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REMARK ON THE NUMBER OF COMPONENTS OF FINITELY GENERATED FUNCTION GROUPS

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1. We begin with recalling some notations and definitions. Let SL' be the group of all conformal self-maps of $\hat{C}$, which is the Aleksandrov compactification of the complex plane C. Each $f \in SL'$ has the form

$$f: z \longmapsto \frac{az+b}{cz+d},$$

where $a, b, c, d \in C$ and $ad - bc = 1$. If $\mathrm{tr}^2 f = (a+d)^2 = 4$, then $f \in SL'$ is parabolic; f is elliptic if $\mathrm{tr}^2 f \in [0\ 4)$; in all other cases f is loxodromic. If f is loxodromic and if $\mathrm{tr}^2 f$ is real with $\mathrm{tr}^2 f > 4$, then f is hyperbolic.

Let G be a subgroup of SL'. If there is a neighborhood V of $z \in \hat{C}$ such that $f(V) \cap V = \emptyset$ for all $f \in G$, $f \neq id$, then the point z is a regular point of G and we write as $z \in \Omega(G)'$. A point $z \in \hat{C}$ is a limit point of G if there are a sequence $\{f_n\}$ of distinct elements of G and a point $w \in \hat{C}$ such that $f_n(w) \to z$ as $n \to \infty$. The set of all limit points is denoted by $\Lambda(G)$. The complement of $\Lambda(G)$ is represented by $\Omega(G)$. A group G is called Kleinian if $\Omega(G) \neq \emptyset$. If $\Lambda(G)$ is a finite set, then the Kleinian group is called elementary. In the following the group G means Kleinian. Note that $\Omega(G) - \Omega(G)'$ consists of isolated fixed points of elliptic elements of G. The connected components of $\Omega(G)$ are called components of G. A component Δ of G is called invariant if $f(\Delta) = \Delta$ for all $f \in G$.

Now we call G a function group if G is a Kleinian group with an invariant component Δ. In particular, if G is a finitely generated non-elementary function group and if Δ is simply connected, then G is called a B-group (see [6]).

Let G be a Kleinian group and let Δ_1, Δ_2 be two distinct components of G. Then Δ_1, Δ_2 are called equivalent if there is an $f \in G$ such that $f(\Delta_1) = \Delta_2$.

2. We are interested in the relation between the number N of generators of a finitely generated function group G and the number p of non-equivalent components of G. We always assume that G is non-elementary.

In general, for any finitely generated Kleinian group G with N generators, Bers' area theorem implies the inequality $p \leqq 84(N-1)$ (see [2]) and L. V. Ahlfors sharpened this inequality and proved $p \leqq 18(N-1)$ (see [1]). B. Maskit conjectured that the inequality $p \leqq 2(N-1)$ holds (cf. L. Bers [3], I. Kra [5]).

In this note, concerning with the above Maskit's conjecture, we shall prove the following

THEOREM. *Let G be a finitely generated non-elementary function group with an invariant component Δ_0 and let N be the number of generators of G. Then the following inequalities hold:*

$$p \leqq \left[\frac{3}{2}(N-1)-(g_0-2)\right] \quad \textit{for a B-group } G$$

and

$$p \leqq 3(N-1)-(g_0-2) \qquad \textit{for } G \textit{ not being a B-group},$$

where g_0 is the genus of Δ_0/G and $[x]$ is the integral part of x.

3. In this section we shall give the proof of Theorem.

Since G is non-elementary we see that $N \geqq 2$.

Now from Ahlfors' finiteness theorem we have $p < \infty$. Let $\{\Delta_0, \Delta_1, \cdots, \Delta_{p-1}\}$ be the complete list of non-equivalent components of $\Omega(G)$. We set $G_i = \{f \in G; f(\Delta_i) = \Delta_i\}$ $(0 \leqq i \leqq p-1)$, where $G_0 = G$. Note that every G_i is non-elementary.

We denote by Π_{2q-2} the vector space of complex polynomials in one variable of degree at most $2q-2$, where $q(\geqq 2)$ is an integer. Let $H^1(G, \Pi_{2q-2})$ be the first cohomology group of G with coefficients in Π_{2q-2}. Let $B_q(\Delta_i, G_i)$ denote the space of bounded holomorphic automorphic forms of weight $-2q$ for the Kleinian group G_i operating on Δ_i.

Let $(g_i, n_i; \nu_{i1}, \cdots, \nu_{in_i})$ be the signature of G_i, that is, g_i is the genus of Δ_i/G_i and $\nu_{ij}(2 \leqq \nu_{ij} \leqq \infty,\ j=1, \cdots, n_i)$ is the ramification number of $x_{ij} \in \overline{\Delta_i/G_i} - \Delta_i'/G_i$ induced by the natural projection $\Delta_i \to \Delta_i/G_i$, where $\overline{\Delta_i/G_i}$ is a compact Riemann surface obtained by attaching punctures to Δ_i/G_i and $\Delta_i' = \Delta_i \cap \Omega(G)'$.

Then it is well known that

$$\text{(1)} \qquad \dim B_q(\Delta_i, G_i) = (2q-1)(g_i-1) + \sum_{j=1}^{n_i}\left[q - \frac{q}{\nu_{ij}}\right],$$

where $[q - q/\nu_{ij}] = q-1$ for $\nu_{ij} = \infty$.

Further, L. Bers [2] proved the inequality

$$\text{(2)} \qquad (2q-1)(N-1) \geqq \dim H^1(G, \Pi_{2q-2}) \geqq \sum_{i=0}^{p-1} \dim B_q(\Delta_i, G_i).$$

Therefore we have

$$(3) \qquad (2q-1)(N-1) \geqq \sum_{i=0}^{p-1} \left\{(2q-1)(g_i-1) + \sum_{j=1}^{n_i}\left[q - \frac{q}{\nu_{ij}}\right]\right\} .$$

In particular, for a B-group G, I. Kra [4] proved the following inequality

$$\dim H^1(G, \Pi_{2q-2}) \geqq 2 \dim B_q(\Delta_0, G) .$$

So, in this case, we have

$$(4) \qquad (2q-1)(N-1) \geqq 2\left\{(2q-1)(g_0-1) + \sum_{j=1}^{n_0}\left[q - \frac{q}{\nu_{0j}}\right]\right\} .$$

If $p \leqq 2$, our theorem is obvious. For, in general, (3) implies $N \geqq g_0 - 1$, which shows that

$$3(N-1) - (g_0 - 2) \geqq 2(N-1) \geqq 2 .$$

Further, for a B-group G, (4) implies $N - 1 \geqq 2(g_0 - 1)$ which gives

$$\frac{3}{2}(N-1) - (g_0 - 2) \geqq N \geqq 2 .$$

Hence we may assume $p \geqq 3$.

Now we consider the case where G is a B-group. If G has no accidental parabolic transformation, then, from Maskit's theorem in [6], the group G is degenerate or quasifuchsian. Hence $p \leqq 2$. Thus in this case we have nothing to prove.

Next we assume that G has accidental parabolic transformations.

We see that the permissible signature of G which minimizes the right hand side of (4) is $(g_0, n_0; 2, \cdots, 2)$. We set $q = 2$ in (4). Since G has accidental parabolic transformations, Maskit's result [6] gives the inequality $p - 1 \leqq 2g_0 - 2 + n_0$. Hence we have

$$p \leqq \frac{3}{2}(N-1) - (g_0 - 2) .$$

Therefore we conclude for any B-group that the following inequality holds:

$$p \leqq \left[\frac{3}{2}(N-1) - (g_0 - 2)\right] .$$

Next we consider the case where G is not a B-group.

Since every G_i is non-elementary we see that, for $1 \leqq i \leqq p-1$, the permissible signatures of G_i that minimize the right hand side of (3) are $(0, 3; 2, 3, 7)$.

For the group G we see that the permissible signature is $(g_0, n_0; 2,$

$\cdots, 2)$. Setting $q=2$ and using the inequality $p-1 \leqq 2g_0 - 2 + n_0$ (see [7]), we have the inequality

$$p \leqq 3(N-1) - (g_0 - 2) .$$

Thus, for any finitely generated function group which is not a B-group, we conclude that the following inequality holds:

$$p \leqq 3(N-1) - (g_0 - 2) .$$

The proof of Theorem is completed.

REMARK. There are some cases where the inequalities in Theorem can be sharpened. For instance, assume that G is a B-group and that $g_0 = 0$, $n_0 \geqq 5$ and the signature of G is $(0, n_0; 2, \cdots, 2)$. In this case we can easily see that

$$p \leqq \left[\frac{3}{2}(N-1)\right] .$$

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