r)$, let U_i^r be an open set about X_i such that $\exp_p\colon U_i^r \to M$ is a diffeomorphism onto an open set about r. We may assume that $\exp_p(U_i^r) = U_r$ for all i, where U_r is a fixed convex normal neighborhood of r.

Suppose there is a sequence $q_j \in U_r$ converging to r having a sequence of minimizing geodesics $\gamma_j(t) = \exp_p(tX^{q_j}/\|X^{q_j}\|)$ which satisfy the following conditions: (i) $X^{q_j} \in T_pM$, $\|X^{q_j}\| = \rho(p, q_j)$, (ii) $0 \leqq t \leqq \rho(p, q_j)$, (iii) $\exp_p X^{q_j} = q_j$, (iv) $X^{q_j} \notin \bigcup\{U_i^r \mid 1 \leqq i \leqq k_p(r)\}$ for all j. Then by choosing subsequences if necessary, we may assume that $X^{q_j} \to X^r \in S_\lambda(0)$. But then X^r is one of the X_i, so the vectors X^{q_j} eventually lie in U_i^r contrary to hypothesis. Therefore, by shrinking U_r if necessary (and also shrinking the U_i^r so $\exp_p U_i^r = U_r$ still holds for all i), we may assume that every minimizing geodesic from p to a point of U_r has the form $t \rightarrow \exp_p(tZ/\|Z\|)$ for some $Z \in U_i^r$, $1 \leqq i \leqq k_p(r)$. This implies that if we follow through the above constructions for each point $q \in C(p) \cap U_r$, then $k_p(q) \leqq k_p(r)$. Since $C(p) - Q_0(p)$ is open in $C(p)$, $k_p\colon C(p) - Q_0(p) \rightarrow Z^+$ is upper semi-continuous.

For each $r \in C(p) - Q_0(p)$ let $\{Y_i^r \mid 1 \leqq i \leqq k_p(r)\}$ be the set of distance vector fields on U_r determined by p and the open sets $U_i^r \subset T_pM$. Then $\|(Y_i^r)_r\| = \rho(p, r)$ for all i; and the vectors $(Y_i^r)_r$ are distinct since if two coincided then their geodesics would coincide, implying that the corresponding X_i coincide. Therefore, the vectors $(Y_i^r)_r$ are either pairwise independent, or certain pairs occur as negatives of each other. We may assume, by shrinking U_r further if necessary, that throughout U_r the vectors Y_i^r are either pairwise independent or certain pairs occur as negative multiples of each other (where the numbers in the multiples may be restricted to lie as near -1 as we like by choosing U_r sufficiently small).

Define functions

$$g_{ij}^r(q) = \|(Y_i^r)_q\| - \|(Y_j^r)_q\|,\ 1 \leqq i, j \leqq k_p(r)\ .$$

These are C^∞ functions on $U_r - \{p\}$, and clearly $g_{ij}^r = -g_{ji}^r$ for all i, j. For each pair $i \neq j$, let:

$$\begin{aligned} K_{ij}^r &= \{q \in U_r \mid g_{ij}^r(q) = 0\} \\ H_{ij}^r &= \{q \in U_r \mid g_{ij}^r(q) > 0\} \\ C_{ij}^r &= K_{ij}^r \cap \left(\bigcap \{\bar{H}_{li}^r \mid 1 \leqq l \leqq k_p(r)\}\right)\ . \end{aligned}$$

PROPOSITION 2.3. $C(p) \cap U_r = \bigcup \{C_{ij}^r \mid i < j\}$.

PROOF. If $q \in C(p) \cap U_r$ then since $U_r \subset M - Q_0(p)$, it follows that there are at least two minimizing geodesics from p to q. If Y_i^r, Y_j^r are the corresponding distance vector fields then $\rho(p, q) = \|(Y_i^r)_q\| = \|(Y_j^r)_q\|$, and all other geodesics from p to q, have length $\geqq \rho(p, q)$. This proves that $q \in C_{ij}^r$. Conversely, if $g_{i_1 i_2}(q) = 0$ for $i_1 \neq i_2$, and $g_{i i_1}(q) \geqq 0$ for all $1 \leqq i \leqq k_p(r)$, then $\|(Y_{i_1}^r)_q\| = \rho(p, q)$ since this is the shortest of the $(Y_i^r)_q$ and one of them must have length $\rho(p, q)$. The geodesics corresponding to $Y_{i_1}^r$, $Y_{i_2}^r$ are distinct and minimizing so $q \in C(p)$. q.e.d.

PROPOSITION 2.4. *For each pair $i \neq j$, the set K_{ij}^r is a smooth submanifold of dimension $n - 1$.*

PROOF. Let $q \in K_{ij}^r$ and $X_q \in T_qM$ any vector. Then

$$X_q g_{ij}^r = X_q\|Y_i^r\| - X_q\|Y_j^r)\| = \|(Y_i^r)_q\|^{-1}\langle X_q, (Y_i^r)_q - (Y_j^r)_q\rangle\ .$$

Since $(Y_i^r)_q \neq (Y_j^r)_q$, there is $X_q \in T_qM$ such that $\langle X_q, (Y_i^r)_q - (Y_j^r)_q\rangle \neq 0$ so $g_{ij}^r\colon U_r \rightarrow \boldsymbol{R}^1$ has maximal rank at q. The result then follows from the implicit function theorem. q.e.d.

REMARK 2.5. $T_q K^r_{ij} = ((Y^r_i)_q - (Y^r_j)_q)^\perp$ since the right side is the kernel of dg^r_{ij} at q.

PROPOSITION 2.6. *If* $q \in K^r_{i_1 j} \cap K^r_{i_2 j} - Q_0(p)$, *for* $i_1 \neq i_2$, $j \neq i_1, i_2$, *then the intersection is transverse at* q.

PROOF. Let Y^r_j, $Y^r_{i_1}$, $Y^r_{i_2}$ be the distance vector fields determined by U^r_j, $U^r_{i_1}$, $U^r_{i_2}$. Then they all have the same length at q. If they span a three-dimensional space at q, then the vectors $(Y^r_j)_q$, $(Y^r_{i_1})_q - (Y^r_j)_q$, $(Y^r_{i_2})_q - (Y^r_j)_q$ also span a three-dimensional space. Therefore $((Y^r_{i_1})_q - (Y^r_j)_q)^\perp$ and $((Y^r_{i_2})_q - (Y^r_j)_q)^\perp$ are transverse. If $(Y^r_j)_q$, $(Y^r_{i_1})_q$, $(Y^r_{i_2})_q$, span a two-dimensional space then two of the vectors are negatives of each other and the third is independent of both. If $(Y^r_j)_q = -(Y^r_{i_1})_q$ then $(Y^r_{i_1})_q - (Y^r_j)_q = -2(Y^r_j)_q$ and $(Y^r_{i_2})_q - (Y^r_j)_q$ are independent so their normal spaces are transverse. The same argument applies if $(Y^r_{i_2})_q = -(Y^r_{i_1})_q$. If $(Y^r_{i_2})_q = -(Y^r_{i_1})_q$, then $(Y^r_j)_q$ is independent of both others so $(Y^r_{i_1})_q - (Y^r_j)_q$ and $(Y^r_{i_2})_q - (Y^r_j)_q = -((Y^r_{i_1})_q) + (Y^r_j)_q)$ are independent. q.e.d.

REMARK. It is not clear whether higher numbers of intersections are transverse, or whether intersections $K^r_{i_1 j_1} \cap K^r_{i_2 j_2}$ are transverse if all the indices i_1, j_1, i_2, j_2 are distinct.

All the previous constructions obviously carry over to the case of a real analytic Riemannian manifold with an analytic metric. In particular, the functions: $g^r_{ij}: U_r \to R^1$ are analytic.

Suppose M is a real analytic manifold, and $S \subset M$ is a subset. If $U \subset M$ is any open subset, and if $f_1, \cdots, f_k$ are real-valued functions defined on U, then we say that S is *described* in U by the functions $f_1, \cdots, f_k$ if $S \cap U$ is a finite union of finite intersections of sets of the form: $\{x \in U \mid f_i(x) > 0\}$ or $\{x \in U \mid f_i(x) = 0\}$.

A subset $S \subset M$ is *semi-analytic* if and only if for each point $x \in M$ (not necessarily in S) there is an open set $U_x \subset M$ about x, and a finite set of real-analytic functions $f_1, \cdots, f_k$ defined on U_x such that S is described in U_x by these functions.

Then we have:

THEOREM 2.7. *Every relatively open subset* V *of* $C(p) - Q_0(p)$ *whose closure in* M *is disjoint from* $Q_0(p)$ *lies in an open semi-analytic subset of* $C(p) - Q_0(p)$.

PROOF. Cover $Q_0(p)$ by a locally finite collection $\{B_\alpha\}$ of closed metric balls having centers q_α and radii r_α, such that (i) $\bar{V} \cap (\bigcup_\alpha B_\alpha) = \varnothing$, (ii)

$Q_0(p) \subset \bigcup_\alpha (B_\alpha^0)$, where B_α^0 is the interior of B_α. Let $S = C(p) - (\bigcup_\alpha B_\alpha)$, so $\bar{S} \subset C(p) - Q_0(p)$. Define functions $h_\alpha : M \to \boldsymbol{R}^1$ by $h_\alpha(q) = \rho(q_\alpha, q) - r_\alpha$. Then $S = \{q \in C(p) \mid h_\alpha(q) > 0, \text{ all } \alpha\}$. Now if $r \in \bar{S}$, then let the open set U_r and the functions g_{ij}^r be constructed as before. By shrinking U_r if necessary we may assume it meets only a finite number of the balls B_α. Then the functions g_{ij}^r together with those h_α such that $U_r \cap B_\alpha \neq \emptyset$, describe S in U_r. If $r \notin \bar{S}$ there is an open set U about r disjoint from S, so any non-zero constant describes S in U. q.e.d.

COROLLARY 2.8. *If M is a complete real analytic Riemannian manifold and $V \subset C(p) - Q_0(p)$ is a relatively open subset whose closure in M is disjoint from $Q_0(p)$, then V lies in an open subset of $C(p) - Q_0(p)$ which has an analytic triangulation.*

PROOF. This is an immediate consequence of Theorem (2.7) and a theorem of S. Łojasiewicz ([3]). q.e.d.

REMARK 2.9. Łojasiewicz also showed ([4]) that a semi-analytic set has a Whitney stratification.

COROLLARY 2.10. *In an analytic manifold, if $C(p) \cap Q_0(p) = \emptyset$ then $C(p)$ is a semi-analytic set and is therefore stratifiable and triangulable.*

REMARK 2.11. (1) If we assume that the sets K_{ij}^r, and all their intersections, are transverse to each other in $C(p) - Q_0(p)$, then $C(p) \cap U_r$ is a finite union of C^∞ submanifolds of U_r. It is easy to see that the conditions for a Whitney stratification are then satisfied. It is not known, however, whether this implies triangulability. (2) In [8], A. Weinstein proved that if M is a compact C^∞ manifold not homeomorphic to S^2 then M has a Riemannian metric and a point p such that $\tilde{C}(p) \cap \tilde{Q}(p) = \emptyset$. This implies that $C(p) \cap Q_0(p) = \emptyset$ so our local structure theorems (Propositions 2.3 – 2.6) for $C(p)$ apply to all of $C(p)$. The same result holds in the real analytic case.

3. Cut loci and Riemannian coverings. Next we will consider the relation between the cut locus $C(p)$ of a point $p \in M$ and the cut locus $C(\pi p)$ of $\pi p \in M/\Gamma$, where Γ is a group of isometries of M acting properly discontinuously, and $\pi : M \to M/\Gamma$ is the Riemannian covering projection. (There seems to be some disparity in the use of the term "properly discontinuous". We have followed the definition in Spanier [7]: Γ is *properly discontinuous* if for each $p \in M$ there is an open set U about p such that if $gU \cap g'U \neq \emptyset$ for any two $g, g' \in \Gamma$ then $g = g'$).

DEFINITION 3.1.

(i) For each pair of points $p, q \in M$ with $p \neq q$ let

$$H_{p,q} = \{r \in M \mid \rho(p, r) < \rho(q, r)\}$$
$$A_{p,q} = \{r \in M \mid \rho(p, r) = \rho(q, r)\} = A_{q,p} .$$

(ii) If Γ is a group of isometries acting properly discontinuously on M, let

$$\Delta_p = \bigcap \{H_{p,gp} \mid g \in \Gamma,\ g \neq e\} .$$

Δ_p is the *normal fundamental domain* of Γ centered at p.

The following facts about these sets are well-known (see for example, H. Busemann [1]).

PROPOSITION 3.2.

(1) $H_{p,q}, \Delta_p$ *are open and star-like with respect to* p (*i.e. they contain all minimizing geodesic segments from* p *to any of their points*);

(2) *Every geodesic segment emanating from* p *which minimizes arc-length between its end-points intersects* $\partial\Delta_p$ *in at most one point*;

(3) $g\Delta_p = \Delta_{gp}$ *for all* $g \in \Gamma$, *and* $g_1\Delta_p \cap g_2\Delta_p = \varnothing$ *if* $g_1 \neq g_2$;

(4) $\bigcup \{\bar{\Delta}_{gp} \mid g \in \Gamma\} = M$;

(5) *The collection of sets* $\bar{\Delta}_{gp}$ *is locally finite*;

(6) Γ *is generated by the positive powers of those* $g \in \Gamma$ *such that* $\bar{\Delta}_p \cap \bar{\Delta}_{gp} \neq \varnothing$;

(7) *Let* $E_{\pi_p} = M/\Gamma - C(\pi p)$. *Then* $E_{\pi_p} \subset \pi\Delta_p$.

Suppose $g \in \Gamma$, $g \neq e$, and $q \in A_{p,gp} - (C(p) \cap C(gp))$. Then there are unique minimizing geodesics γ_1, γ_2 from p to q, gp to q respectively. Let X_1, X_2 be tangent vectors to γ_1, γ_2 at p, gp such that $\|X_1\| = \|X_2\| = \rho(p, q)$; and let U_1, U_2 be open sets about X_1, X_2 on which $\exp_p, \exp_{gp}$ are diffeomorphisms onto open subsets of M. Let Y_1, Y_2 be the distance vector fields determined by these objects. We may assume that $U = \exp_p U_1 = \exp_{gp} U_2$, and U is so small that $U \cap (C(p) \cup C(gp)) = \varnothing$. Since $p \neq gp$, it follows that $(Y_1)_q \neq (Y_2)_q$ so by the same argument as in Proposition 2.4, we see that the set $\{r \in U \mid \|(Y_1)_r\| = \|(Y_2)_r\|\}$ is a smooth submanifold of dimension $n - 1$, with tangent space $((Y_1)_r - (Y_2)_r)^\perp$ at r. Since

$$\|(Y_2)_r\| = \rho(gp, r) ,$$

this submanifold is $A_{p,gp} \cap U$. Summarizing:

PROPOSITION 3.3. *For each* $p \in M$ *and each* $g \in \Gamma$, $g \neq e$, $A_{p,gp} - (C(p) \cup C(gp))$ *is a smooth submanifold of dimension* $n - 1$, *having tangent space* $((Y_1)_q - (Y_2)_q)^\perp$ *at each* q.

It is easy to see that for each $g \neq e$, we have the disjoint union: $M = A_{p,gp} \cup H_{p,gp} \cup H_{gp,p}$. Since $\Delta_p = \bigcap \{H_{p,gp} \mid g \neq e\}$ and the Δ_{gp} are locally finite, it follows that $\bar{\Delta}_p = \bigcap \{A_{p,gp} \cup H_{p,gp} \mid g \neq e\}$.

PROPOSITION 3.4. *For each* $g \neq e$, $\bar{\varDelta}_p \cap \bar{\varDelta}_{gp} \subset A_{p,gp}$.

PROOF. Let γ_1, γ_2 be minimizing geodesics from p to q and gp to q, where $q \in \bar{\varDelta}_p \cap \bar{\varDelta}_{gp}$. Suppose $\rho(p, q) < \rho(gp, q)$. Then all points q' on γ_2 sufficiently near q also satisfy $\rho(p, q') < \rho(gp, q')$. But we must have $q' \in \varDelta_{gp}$, which is a contradiction. The opposite inequality is proved impossible by the same argument. q.e.d.

PROPOSITION 3.5.

$$\begin{aligned}\partial\varDelta_p &= \bigcup \{\bar{\varDelta}_p \cap \bar{\varDelta}_{gp} \mid g \neq e\} \\ &= \bigcup \{\partial\varDelta_p \cap \partial\varDelta_{gp} \mid g \neq e\}\end{aligned}$$

and these unions are locally finite.

PROOF. Local finiteness follows from Proposition (3.2) (5), and $\bar{\varDelta}_p \cap \bar{\varDelta}_{gp} \subset \partial\varDelta_p$ is clear if $g \neq e$. Suppose $q \in \partial\varDelta_p$ and U is an open set about q which meets only finitely many $\bar{\varDelta}_{gp}$. Since q is a boundary point, there is a sequence $q_i \in M - \bar{\varDelta}_p$ converging to q. We may assume this lies in U, and then by choosing a subsequence if necessary we may assume that $q_i \in \bar{\varDelta}_{gp}$ for some fixed $g \neq e$. Then $q \in \bar{\varDelta}_{gp} \cap \bar{\varDelta}_p$. q.e.d.

Let $g_0 \neq e$ be fixed. Then by the same argument as in Proposition (3.2) (1), one sees that $\bigcap \{H_{p,gp} \mid g \neq e, g_0\}$ is an open set about p. Denote by int $(\bar{\varDelta}_p \cap \bar{\varDelta}_{gp})$ the set $A_{p,gp} \cap [\bigcap \{H_{p,g'p} \mid g' \neq e, g\}]$. We will call these the *faces* of $\varDelta_p$. The following is easy to verify:

PROPOSITION 3.6. $\bar{\varDelta}_p \cap \bar{\varDelta}_{gp} = A_{p,gp} \cap [\bigcap \{\bar{H}_{p,g'p} \mid g' \neq e, g\}]$ *and* $\bar{\varDelta}_p \cap \bar{\varDelta}_{gp}$ *is the closure of* int $(\bar{\varDelta}_p \cap \bar{\varDelta}_{gp})$.

PROPOSITION 3.7. *For each* $g \neq e$, int $(\bar{\varDelta}_p \cap \bar{\varDelta}_{gp}) - (C(p) \cup C(gp))$ *is either empty or a smooth submanifold of dimension* $n - 1$.

PROPOSITION 3.8. *If* $q \in A_{p,gp} \cap A_{p,g'p} - (C(p) \cup C(gp) \cup C(g'p))$ *then the intersection is transverse at* q.

PROOF. The proof is the same as the proof of Proposition 2.6.
q.e.d.

COROLLARY 3.9. *If* $q \in \text{int}\,(\bar{\varDelta}_p \cap \bar{\varDelta}_{gp}) \cap \text{int}\,(\bar{\varDelta}_p \cap \bar{\varDelta}_{g'p})$ *and*

$$q \notin C(p) \cup C(gp) \cup C(g'p),\ e \neq g \neq g' \neq e\,,$$

then in a neighborhood of q, int $(\bar{\varDelta}_p \cap \bar{\varDelta}_{gp}) \cap$ int $(\bar{\varDelta}_p \cap \bar{\varDelta}_{g'p})$ *is a smooth* $(n - 2)$*-dimensional submanifold.*

For each $p \in M$, let $E_p = M - C(p)$. It is well-known ([2], [8]) that E_p is a cell diffeomorphic to an open cell in $\boldsymbol{R}^n$. Note that $\partial E_p = C(p)$; and

$$\partial(\Delta_p \cap E_p) = (\partial\Delta_p \cap E_p) \cup (\Delta_p \cap C(p)) \cup (\partial\Delta_p \cap C(p)) .$$

PROPOSITION 3.10. $C(\pi p) = \pi(\partial(\Delta_p \cap E_p))$.

PROOF. If $q \in \partial\Delta_p \cap E_p$ then there is a (unique) minimizing geodesic γ from p to q. If $q \in \bar{\Delta}_p \cap \bar{\Delta}_{gp}$, $g \neq e$, let γ' be a minimizing geodesic from gp to q. Then $\pi\gamma$ and $\pi\gamma'$ are minimizing geodesics from πp to πq; but we cannot have $\pi\gamma = \pi\gamma'$ since then we would have $g\gamma = \gamma'$ so that $gq = q$, and $g \neq e$ has no fixed points. Therefore $\pi q \in C(\pi p)$. If $q \in \Delta_p \cap C(p)$ then since $\pi: \Delta_p \to M/\Gamma$ is an isometry onto an open subset, $\pi(q) \in C(\pi p)$. If $q \in \partial\Delta_p \cap C(p)$ then either: (i) q is conjugate to p along a minimizing geodesic, so the same holds for πq and πp; or (ii) there are two distinct minimizing geodesics from p to q, so the same is true for $\pi(p)$ and $\pi(q)$. This proves that $\pi(\partial(\Delta_p \cap E_p)) \subset C(\pi p)$. Conversely, by Proposition (3.2) (7), we have $E_{\pi_p} \subset \pi(\Delta_p)$. If $\bar{q} \in C(\pi p) \cap \pi(\Delta_p)$ then since $\pi: \Delta_p \to M/\Gamma$ is an isometry onto an open set, there is $q \in C(p) \cap \Delta_p$ with $\bar{q} = \pi(q)$ (see this by lifting geodesics from πp to $\bar{q}$ up to M). If $\bar{q} \in C(\pi p) \cap \pi(\partial\Delta_p)$, then $\bar{q} = \pi q$ for some $q \in \partial\Delta_p = (\partial\Delta_p \cap C(p)) \cup (\partial\Delta_p \cap E_p)$. Thus $C(\pi p) \subset \pi(\partial(\Delta_p \cap E_p))$. q.e.d.

COROLLARY 3.11. *If $\bar{\Delta}_p \subset E_p$ then $C(\pi p) = \pi(\partial\Delta_p)$.*

COROLLARY 3.12. *If $\bar{\Delta}_p \subset E_p$ then the faces* $\operatorname{int}(\bar{\Delta}_p \cap \bar{\Delta}_{gp})$ *and "edges"* $\operatorname{int}(\bar{\Delta}_p \cap \bar{\Delta}_{gp}) \cap \operatorname{int}(\bar{\Delta}_p \cap \bar{\Delta}_{g'p})$ *are smooth submanifolds of dimension $n-1$ and $n-2$ respectively.*

PROOF. In view of the previous propositions, it suffices to show that the points q in these faces and edges lie outside the cut loci involved. But since $C(gp) = gC(p)$, it follows that if $\bar{\Delta}_p \subset E_p$ then $\bar{\Delta}_{gp} \subset E_{gp}$ for all $g \in \Gamma$. q.e.d.

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