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PLURIHARMONIC BOUNDARY VALUES

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Abstract. Let $\Omega = \{\rho < 0\}$ be a domain with C^3-boundary $\Gamma = \{\rho = 0\}$. For a large class of domains, the functions $u \in C^3(\Gamma)$ which are the restrictions of pluriharmonic functions on Ω are characterized as the solutions of a system of partial differential equations.

I. Introduction. Let $\Omega = \{\rho < 0\}$ be a bounded domain in $C^n (n \geqq 2)$ with connected C^3-boundary $\Gamma = \{\rho = 0\}$, grad $\rho \neq 0$ on Γ. A function $f \in C^1(\Gamma)$ can be extended to an analytic function F on Ω if and only if it satisfies the tangential Cauchy-Riemann equations:

$$\bar{\partial}\rho \wedge \bar{\partial} f = 0 \tag{1}$$

on Γ (see [1], [3]). We will give an analogous system for pluriharmonic functions. It will also be pointed out that the Neumann conditions for the $\partial\bar{\partial}$-operator give a simple characterization of pluriharmonic functions although these conditions involve derivatives normal to Γ.

Let $d = 1/2(\bar{\partial} + \partial)$ and $d^c = 1/2i(\bar{\partial} - \partial)$ denote the real and imaginary parts of $\bar{\partial}$. It will be shown here that for certain domains Ω, a function $u \in C^3(\Gamma)$ can be extended to a pluriharmonic function U on Ω if and only if there exists a function $\alpha \in C^1(\Gamma)$ such that:

$$d\rho \wedge d^c\rho \wedge dd^c u = \alpha d\rho \wedge d^c\rho \wedge dd^c\rho \tag{2}$$

$$d\rho \wedge dd^c u = d\rho \wedge d\alpha \wedge d^c\rho + \alpha d\rho \wedge dd^c\rho\ . \tag{3}$$

Since the expression $dd^c u$ does not depend only on the values of u on Γ, the equations (2) and (3) are to be interpreted in the following sense. A C^3 extension u_1 of u to a neighborhood of Γ is picked, and the same extension u_1 is substituted into both equations (2) and (3).

If u extends to a pluriharmonic function U on Ω, then (2) and (3) are satisfied. For any extension u_1 will have the form $u_1 = U + a\rho$, and equation (2) will yield $a = \alpha$ since $dd^c U = 0$. With $a = \alpha$, equation (3)

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holds. From this argument it follows that the system (2) and (3) depends only on the values of u on Γ.

In fact (2) and (3) can be rewritten in terms of operators tangential to Γ and such that the extraneous function α does not appear. This will be done in the special case where Ω is the unit ball in C^n. It was observed by L. Nirenberg that there is no second order system of operators tangential to the unit ball in C^n that annihilates exactly the pluriharmonic functions.

We have seen that conditions (2) and (3) are necessary for pluriharmonic continuation. The next two sections of this paper are devoted to showing that, for certain domains Ω, (2) and (3) are also sufficient to guarantee a pluriharmonic extension. The basic tool is the following result.

THEOREM 1. *Suppose* $u \in C^3(\Gamma)$ *is given. Then* u *satisfies* (2) *and* (3) *if and only if for each* $p \in \Gamma$ *there exists a function* v_p *and an open set* $\mathcal{O}_p$ *containing* p *such that* $u + iv_p$ *satisfies* (1) *on* $\Gamma \cap \mathcal{O}_p$.

II. Proof of Theorem 1. The equation (1) is equivalent to

$$\bar{\partial}u + i\bar{\partial}v = (\alpha + i\beta)\bar{\partial}\rho$$

on Γ, where u and v are defined in a neighborhood of Γ so that du and dv are defined on Γ. When this is divided into real and imaginary parts, it becomes:

$$dv = -d^c u + \alpha d^c \rho + \beta d\rho \tag{4}$$

$$d^c v = du - \alpha d\rho + \beta d^c \rho \,. \tag{5}$$

LEMMA 1. *The systems* (4) *and* (5) *can be solved locally if and only if there is a function* v *defined locally on* Γ *such that*:

$$d\rho \wedge dv = -d\rho \wedge d^c u + d\rho \wedge \alpha d^c \rho \,. \tag{6}$$

PROOF. We first show that (4) and (5) are equivalent. Consider the mapping of 1-forms $\sigma: \Lambda^1 \to \Lambda^1$ defined by:

$$\sigma(dx_j) = dy_j$$
$$\sigma(dy_j) = -dx_j \,.$$

We observe that

$$\sigma d = \sigma\Big(\sum \frac{\partial}{\partial x_j} dx_j + \sum \frac{\partial}{\partial y_j} dy_j\Big)$$
$$= -\sum \frac{\partial}{\partial y_j} dx_j + \sum \frac{\partial}{\partial x_j} dy_j = -d^c \,.$$

Similarly, $\sigma d^c = d$. Thus σ applied to (4) yields (5).

If v is a solution of (4), then it also solves (6). Conversely, if v is a solution of (6), then

$$dv = -d^c u + \alpha d^c \rho + \beta d\rho + \gamma d\rho .$$

Thus $v_0 = v + c$ will solve (4) if c is chosen so that $c = 0$ on Γ and $\partial c/\partial\rho = \gamma$.

LEMMA 2. *Given a 1-form ω on Γ, there exists a function v locally on Γ such that*

$$d\rho \wedge dv = d\rho \wedge \omega \tag{7}$$

if and only if $d\rho \wedge d\omega = 0$.

PROOF. Taking coordinates $(\rho, x_2, \cdots, x_{2n})$, we may write:

$$\omega = a d\rho + \sum_{j\geq 2} f_j dx_j .$$

Then (7) becomes:

$$d\rho \wedge dv = \sum_{j\geq 2} f_j d\rho \wedge dx_j .$$

By Poincare's lemma, one can solve locally for v if and only if

$$0 = d\Big(\sum_{j\geq 2} f_j dx_j\Big) \wedge d\rho = d\rho \wedge d\omega .$$

PROOF OF THEOREM 1. By Lemma 1 we can find v locally if and only if (6) holds. If we set $\omega = -d^c u + \alpha d^c \rho$, then by Lemma 2, (6) can be solved if and only if:

$$0 = d\rho \wedge d\omega = d\rho \wedge (-dd^c u) + d\rho \wedge d\alpha \wedge d^c\rho + d\rho \wedge \alpha dd^c \rho .$$

One obtains (2) by applying $d\rho \wedge d^c\rho \wedge d$ to (4):

$$d\rho \wedge ddv = 0 = -d\rho \wedge d^c\rho \wedge dd^c u + \alpha d\rho \wedge d^c\rho \wedge dd^c\rho .$$

III. Proof of sufficiency for certain domains.

THEOREM 2. *If $H^1(\Gamma, C) = 0$, and if $u \in C^3(\Gamma)$ satisfies (2) and (3), then u can be extended to be the real part of an analytic function on Ω.*

PROOF. For each $p \in \Gamma$, we can pick β so that the 1-form defined by (4) is tangential to Γ. Since u satisfies (2) and (3), v_p can be chosen to satisfy (1) on Γ near p. Suppose that v_1 and v_2 are defined in this manner on $\mathcal{O}_1$ and $\mathcal{O}_2$. Then $dv_1 = dv_2$ on $\mathcal{O}_1 \cap \mathcal{O}_2$. Thus dv_p defines a global section of $T^*(\Gamma)$, the cotangent boundle of Γ.

Let d' be the exterior derivative on Γ, where Γ is given its Riemannian structure as a submanifold of C^n. Then $d'dv_p = 0$. By the de Rham isomorphism, there is a function v on Γ such that $d'v = d'v_p$. Thus $d\rho \wedge (dv - dv_p) = 0$ on Γ. From this it follows that

$$\bar{\partial}\rho \wedge \bar{\partial}(u + iv) = \bar{\partial}\rho \wedge \bar{\partial}(u + iv_p) = 0 .$$

Thus $f = u + iv$ can be extended to an analytic function on Ω.

THEOREM 3. *Suppose the Levi form of Γ is nonvanishing and suppose Γ is connected. If $u \in C^3(\Gamma)$ satisfies* (2) *and* (3), *then u may be extended to a pluriharmonic function in Ω.*

PROOF. Let p be a point of Γ. By Theorem 1, there exists a function v_p in some neighborhood $\mathcal{O}_p \cap \Gamma$ of p such that (1) is satisfied. Since the Levi form does not vanish it must have either a positive or a negative eigenvalue. Thus $u + iv_p$ can be continued analytically to one side or the other (or both) of Γ.

We claim that u can be extended to a pluriharmonic function on an open set $\mathcal{O}$ which disconnects C^n and such that $\overline{\mathcal{O}} \supset \Gamma$. Suppose that u_1 and u_2 are extensions of u to open sets $\mathcal{O}_1$ and $\mathcal{O}_2$. Let $\mathcal{O}_3$ be a connected component of $\mathcal{O}_1 \cap \mathcal{O}_2$ such that $\overline{\mathcal{O}_3} \cap \Gamma$ contains an open subset of Γ. Since the Levi form is nonzero and $u_1 = u_2$ on $\overline{\mathcal{O}_3} \cap \Gamma$, it follows that $u_1 = u_2$ on $\mathcal{O}_3$. Thus u can be extended in a single-valued manner to one-sided neighborhoods $\mathcal{O}_+$ and $\mathcal{O}_-$ of Γ. If $p \in \Gamma$ is an interior point of $\mathcal{O}_+ \cup \Gamma \cup \mathcal{O}_-$, then u is pluriharmonic at p. Thus u can be extended to an open set $\mathcal{O}$ which disconnects C^n.

Since the boundary Γ of Ω is connected, we may apply Hartogs' theorem for pluriharmonic functions and obtain a pluriharmonic extension to Ω.

REMARK. The theorems above remain valid if Ω is a relatively compact domain with connected C^3-boundary in a Stein manifold X. The same proofs apply in this more general context because a function on $\Gamma = \partial\Omega$ can be extended to be analytic in Ω if and only if (1) holds.

IV. Examples. The equations (2) and (3) may be written in terms of ∂ and $\bar{\partial}$ by substituting $d = 1/2(\bar{\partial} + \partial)$ and $d^c = 1/2i(\bar{\partial} - \partial)$. Selecting the terms in (3) of type (1, 2), one obtains

$$(2)' \qquad \bar{\partial}\rho \wedge \partial\rho \wedge \partial\bar{\partial}u = \alpha\bar{\partial}\rho \wedge \partial\rho \wedge \partial\bar{\partial}\rho .$$

$$(3)' \qquad \bar{\partial}\rho \wedge \partial\bar{\partial}u = \bar{\partial}\rho \wedge \bar{\partial}(\alpha\partial\rho) .$$

Thus the above system is equivalent to the statement that for a given

$u \in C^3(\bar{\Omega})$ there is a modification $\tilde{u} = u - \alpha\rho$ of u such that on Γ,

$$\bar{\partial}\rho \wedge \bar{\partial}\partial\tilde{u} = 0 .$$

In the case where $\Omega = \{|z| < 1\}$ is the unit ball in C^n, we shall calculate (2) and (3) explicitly and show how they can be written in terms of tangential operators.

If $\rho(z) = |z|^2 - 1$, it follows that $\bar{\partial}\rho = \sum z_j d\bar{z}_j$ and $\partial\bar{\partial}\rho = \sum \delta_{jk} dz_j d\bar{z}_k$, where δ_{jk} is 1 if $j = k$ and 0 otherwise. Let us introduce the notation:

$$L_{jk} = \bar{z}_j \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial z_j}$$

$$\bar{L}_{jk} = z_j \frac{\partial}{\partial \bar{z}_k} - z_k \frac{\partial}{\partial \bar{z}_j}$$

$$N_{jk} = z_j \frac{\partial}{\partial z_j} + z_k \frac{\partial}{\partial z_k} .$$

Then (2)′ and (3)′ can be rewritten as

$$(2)'' \qquad (|z_j|^2 + |z_k|^2)\alpha = (\bar{L}_{jk} L_{jk} + N_{jk})u$$

$$(3)'' \qquad \bar{L}_{jk}[u_{z_l} - \bar{z}_l \alpha] = 0$$

for $1 \leqq j, k, l \leqq n$ and $|z_1|^2 + \cdots + |z_n|^2 = 1$.

Solving (2)″ for α and substituting into (3)″ one can show that (3)″ is equivalent to:

$$(8) \qquad \bar{L}_{jk} \bar{L}_{lm} L_{lm} u = 0$$

for $1 \leqq j, k, l, m \leqq n$ and $|z_1|^2 + \cdots + |z_n|^2 = 1$. The advantage of this formulation is that the operators L_{jk} and $\bar{L}_{jk}$ are tangential to the unit sphere. The equations (8) were derived in [4] along with consistency conditions for more general overdetermined systems on the unit ball in C^n.

V. $\partial\bar{\partial}$-Neumann conditions. An alternate approach to the problem of finding boundary conditions for pluriharmonic functions is to compute the Neumann conditions for the operator $\partial\bar{\partial}$. The equations obtained in this manner are much simpler, although they cannot be written as operators tangential to Γ.

Let $\langle\ ,\ \rangle$ be the standard hermitian inner product on the space $\Lambda^{p,q}$ of forms of type (p, q), and let $(\ ,\ )$ be the inner product on $C^\infty_{p,q}(\bar{\Omega})$ given by

$$(\varphi, \psi) = \int_\Omega \langle \varphi, \psi \rangle .$$

We define the contraction operation "$\vee$" by $\langle \omega \vee \alpha, \beta \rangle = \langle \alpha, \omega \wedge \beta \rangle$.

If ϑ is the formal adjoint of $\bar{\partial}$, then it follows (Proposition 1.3.1 of [5]) that for all $\varphi \in C^\infty_{p,q+1}(\bar{\Omega})$, $\psi \in C^\infty_{p,q}(\bar{\Omega})$,

$$(\vartheta\varphi, \psi) = (\varphi, \bar{\partial}\psi) - \int_\Gamma \langle \bar{\partial}\rho \vee \varphi, \psi \rangle \tag{9}$$

$$(\bar{\partial}\psi, \varphi) = (\psi, \vartheta\varphi) + \int_\Gamma \langle \bar{\partial}\rho \wedge \psi, \varphi \rangle \tag{10}$$

where the integrals are taken with respect to surface area on Γ.

A function $\varphi \in C^\infty_{(0,0)}(\bar{\Omega})$ satisfies the Neumann conditions for the $\partial\bar{\partial}$-complex if

$$(\vartheta\bar{\vartheta}\partial\bar{\partial}\varphi, \psi) = (\partial\bar{\partial}\varphi, \partial\bar{\partial}\psi)$$

for all functions $\psi \in C^\infty_{(0,0)}(\bar{\Omega})$. Using (9) and (10), one obtains for arbitrary $\varphi, \psi \in C^\infty_{(0,0)}(\bar{\Omega})$

$$(\vartheta\bar{\vartheta}\partial\bar{\partial}\varphi, \psi) = (\partial\bar{\partial}\varphi, \partial\bar{\partial}\psi) - \int_\Gamma \langle \bar{\partial}\rho \vee \bar{\vartheta}\partial\bar{\partial}\varphi, \psi \rangle - \int_\Gamma \langle \partial\rho \vee \partial\bar{\partial}\varphi, \bar{\partial}\psi \rangle .$$

Thus we obtain the $\partial\bar{\partial}$-Neumann conditions:

$$\bar{\partial}\rho \vee \partial\rho \vee \partial\bar{\partial}\varphi = 0 \tag{11}$$

$$\bar{\partial}\rho \vee \vartheta\bar{\partial}\varphi + \vartheta(\partial\rho \vee \partial\bar{\partial}\varphi) = 0 \quad \text{on} \quad \Gamma . \tag{12}$$

To see this, set $\partial_b = \bar{\partial}\rho \vee \bar{\partial}\rho \wedge \bar{\partial}$ and $\vartheta_b = \vartheta(\bar{\partial}\rho \vee \bar{\partial}\rho \wedge \cdot)$, then $\bar{\partial}_b$ and ϑ_b are formally adjoint on $C^\infty_{(p,q)}(\Gamma)$. Condition (11) allows us to replace $\bar{\partial}\psi$ by $\bar{\partial}_b\psi$ in the second integral. Integration by parts and (11) then produce (12).

We thus have the following result, which resembles Proposition 1.3.7 in [5].

PROPOSITION. *A function $\varphi \in C^\infty(\bar{\Omega})$ is pluriharmonic on Ω if and only if φ is biharmonic and satisfies* (11) *and* (12).

PROOF. If $\partial\bar{\partial}\varphi = 0$, then (11) and (12) are satisfied. Conversely, if φ is biharmonic, then

$$\vartheta\bar{\vartheta}\partial\bar{\partial}\varphi = \sum_{j,k=1}^{n} \frac{\partial^4\varphi}{\partial z_j \partial\bar{z}_j \partial z_k \partial\bar{z}_k} = \Delta^2\varphi = 0 .$$

If φ satisfies the Neumann conditions, then

$$0 = (\vartheta\bar{\vartheta}\partial\bar{\partial}\varphi, \varphi) = (\partial\bar{\partial}\varphi, \partial\bar{\partial}\varphi) .$$

Thus $\partial\bar{\partial}\varphi = 0$ completing the proof.

Using the formula

$$\vartheta(\alpha \vee \beta) = (-1)^{\deg \alpha}(\alpha \vee \vartheta\beta - \bar{\partial}\alpha \vee \beta)$$

one may rewrite (12) as

$$(\bar{\partial}\rho \vee \bar{\vartheta} - \partial\rho \vee \vartheta)\partial\bar{\partial}\varphi + \partial\bar{\partial}\rho \vee \partial\bar{\partial}\varphi = 0 ,$$

which is the same as

$$2(d^c\rho \vee d^c + d\rho \vee d)\Delta\varphi - \langle \partial\bar{\partial}\varphi, \partial\bar{\partial}\rho \rangle = 0 .$$

Thus equations (11) and (12) become

$$\sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k} \frac{\partial\rho}{\partial \bar{z}_j} \frac{\partial\rho}{\partial z_k} = 0 \tag{13}$$

$$X(\Delta\varphi) = \sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k} \frac{\partial^2\rho}{\partial \bar{z}_j \partial z_k} , \tag{14}$$

where X is the vector field given by grad ρ.

From this we conclude that within the class of biharmonic functions on $\bar{\Omega}$, a function is actually pluriharmonic if and only if it satisfies (13) and (14). One can easily see, however, that these equations do not depend only on the values of φ on Γ. If we further restrict ourselves to harmonic functions, then we actually have a pair of second order operators.

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