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COMPACT LEAVES WITH ABELIAN HOLONOMY

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1. Introduction and statement of the results. In this paper we investigate the behaviour of codimension-one foliations in neighborhoods of their compact leaves with abelian holonomy. If the foliations are C^2 class, we have three cases which occur independently. The following theorem is the precise formulation.

THEOREM 1. *Let $\mathscr{F}$ be a transversely orientable codimension-one C^r foliation on an orientable C^r manifold M and F_0 a compact leaf of $\mathscr{F}$. Suppose that $2 \leqq r \leqq \infty$. Let T be a tubular neighborhood of F_0 and U_+ the union of F_0 and a connected component of $T - F_0$. We denote by $\Phi_+(F_0)$ the one-sided holonomy group of F_0 which is defined by the restricted foliation $\mathscr{F} \mid U_+$. Suppose that $\Phi_+(F_0)$ is an abelian group, then only one of the following three cases occurs.*

(1) *For all neighborhoods V of F_0, the restricted foliation $\mathscr{F} \mid V \cap U_+$ has a compact leaf which is not F_0.*

(2) *There is a neighborhood V of F_0 such that all leaves of $\mathscr{F} \mid V \cap U_+$ except F_0 are dense in $V \cap U_+$. In this case $\Phi_+(F_0)$ is a free abelian group of the rank $\geqq 2$.*

(3) *There are a neighborhood V of F_0 and a connected oriented codimension-one submanifold N of F_0 as follows. We denote by F_* the compact manifold with boundary obtained by attaching two copies N_1 and N_2 of N to $F_0 - N$, so that $\partial F_* = N_1 \cup N_2$. Let $f: [0, \varepsilon) \rightarrow [0, \delta)$ be a contracting C^r diffeomorphism satisfying $f(0) = 0$. We denote by X_f the quotient manifold obtained from $F_* \times [0, \varepsilon)$ by identifying $(x, t) \in N_1 \times [0, \varepsilon)$ and $(x, f(t)) \in N_2 \times [0, \delta)$. We have a foliation $\mathscr{F}_f$ on X_f whose leaves consist of some family of the sets $F_* \times \{t\}$, $t \in [0, \varepsilon)$. Then for some f as above, there is a C^r diffeomorphism $h: V \cap U_+ \rightarrow X_f$ which maps each leaf of $\mathscr{F} \mid V \cap U_+$ onto some leaf of $\mathscr{F}_f$. $\mathscr{F} \mid V \cap U_+$ determines the homology class $[N] \in H_{n-2}(F_0^{n-1}, Z)$ uniquely and the germ at 0 of f uniquely up to conjugation. In this case $\Phi_+(F_0)$ is an infinite cyclic group.*

We can reformulate Theorem 1 in a following weaker form. Foliations of the type in Theorem 1* were studied by Reeb [8] and Theorem 1* can be considered as a strengthened form of the results of [8]. Theorem 1* says that Theorem 5 of [8] is valid even if supposing C^2 in place of analyticity of foliations.

THEOREM 1*. *Let $\mathscr{F}$ be a transversely orientable codimension-one C^r foliation on $M \times [0, 1]$ where M is a connected orientable closed C^r manifold, $2 \leqq r \leqq \infty$. Suppose that all leaves of $\mathscr{F}$ are transverse to $\{x\} \times [0, 1]$ for all $x \in M$ and that $\mathscr{F}$ has only two compact leaves $M \times \{0\}$ and $M \times \{1\}$. As usual we have a homomorphism $\Phi: \pi_1(M, x_0) \to \mathrm{Diff}_0^r([0, 1])$ where $x_0 \in M$ and $\mathrm{Diff}_0^r([0, 1])$ is the group of all orientation-preserving C^r diffeomorphisms: $[0, 1] \to [0, 1]$. [Let ω: $([0, 1], \{0, 1\}) \to (M, x_0)$ be a closed path, then we have a foliation $\mathscr{F}_\omega$ on $[0, 1] \times [0, 1]$ induced from $\mathscr{F}$ by the map $\omega \times id$: $[0, 1] \times [0, 1] \to M \times [0, 1]$. We define $f_\omega(t) \in [0, 1]$ so that $(t, 0)$ and $(f_\omega(t), 1)$ are the endpoints of a leaf of $\mathscr{F}_\omega$. Let $\Phi([\omega]) = f_\omega: [0, 1] \to [0, 1]$ where $[\omega]$ means the homotopy class of ω, then Φ is a well-defined homomorphism.] If we suppose that the image of Φ is abelian, then only one of the following occurs.*

(1) *All leaves except the compact leaves are dense and Image (Φ) is a free abelian group of the rank $\geqq 2$.*

(2) *All leaves are proper. All leaves except the compact leaves are mutually C^r diffeomorphic and are covering spaces of M whose covering transformation groups are infinite cyclic groups. Image (Φ) is an infinite cyclic group.*

(1) and (3) of Theorem 1 clearly occur for some foliations. In § 4 we construct a C^∞ foliation which admits (2) of Theorem 1. Pay attention to the example of Sacksteder [9]. It shows that the hypothesis that the holonomy group is abelian is a necessary condition. Furthermore we can construct a topological foliation as to which Theorem 1 and 1* are not valid (Theorem 2). But we can say nothing about the C^1 case.

THEOREM 2. *Let S_g be the closed orientable surface of genus g, then there is a transversely orientable codimension-one topological foliation $\mathscr{F}$ on $S_g \times [0, 1]$ as follows.*

(1) *All leaves of $\mathscr{F}$ are C^∞ submanifolds and transverse to $\{x\} \times [0, 1]$ for all $x \in S_g$ and $\mathscr{F}$ has only two compact leaves $S_g \times \{0\}$ and $S_g \times \{1\}$.*

(2) *Image (Φ) (defined as in Theorem 1*) is a free abelian group of rank g.*

(3) *All leaves are proper.*

(4) *There is a sequence $F_0, F_1, \cdots, F_g$ of the leaves of $\mathscr{F}$ such that F_0 is $S_g \times \{0\}$ and $\bar{F}_0 \subsetneqq \bar{F}_1 \subsetneqq \cdots \subsetneqq \bar{F}_g$.*

2. Definitions and notations.

2.1. The fundamental references on foliations are Reeb [7], Haefliger [2] and Lawson [5]. In spite of their existence, we will start from the definitions for the sake of the self-containedness and technical advantage.

Let M be a C^r n-manifold, $0 \leqq r \leqq \omega$. A subset $\mathscr{F}$ of the C^r structure $\mathscr{S} \subset \{(U, f) \mid U$ is an open set of M. $f\colon U \to R^n$ is an injective homeomorphism.$\}$ of M is called a *codimension q C^r foliation* if $\mathscr{F}$ satisfies the following conditions (1) and (2).

(1) $\{U \mid (U, f) \in \mathscr{F}\}$ is an open covering of M and $f(U) = R^n$ for all $(U, f) \in \mathscr{F}$.

(2) For all pairs (U_1, f_1), $(U_2, f_2) \in \mathscr{F}$ with $U_1 \cap U_2 \neq \varnothing$, the C^r diffeomorphism $f_1 \circ f_2^{-1}\colon f_2(U_1 \cap U_2)(\subset R^n = R^p \times R^q) \to R^n = R^p \times R^q$ has the form

$$f_1 \circ f_2^{-1}(x, y) = (\alpha(x, y), \beta(y))$$

where $x \in R^p$, $y \in R^q$, $\alpha\colon R^p \times R^q \to R^p$ is a C^r local submersion and $\beta\colon R^q \to R^q$ is a C^r local diffeomorphism.

Two foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ are *equivalent* if $\mathscr{F}_1 \cup \mathscr{F}_2$ satisfies the condition (2). A foliation $\mathscr{F}$ is called *transversely orientable* if $\mathscr{F}$ is equivalent to another foliation $\mathscr{F}_1$ such that in the definition of $\mathscr{F}_1$ all β of the condition (2) are orientation-preserving.

Let $\mathscr{F}$ be a foliation on M. The condition (2) guarantees that

$$\begin{aligned} f_2^{-1}(R^p \times \{y\}) \cap f_1^{-1}(R^p \times \{\beta(y)\}) &= U_1 \cap f_2^{-1}(R^p \times \{y\}) \\ &= U_2 \cap f_1^{-1}(R^p \times \{\beta(y)\}) \end{aligned}$$

and $f_2^{-1}(R^p \times \{y\}) \cup f_1^{-1}(R^p \times \{\beta(y)\})$ is a submanifold of $U_1 \cup U_2$. We say that $f_2^{-1}(R^p \times \{y\})$ and $f_1^{-1}(R^p \times \{\beta(y)\})$ are *adjacent*. Then we have an equivalence relation on the family

$$\{f^{-1}(R^p \times \{y\}) \mid (U, f) \in \mathscr{F}, y \in R^q\}$$

defined by the relation adjacent. The union F of all subsets in an equivalence class is called a *leaf* of $\mathscr{F}$ and F has naturally a structure of C^r manifold of dimension p.

If the topology of a leaf F as a subset of M and the topology as a p-manifold coincide, F is called *proper*.

If $\mathscr{F}$ is a codimension q C^r foliation on M_2 and $f\colon M_1 \to M_2$ is a C^r map which is transverse to all leaves of $\mathscr{F}$, then there is a codimension q C^r foliation $f^*\mathscr{F}$ whose leaves are connected components of $f^{-1}(F)$ where F runs through the leaves of $\mathscr{F}$. We call $f^*\mathscr{F}$ the foliation induced by f from $\mathscr{F}$. If f is an inclusion, we call $f^*\mathscr{F}$ the restricted foliation and we write $f^*\mathscr{F} = \mathscr{F} \mid M$.

2.2. Now we restrict our attention to transversely orientable codimension-one C^r foliation $\mathscr{F}$ on an orientable manifold M. We give a definition of holonomy groups in a style which is useful in the following arguments and which is a slight modification of that of Hirsch [3].

We suppose $r \geqq 1$ and we remark that in the C^0 case the following arguments can be suitably modified by using Siebenmann [10] and Hirsch [3].

Let $\mathscr{F}$ be a transversely orientable codimension-one C^r foliation on an orientable C^r n-manifold, $r \geqq 1$. Therefore we may suppose that all β of 2.1 (2) are orientation-preserving. Transverse orientability of $\mathscr{F}$ and the partition of unity imply the existence of a non-singular vector field X which is transverse to all leaves of $\mathscr{F}$. Let $\varphi: M \times R \to M$ be the associated flow of X. Then $\varphi \mid \varphi(\{x\} \times R)$ and $f^* \mid \varphi(\{x\} \times R)$ have derivatives at $0 \in R$ of the same sign for all $x \in M$ and $(U, f) \in \mathscr{F}$ satisfying $x \in U$ where f^* is the composition of f and the projection of $R^{n-1} \times R$ to the last factor R, or of the opposite sign. We suppose the former without the loss of generality.

Let F be a compact leaf of $\mathscr{F}$. Triangulate F and take its duall cell decomposition $F = \bigcup_{\lambda \in \Lambda} A_\lambda$. By restarting from another finer triangulation of F if necessary, we can suppose that for all $\lambda \in \Lambda$ there are a positive number ε_λ and a chart $(U_\lambda, f_\lambda) \in \mathscr{F}$ such that $\varphi(B_\lambda \times [-\varepsilon_\lambda, \varepsilon_\lambda]) \subset U_\lambda$ where $B_\lambda = \bigcup \{A_{\lambda'} \mid A_{\lambda'} \cap A_\lambda \neq \emptyset\}$. We denote by Ω_λ the subset

$$\varphi(A_\lambda \times [0, \varepsilon_\lambda]) \cap f_\lambda^{*-1}([a_\lambda, b_\lambda])$$

where

$$a_\lambda = \mathrm{Max}\,\{f_\lambda^* \circ \varphi(x, -\varepsilon_\lambda) \mid x \in A_\lambda\} \quad \text{and} \quad b_\lambda = \mathrm{Min}\,\{f_\lambda^* \circ \varphi(x, \varepsilon_\lambda) \mid x \in A_\lambda\}\,.$$

Let $S = \{\Omega_\lambda \mid \lambda \in \Lambda\}$ and $m = \sharp(\Lambda) < \infty$. An S *chain* means a sequence $\omega = (\Omega_0, \cdots, \Omega_k)$ such that $\Omega_i \in S$ and $\Omega_i \cap \Omega_{i-1} \neq \emptyset$, $i = 1, \cdots, k$. We call k the *length* of ω. An Ω_λ *plaque* means a set of the form $\Omega_\lambda \cap f_\lambda^{-1}(R^{n-1} \times \{y\})$, $y \in R$. We denote by $\Omega_\lambda/\mathscr{F}$ the set of all Ω_λ plaques. $\Omega_\lambda/\mathscr{F}$ can be identified with an interval by the map $\bar{f}_\lambda: \Omega_\lambda/\mathscr{F} \to R$ induced from $f_\lambda^*: \Omega_\lambda \to R$. An Ω_0 plaque P_0 *admits* an S chain $\omega = (\Omega_0, \cdots, \Omega_k)$ if there are Ω_i plaques P_i, $i = 0, \cdots, k$, such that $P_{i-1} \cap P_i \neq \emptyset$, $i = 1, \cdots, k$. In this case $\Omega_{i-1} \cap \Omega_i$ is a disk and P_{i-1} determines P_i uniquely by the condition $P_{i-1} \cap P_i \neq \emptyset$. So P_0 determines P_k uniquely. Thus an S chain $\omega = (\Omega_0, \cdots, \Omega_k)$ defines a local C^r map

$$G(\omega): \Omega_0/\mathscr{F} \to \Omega_k/\mathscr{F}$$

whose domain $DG(\omega) \subset \Omega_0/\mathscr{F}$ is the set of all Ω_0 plaques admitting ω.

We call $G(\omega)$ the *geometric holonomy* of ω. If $\omega_1 = (\Omega_0, \cdots, \Omega_k)$ and $\omega_2 = (\Omega'_0, \cdots, \Omega'_l)$ are S chains and $\Omega_k = \Omega'_0$, let $\omega_1 \# \omega_2 = (\Omega_0, \cdots, \Omega_k, \Omega'_1, \cdots, \Omega'_l)$. Then $G(\omega_1 \# \omega_2) = G(\omega_2) \circ G(\omega_1)$ and $DG(\omega_1 \# \omega_2) = DG(\omega_1) \cap G(\omega_1)^{-1}(DG(\omega_2))$.

A *homotopy* in S is a sequence of S chains $\omega_0, \cdots, \omega_q$ such that ω_i is obtained from ω_{i-1} by inserting an element $\Omega \in S$ or ω_{i-1} from ω_i for $i = 1, \cdots, q$. We call ω_0 and ω_q *homotopic*. Note that ω_0 and ω_q are coterminous. In this case $G(\omega_0) = G(\omega_q)$ on $\bigcap_{i=0}^q DG(\omega_i)$.

Choose an element $\Omega_* \in S$. Let $\pi_1(S, \Omega_*)$ be the set of homotopy classes of all S chains $\omega = (\Omega_0, \cdots, \Omega_k)$ such that $\Omega_0 = \Omega_k = \Omega_*$. We can induce a group operation on $\pi_1(S, \Omega_*)$ from the operation $\#$.

Let $G^r_{P_*}$ be the group of the germs f_{P_*} at the plaque $P_* = \Omega_* \cap F$ of all orientation-preserving local C^r diffeomorphisms $f: \Omega_*/\mathscr{F} \to \Omega_*/\mathscr{F}$ satisfying $f(P_*) = P_*$. Note that $\Omega_*/\mathscr{F}$ is a compact connected one-dimensional manifold and P_* is an interior point of $\Omega_*/\mathscr{F}$. If ω_1 and ω_2 are homotopic closed S chains at Ω_*, the germs $(G(\omega_1))_{P_*}$ and $(G(\omega_2))_{P_*}$ coincide as already seen. Thus we have a map

$$\bar{G}: \pi_1(S, \Omega_*) \to G^r_{P_*}$$

which is clearly a homomorphism. Let H^r_0 be the group of the germs f_0 at 0 of all orientation-preserving local C^r diffeomorphism $f: R \to R$ satisfying $f(0) = 0$ and $\iota: \Omega_*/\mathscr{F} \to R$ a C^r imbedding satisfying $\iota(P_*) = 0$. We define a homomorphism

$$\bar{H}: \pi_1(S, \Omega_*) \to H^r_0$$

by $H([\omega]) = (\iota \circ G(\omega) \circ \iota^{-1})_0$. The image of the homomorphism H is called the *holonomy group* of F, which is unique up to conjugation. Let H^r_+ be the group of the germs f_0 at 0 of all local C^r diffeomorphisms $f: [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$ and $r: H^r_0 \to H^r_+$ the homomorphism induced by the restriction. The image of $r \circ \bar{H}$ is called a *one-sided holonomy group* of F and denoted by $\Phi_+(F)$.

Choose a point $x_* \in \Omega_* \cap F$. A path $u: [0, 1] \to F$ is *contained* in an S chain $\omega = (\Omega_0, \cdots, \Omega_k)$ if there is a subdivision $0 = t_0 < \cdots < t_{k+1}$ such that $u([t_i, t_{i+1}]) \subset \Omega_i$, $i = 0, \cdots, k$.

LEMMA 1. (1) *For all $\alpha \in \pi_1(F, x_*)$, we can find a closed path $u: ([0, 1], \{0, 1\}) \to (F, x_*)$ representing α which is contained in some S chain $\omega = (\Omega_0, \cdots, \Omega_k)$ with $\Omega_0 = \Omega_k = \Omega_*$.*

(2) *If a path u_i is contained in an S chain ω_i, $i = 1, 2$, then u_1 and u_2 are homotopic relative to $\{0, 1\}$ if and only if ω_1 and ω_2 are homotopic.*

The proof of Lemma 1 is easy and we omit it. According to Lemma

1, we have naturally an isomorphism $i\colon \pi_1(F, x_*) \to \pi_1(S, \Omega_*)$.

3. The proof of Theorem 1 and 1*. We will prove only Theorem 1 since Theorem 1* can be analogically proved. We consider the case when (1) of Theorem 1 does not occur and suppose that $\mathscr{F} \mid U_+$ has the only compact leaf F_0 throughout this section. We use the notation of 2.2 with F_0 in the place of F and we may assume that $\varphi(A_\lambda \times [0, \varepsilon_\lambda)) \subset U_+$. Let $\Omega_\lambda^+ = \Omega_\lambda \cap U_+$.

At first we show the following.

LEMMA 2. *The one-sided holonomy group $\Phi_+(F_0)$ of F_0 is a non-trivial free abelian group.*

PROOF. $\Phi_+(F_0)$ is free abelian since $\Phi_+(F_0)$ is abelian by assumption and the group H_+^r has no torsion elements. We suppose that $\Phi_+(F_0)$ is a trivial group, out of which we will bring a contradiction. Let Λ_{2m} is the set of all S chains $= (\Omega_0, \Omega_1, \cdots, \Omega_k)$ such that $k \leqq 2m$ and $\Omega_0 = \Omega_k = \Omega_*$. Recall that $\#(S) = m < \infty$. For all $\omega \in \Lambda_{2m}$ we have a neighborhood W_ω of the plaque P_* in $\Omega_*^+/\mathscr{F}$ such that $G(\omega) \mid W_\omega$ is the identity map since the germ at P_* of $G(\omega) \mid \Omega_*^+/\mathscr{F}$ is the identity. Let

$$W = \bigcap \{W_\omega \mid \omega \in \Lambda_{2m}\} ,$$

then W is a neighborhood of P_* since Λ_{2m} is a finite set. Take a plaque $P \in \Omega_*^+/\mathscr{F} - \{P_*\}$. Let

$$F = \bigcup \{G(\omega)(P) \mid \omega = (\Omega_0, \cdots, \Omega_k),\ \Omega_0 = \Omega_*,\ k \leqq m\} ,$$

F is compact since ω runs through a finite set. Let $Q = G(\omega)(P) \subset F$ and $Q^* \in \Omega_{k+1}/\mathscr{F}$ are adjacent, that is $G(\omega_1)(Q) = Q^*$ where ω_1 is the S chain (Ω_k, Ω_{k+1}). There is an S chain $\omega_2 = (\Omega_0^*, \cdots, \Omega_j^*)$ such that $j < m$, $\Omega_0^* = \Omega_*$ and $\Omega_j^* = \Omega_{k+1}$. Since $\omega \# \omega_1 \# \omega_2^{-1} \in \Lambda_{2m}$ where $\omega_2^{-1} = (\Omega_j^*, \cdots, \Omega_0^*)$,

$$\begin{aligned} P &= G(\omega \# \omega_1 \# \omega_2^{-1})(P) \\ &= G(\omega_2)^{-1} \circ G(\omega_1) \circ G(\omega)(P) \\ &= G(\omega_2)^{-1}(Q^*) \end{aligned}$$

and then $Q^* = G(\omega_2)(P) \subset F$. Therefore F is a leaf of $\mathscr{F} \mid U_+$. Furthermore $F \cap \Omega$ is an Ω plaque for all $\Omega \in S$ and F is C^r diffeomorphic to F_0. This is a contradiction to the assumption that $\mathscr{F} \mid U_+$ has no compact leaf except F_0. This completes the proof of Lemma 2.

We need the theorem of Kopell.

THEOREM (Kopell [4]). *Let $f\colon [0, a_1] \to [0, b_1]$ and $g\colon [0, a_2] \to [0, b_2]$ be orientation-preserving C^2 diffeomorphisms. Suppose that there is $0 < a < \mathrm{Min}\,\{a_1, a_2\}$ satisfying the following.*

(1) *There is* $t_0 \in (0, a)$ *such that* $f(t_0) = t_0$.

(2) $g(t) < t$ *for all* $t \in (0, a)$.

(3) $f \circ g$ *and* $g \circ f$ *can be defined on* $[0, a)$ *and coincide there.*

Then $f | [0, a)$ *is the identity map.*

By using this theorem we prove the following lemma, which shows some character of the elements of abelian holonomy groups.

LEMMA 3. *For all closed* S *chain* $\omega_0 = (\Omega_0, \cdots, \Omega_k)$ *at* Ω_*, *that is* $\Omega_0 = \Omega_k = \Omega_*$, *such that the germ* $(G(\omega_0) | \Omega_*^+/\mathscr{F})_{P_*}$ *is not the identity, there is a neighborhood* W_{ω_0} *of* P_* *in* $\Omega_*^+/\mathscr{F}$ *such that* $G(\omega_0) | W_{\omega_0}$ *has no fixed point except* P_*.

PROOF. Let $\iota: \Omega_*^+/\mathscr{F} \to [0, a)$ be a C^r diffeomorphism and let $H(\omega) = \iota \circ G(\omega) \circ \iota^{-1}$ for all closed S chains ω at Ω_*. Since the group $\Phi_+(F_0)$ is abelian and Λ_{2m}, defined in the proof of Lemma 2, is a finite set, there is a positive number ε such that

(1) $[0, \varepsilon] \subset (DG(\omega))$ for all $\omega \in \Lambda_{2m} \cup \{\omega_0\}$;

(2) $H(\omega) \circ H(\omega_0)$ and $H(\omega_0) \circ H(\omega)$ are defined on $[0, \varepsilon]$ and coincide there for all $\omega \in \Lambda_{2m}$.

We are going to show that $\iota^{-1}([0, \varepsilon))$ has the property of W_{ω_0} in Lemma 3. We suppose that there is $t_0 \in (0, \varepsilon)$ satisfying that $H(\omega_0)(t_0) = t_0$, out of which we will bring a contradiction.

We may suppose that there is t_1 such that $0 < t_1 < t_0$ and $f(t) \neq t$ for all $t \in (t_1, t_0)$, without loss of generality. In fact let

$$K = \{t \in (0, t_0) \mid H(\omega_0)(t) \neq t\} ,$$

then K is a non-empty open subset since the germ at 0 of $H(\omega_0)$ is not the identity. Let (b, c) be a connected component of K. Then $H(\omega_0)(b) = b$ and $H(\omega_0)(c) = c$. It is sufficient to take c in place of t_0 and let $t_1 = b$.

Now we see that there is a closed S chain $\omega_1 \in \Lambda_{2m}$ such that $H(\omega_1)(t_0) \neq t_0$. In fact if not, we can show that

$$F = \bigcup \{G(\omega_0)(P_0) \mid \omega = (\Omega_0, \cdots, \Omega_k),\ \Omega_0 = \Omega_*,\ k \leqq m\}$$

is a compact leaf, where $P_0 = \iota^{-1}(t_0)$, by the same argument as in the proof of Lemma 2. This is a contradiction.

We may suppose that $H(\omega_1)(t_0) < t_0$ by taking ω_1^{-1} in place of ω_1 if necessary. Let $f = H(\omega_0)$ and $g = H(\omega_1)$. Since $f([0, t_0]) = [0, t_0]$ and $g([0, t_0]) = [0, t_0)$, $f^{\mu_1} \circ g^{\nu_1} \circ \cdots \circ f^{\mu_j} \circ g^{\nu_j}(t) \in [0, t_0]$ if $t \in [0, t_0]$, $\mu_i \geqq 0$ and $\nu_i \geqq 0$. Then

$$f^{\mu_1} \circ g^{\nu_1} \circ \cdots \circ f^{\mu_j} \circ g^{\nu_j}(t) = f^{\mu_1 + \cdots + \mu_j} \circ g^{\nu_1 + \cdots + \nu_j}(t)$$

since f and g are commutative on $[0, t_0]$. Let $t_2 = \lim_{n \to \infty} g^n(t_0)$, which exists because the sequence $t_0, g(t_0), \cdots$ is monotonely decreasing and bounded. Then

$$g(t_2) = \lim_{n\to\infty} g(g^n(t_0)) = \lim_{n\to\infty} g^{n+1}(t_0) = t_2 .$$

$$f(t_2) = \lim_{n\to\infty} f(g^n(t_0)) = \lim_{n\to\infty} g^n(f(t_0)) = \lim_{n\to\infty} g^n(t_0) = t_2 .$$

Since $g(t) < t$ for all $t \in (t_2, t_0]$, there is t_3 such that $t_0 < t_3 < \varepsilon$ and $g \mid [t_2, t_3)$ is a contraction. Note that $f(t_0) = t_0$ and f and g are C^2 and commutative on $[t_2, t_3) \subset [0, \varepsilon]$. By applying the theorem of Kopell to f and g in the interval $[t_2, t_3)$, we see that $f \mid [t_2, t_3)$ is the identity map. This is a contradiction, which completes the proof of Lemma 3.

Now we need the well-known theorem of Denjoy-Siegel.

THEOREM (Denjoy [1], Siegel [11]). *Let $f: S^1 \to S^1$ be a C^2 diffeomorphism. Then only one of the following occurs.*

(1) *$\{f^\nu(x) \mid \nu$: positive integer$\}$ is dense in S^1 for all $x \in S^1$.*

(2) *There are $x \in S^1$ and a positive integer ν such that $f^\nu(x) = x$.*

By using this theorem we show the following.

LEMMA 4. *Let ω_1 and ω_2 be closed S chains at Ω_* such that the germs at P_* of $G(\omega_1)$ and $G(\omega_2)$ are not the identity. Then only one of the following occurs.*

(1) *There is a neighborhood V of F_0 such that all leaves of $\mathscr{F} \mid V \cap U_+$ except F_0 are dense in $V \cap U_+$.*

(2) *There are integers μ and ν such that the germs at P_* of $G(\omega_1)^\mu$ and $G(\omega_2)^\nu$ coincide.*

PROOF. Let $f = H(\omega_1)$ and $g = H(\omega_2)$. By Lemma 3, there is $\varepsilon > 0$ such that

(1) $f \mid [0, \varepsilon]$ and $g \mid [0, \varepsilon]$ have no fixed points except 0.

(2) $f \circ g$ and $g \circ f$ are defined on $[0, \varepsilon]$ and coincide there.

It is sufficient to consider the case where f and g are contractions. Since $f([0, \varepsilon]) \subset [0, \varepsilon)$ and $g([0, \varepsilon]) \subset [0, \varepsilon)$, $f^{\mu_1} \circ g^{\nu_1} \circ \cdots \circ f^{\mu_j} \circ g^{\nu_j}$ are defined on $[0, \varepsilon]$ if $\mu_i \geqq 0$ and $\nu_i \geqq 0$, and coincide there with $f^{\mu_1 + \cdots + \mu_j} \circ g^{\nu_1 + \cdots + \nu_j}$.

Consider the equivalence relation $\sim$ on $(0, \varepsilon)$ defined by $t \sim f(t)$, $t \in (0, \varepsilon)$. The quotient space $(0, \varepsilon)/\sim$ is a C^r manifold and C^r diffeomorphic to S^1. We denote by $[t]$ the equivalence class of $t \in (0, \varepsilon)$. If $t_1 \sim t_2$, then $g(t_1) \sim g(t_2)$ because of the commutativity of f and g. Thus we have a C^r diffeomorphism

$$g_*: (0, \varepsilon)/\sim \ \to (0, \varepsilon)/\sim$$

such that $g_*([t]) = [g(t)]$.

By the theorem of Denjoy-Siegel, only one of the following occurs.

(1) $\{g_*^\nu(x) \mid \nu \geqq 0\}$ is dense for all $x \in (0, \varepsilon)/\sim$.

(2) There are $x \in (0, \varepsilon)/\sim$ and a positive number ν such that $g_*^\nu(x) = x$.

At first we consider the case (1). Let V be a neighborhood of F_0 such that $V \cap U_+$ consists of the leaves of $\mathscr{F} \mid U_+$ which contains a plaque $\in W = \iota^{-1}([0, \varepsilon]) \subset \Omega_*/\mathscr{F}$. We are going to show all leaves of $\mathscr{F} \mid V \cap U_+$ except F_0 are dense in $V \cap U_+$. Let $P = \iota^{-1}(t)$, $t \in (0, \varepsilon)$ and F be a leaf of $\mathscr{F} \mid V \cap U_+$ which contains P. Let $K = \{Q \in W \mid Q \subset F\}$. It is sufficient to show that K is dense in W. Let W^* be an arbitrary open set of W. There is a positive integer ν such that $g^\nu(t) \sim t^* \in \iota(W^*)$. There is an integer μ such that $t^* = f^\mu \circ g^\nu(t)$ by the definition of $\sim$. Then

$$\begin{aligned} P^* &= \iota^{-1}(t^*) \\ &= G(\omega_1)^\mu \circ G(\omega_2)^\nu(P) \\ &= G(\omega_1^\mu \# \omega_2^\nu)(P) \end{aligned}$$

and $P^* \subset F$. Therefore $P^* \in K \cap W^*$, which implies that K is dense in W.

Secondly we consider the case (2). Let $x = [t]$. Since f and g are contractions and $g^\nu(t) \sim t$, there is a positive integer μ such that $g^\nu(t) = f^\mu(t)$. Note that $f^{-\mu} \circ g^\nu$ is defined at least on $[0, t]$. Since $g^\nu(f^k(t)) = f^k(g^\nu(t)) = f^k(f^\mu(t)) = f^\mu(f^k(t))$, there is a sequence $t, f(t), f^2(t), \cdots$ of fixed points of $f^{-\mu} \circ g^\nu$ which converges to 0. Therefore by Lemma 3, the germ at P_* of $G(\omega_1^{-\mu} \# \omega_2^\nu)$ is the identity. This completes the proof of Lemma 4.

If the one-sided holonomy group $\Phi_+(F_0)$ is a free abelian group of rank $\geqq 2$, there are $G(\omega_1)_{P_*}$ and $G(\omega_2)_{P_*}$ which do not admit (2) of Lemma 4. Then (2) of Theorem 1 occurs.

The rest is the case where (1) of Theorem 1 does not occur and $\Phi_+(F_0)$ is an infinite cyclic group. Let $j: \Phi_+(F_0) \to Z$ be the isomorphism which maps contractions to positive integers. Since $\Phi_+(F_0)$ is abelian, there is an epimorphism $\alpha: H_1(F_0, Z) \to \Phi_+(F_0)$ such that $\alpha \circ h = r \circ \bar{H} \circ i$ where $h: \pi_1(F_0, x_*) \to H_1(F_0, Z)$ is the Hurewicz homomorphism. Thus we have a diagram

$$\begin{array}{ccccccc} \pi_1(F_0, x_*) & \xrightarrow[\cong]{i} & \pi_1(S, \Omega_*) & \xrightarrow{r \cdot \bar{H}} & \Phi_+(F_0) & \xrightarrow[\cong]{j} & Z . \\ \Big\downarrow h & & & \nearrow \alpha & & & \\ H_1(F_0, Z) & & & & & & \end{array}$$

In the exact sequence

$$0 \to \operatorname{Ext}(H_0(F_0, Z), Z) \to H^1(F_0, Z) \xrightarrow{\beta} \operatorname{Hom}(H_1(F_0, Z), Z) \to 0$$

β is an isomorphism since $H_0(F_0, Z)$ is free and $\operatorname{Ext}(H_0(F_0, Z), Z)$ is trivial. Let $\theta = d \circ \beta^{-1}(j \circ \alpha) \in H_{n-2}(F_0, Z)$ where $d: H^1(F_0, Z) \to H_{n-2}(F_0, Z)$ is the Poincaré duality isomorphism.

Now we need the theorem of Nakatsuka which is on the line of Thom's representation theorem of codimension-one integral homology classes [12].

THEOREM (Nakatsuka [6]). *Let M be a compact connected orientable manifold of dimension $n \geqq 3$ and $\theta \in H_{n-1}(M, Z)$. Then there is a connected orientable $(n-1)$-submanifold $N \subset M$ such that $\theta = [N]$ if and only if there is a homology class $\alpha \in H_1(M, Z)$ such that the intersection number $\theta \cdot \alpha$ is* 1.

Since $j \circ \alpha \in \operatorname{Hom}(H_1(F_0, Z), Z)$ is an epimorphism, θ satisfies the condition of the theorem of Nakatsuka. Thus we have a connected oriented submanifold $N \subset F_0$ such that $\theta = [N] \in H_{n-2}(F_0, Z)$.

Retriangulate F_0 so that N is a subcomplex. We use the same notation as 2.2. Let $S' = \{\Omega_\lambda \mid \Omega_\lambda \cap N \neq \emptyset\}$ and $E = \bigcup \{A_\lambda \mid \Omega_\lambda \in S'\}$. Clearly E is homotopy equivalent to N. We may assume that $\Omega_* \in S'$ and $x_* \in N$. Let $\omega = (\Omega_0, \cdots, \Omega_k)$ be a closed S' chain at Ω_*, that is $\Omega_0 = \Omega_k = \Omega_*$ and all $\Omega_i \in S'$. There is a closed path $c: ([0, 1], \{0, 1\}) \to (N, x_*)$ contained by ω. Since $j \circ \alpha \circ h([c])$ is the intersection number of $h([c])$ and θ in F_0 and N has a trivial normal bundle in F_0, $j \circ \alpha \circ h([c]) = 0$. Then

$$\begin{aligned} H(\omega)_0 &= r \circ \bar{H} \circ i([c]) \\ &= \alpha \circ h([c]) \\ &= j^{-1} \circ j \circ \alpha \circ h([c]) \\ &= j^{-1}(0) \\ &= \text{the identity} . \end{aligned}$$

Let Λ'_{2m} be the set of all closed S' chains at Ω_* of length $\leqq$2m. For each $\omega \in \Lambda'_{2m}$, there is a neighborhood D_ω of P_* in $\Omega_*^+/\mathscr{F}$ such that $G(\omega) \mid D_\omega$ is the identity. Let $D = \bigcap_{\omega \in \Lambda'_{2m}} D_\omega$. By the same argument as the proof of Lemma 2, we can show that $F = \bigcup_{\omega \in J} G(\omega)(P)$ is a compact leaf which is diffeomorphic to E for each $P \in D$ where J is the set of all S' chains $\omega = (\Omega_0, \cdots, \Omega_k)$ satisfying that $k \leqq m$ and $\Omega_0 = \Omega_*$. Furthermore there is a C^r diffeomorphism $\xi: E \times [0, 1] \to K = \bigcup_{\omega \in J} G(\omega)(D)$ such that

(1) $\xi \mid E \times \{0\}$ is the identity,

(2) $\xi(\{x\} \times [0, 1]) \subset \varphi(\{x\} \times R)$ for all $x \in E$,

(3) $\xi(E \times \{t\})$ is a leaf of $\mathscr{F} \mid K$ for all $t \in [0, 1]$.

Since for all closed path c in $F_0 - \operatorname{Int}(E)$ the intersection number of c and θ is 0, there is a C^r imbedding $\eta: (F_0 - \operatorname{Int}(E)) \times [0, 1] \to U_+$ such that

(1) $\eta \mid (F_0 - \operatorname{Int}(E)) \times \{0\}$ is the identity,

(2) $\eta(\{x\} \times [0, 1]) \subset \varphi(\{x\} \times R)$ for all $x \in F_0 - \operatorname{Int}(E)$,

(3) $\eta((F_0 - \operatorname{Int}(E)) \times \{t\})$ is a leaf of U_+ for all $t \in [0, 1]$.

By using ξ, we can extend η to a C^r imbedding $\bar{\eta}: F_* \to U_+$ satisfying (1), (2) and (3) of η when $F_0 - \operatorname{Int}(E)$ replaced by F_*, where F_* is defined in (3) of Theorem 1. We denote by $x_i \in N_i$ $(i = 1, 2)$ the copies of $x \in N$. We may assume that $\bar{\eta}(N_1 \times [0, 1]) \subset \bar{\eta}(N_2 \times [0, 1])$. We define $f_x(t) \in [0, 1]$ by $\bar{\eta}(x_1, t) = \bar{\eta}(x_2, f_x(t))$ for all $t \in [0, 1]$. f_x does not depend on $x \in N$. Let $f = f_x$. It is clear that f is a C^r contraction. This completes the proof of Theorem 1.

4. The proof of Theorem 2 and examples. At first we give a method to construct a C^r foliation $\mathscr{F}$ on $S_g \times [0, 1]$, whose leaves are all C^∞ submanifold and transverse to $\{x\} \times [0, 1]$ for all $x \in S_g$, from given orientation-preserving C^r diffeomorphisms $f_1, \cdots, f_g$: $[0, 1] \to [0, 1]$, $0 \leqq r \leqq \infty$.

Take disjoint circles $C_1, \cdots, C_g$ in S_g such that $S_g - (C_1 \cup \cdots \cup C_g)$ is connected. Let $U_1, \cdots, U_g$ be disjoint closed tubular neighborhoods of $C_1, \cdots, C_g$. There are C^∞ diffeomorphisms h_i: $C_i \times [-1, 1] \to U_i$, $i = 1, \cdots, g$. Let α: $[-1, 1] \to [0, 1]$ be a C^∞ map such that

(1) $\alpha(t) = 0$ in a neighborhood of -1,

(2) $\alpha(t) = 1$ in a neighborhood of 1.

Let $\mathscr{F}_i$ be the foliation on $U_i \times [0, 1]$ whose leaves are

$$\{(h(x, s), t) \mid x \in C_i,\ s \in [-1, 1],\ t = \alpha(s)t_0 + (1 - \alpha(s))f_i(t_0)\}\ ,$$

$t_0 \in [-1, 1]$. Let $\mathscr{F}_0$ be the foliation on $(S_g - \bigcup_{i=1} \operatorname{Int}(U_i)) \times [0, 1]$ whose leaves are $(S_g - \bigcup_{i=1} \operatorname{Int}(U_i)) \times \{t\}$, $t \in [0, 1]$. By connecting $\mathscr{F}_0, \mathscr{F}_1, \cdots, \mathscr{F}_g$ together, we have a foliation $\mathscr{F}$ on $S_g \times [0, 1]$ which clearly has the desired property. Furthermore Image (Φ) is the subgroup of $\operatorname{Diff}_0^r([0, 1])$ generated by $f_1, \cdots, f_g$.

PROOF OF THEOREM 2. Choose numbers $a_1, \cdots, a_g, b_1, \cdots, b_g$ such that

$$0 = a_g < a_{g-1} < \cdots < a_2 < a_1 < b_1 < b_2 < \cdots < b_g = 1\ .$$

By induction we construct homeomorphisms $f_1, \cdots, f_g$: $[0, 1] \to [0, 1]$ as follows.

(1) Take $f_1 \mid [a_1, b_1]$ such that

$$f_1(a_1) = a_1,\ f_1(b_1) = b_1,\ f_1(t) < t \qquad \text{for all } t \in (a_1, b_1)\ .$$

(2) Suppose that $f_1, \cdots, f_k$ are already defined on $[a_k, b_k]$. Take $f_{k+1} \mid [a_{k+1}, b_{k+1}]$ such that

$$f_{k+1}(a_{k+1}) = a_{k+1},\ f_{k+1}(b_{k+1}) = b_{k+1},\ f_{k+1}(b_k) = a_k,\ f_{k+1}(t) < t \quad \text{for all } t \in (a_{k+1}, b_{k+1}) .$$

For each $t \in (a_{k+1}, b_{k+1}) - [a_k, b_k]$, there is the unique integer ν satisfying $f_{k+1}^{\nu}(t) \in (a_k, b_k]$. Let $f_i(t) = f_{k+1}^{-\nu} \circ f_i \circ f_{k+1}^{\nu}(t)$ and $f_i(a_{k+1}) = a_{k+1}$, $f_i(b_{k+1}) = b_{k+1}$ for $i = 1, \cdots, k$.

Note that $f_1, \cdots, f_g$ are mutually commutative.

Let $\mathscr{F}$ be the foliation on $S_g \times [0, 1]$ obtained by using $f_1, \cdots, f_g$ and F_t the leaf containing $(S_g - \bigcup_{i=1} \mathrm{Int}\,(U_i)) \times \{t\}$. Image (Φ) is a free abelian group of rank g. By considering the action of $f_1, \cdots, f_g$, we see that

(1) $F_{a_g} = S_g \times \{0\}$, $F_{b_g} = S_g \times \{1\}$, $F_{a_{g-1}} = F_{b_{g-1}}, \cdots, F_{a_1} = F_{b_1}$, and F_t, $t \in (a_1, b_1)$, are all leaves of $\mathscr{F}$.

(2) $\bar{F}_{a_{g-1}} = F_{a_{g-1}} \cup F_{a_g} \cup F_{b_g}$,

(3) $\bar{F}_{a_{g-2}} = F_{a_{g-2}} \cup F_{a_{g-1}} \cup F_{a_g} \cup F_{b_g}$,

. . . .

(g) $\bar{F}_{a_1} = F_{a_1} \cup F_{a_2} \cup \cdots \cup F_{a_{g-1}} \cup F_{a_g} \cup F_{b_g}$,

(g + 1) $\bar{F}_t = F_t \cup \bar{F}_{a_1}$ for all $t \in (a_1, b_1)$,

(g + 2) F_t is non-compact and proper for all $t \in (0, 1)$.

This completes the proof of Theorem 2.

REMARK. Although $f_i \mid [a_i, b_i]$ can be taken C^∞, the differentiability may be broken at a_{i+1} and b_{i+1}.

EXAMPLE 1.*) Let the genus $g = 2$. Let $f_1, f_2\colon [0, 1] \to [0, 1]$ be C^∞ diffeomorphisms such that

$$f_1(t) = t/2 \quad \text{and} \quad f_2(t) = \left(\frac{1}{2}\right)^{\alpha} t \qquad \text{for all } t \in [0, 1/2]$$

where α is a positive irrational number $\in (0, 1)$. Let $\mathscr{F}$ be the foliation on $S_2 \times [0, 1]$ obtained by using f_1 and f_2. In a neighborhood of the compact leaf $S_g \times \{0\}$, (2) of Theorem 1 occurs.

REMARK. This foliation may be not an example of (1) of Theorem 1*. For the case (1) of Theorem 1* we need $f \in \mathscr{S}$ such that $\mathscr{C}(f)$ is isomorphic to R^1 where the notations $\mathscr{S}$ and $\mathscr{C}(f)$ are in Lemma 3 of Kopell [4]. We do not know whether such f exists or not.

EXAMPLE 2 (Sacksteder [9]). There are orientation-preserving C^∞ diffeomorphisms $f_1, f_2\colon [0, 1] \to [0, 1]$ such that the C^∞ foliation on $S_2 \times [0, 1]$

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*) The author thanks Professor Y. Saito for his correction.

obtained by using f_1 and f_2 has an exceptional leaf F (that is, neither locally dense nor proper). Of course f_1 and f_2 are not commutative and Image (Φ) is a non-commutative group.

EXAMPLE 3. Denjoy [1] constructed a C^1 diffeomorphism $g: S^1 \to S^1$ such that

(1) $\{g^\nu(x) \mid \nu \in Z\}$ is neither locally dense nor compact for all $x \in S^1$,

(2) $x \in \bigcap_{i=1} \overline{\{g^\nu(x) \mid |\nu| \geqq n\}}$ for some $x \in S^1$,

(3) $x \notin \bigcap_{i=1} \overline{\{g^\nu(x) \mid |\nu| \geqq n\}}$ for some $x \in S^1$.

Let $p: R \to S^1$ be the universal covering and $f: R \to R$ a generator of the covering transformation group of $p: R \to S^1$. Since $(g \circ p)_\# \pi_1(R) = \{1\} \subset p_\# \pi_1(R) = \{1\}$, there is a C^1 diffeomorphism $\tilde{g}: R \to R$ such that $p \circ \tilde{g} = g \circ p$ and $\tilde{g}(0) = \operatorname{Min}\{t \in p^{-1}(g \circ p(0)) \mid 0 < t\}$.

Then $\tilde{g}(s) = \operatorname{Min}\{t \in p^{-1}(g \circ p(s)) \mid s < t\}$ for all $s \in R$. Really let $A = \{s \in R \mid \tilde{g}(s) = \operatorname{Min}\{t \in p^{-1}(g \circ p(s)) \mid s < t\}\}$. Let $s \in R$, then there is a connected neighborhood U of $p(s)$ satisfying $U \cap g(U) = \emptyset$. Let $\tilde{U}$ be the connected component, of $p^{-1}(U)$, containing s. Then $\tilde{g}(\tilde{U})$ is the connected component, of $p^{-1}(g(U))$, containing $\tilde{g}(s)$. If $s \in A$, then $[s, \tilde{g}(s)] \cap (p^{-1}(g(U)) - \tilde{g}(\tilde{U})) = \emptyset$ and $\tilde{U} \subset A$. Therefore A is open. If $s \in R - A$, then $[s, g(s)]$ contains a connected component of $p^{-1}(g(U)) - \tilde{g}(\tilde{U})$ and $\tilde{U} \subset R - A$. Therefore A is closed. Since $0 \in A$, A is a non-empty closed open subset of R. Therefore $A = R$.

Furthermore $f \circ \tilde{g} = \tilde{g} \circ f$. In fact suppose that $f \circ \tilde{g}(s) < \tilde{g} \circ f(s)$ for some $s \in R$. Since $s < \tilde{g}(s)$ and f preserves the relation $<$, $f(s) < f \circ \tilde{g}(s) < \tilde{g}(f(s))$. Since

$$p \circ f \circ \tilde{g}(s) = p \circ \tilde{g}(s) = g \circ p(s) = g \circ p \circ f(s) \,, \quad f \circ \tilde{g}(s) \in p^{-1}(g \circ p(f(s))) \,.$$

This contradicts to the fact that $\tilde{g}(f(s)) = \operatorname{Min}\{t \in p^{-1}(g \circ p(f(s))) \mid f(s) < t\}$. Suppose that $f \circ \tilde{g}(s) > \tilde{g} \circ f(s)$ for some $s \in R$. Since $f(s) < \tilde{g} \circ f(s)$, $s < f^{-1} \circ \tilde{g} \circ f(s) < f^{-1} \circ f \circ \tilde{g}(s) = \tilde{g}(s)$. Since $f^{-1} \circ \tilde{g} \circ f(s) \in p^{-1}(g \circ p(s))$, this is a contradiction.

Choose a homeomorphism $\alpha: (0, 1) \to R$ and let $f_1, f_2: [0, 1] \to [0, 1]$ be the homeomorphisms defined by

(1) $f_1(0) = f_2(0) = 0$, $f_1(1) = f_2(1) = 1$,

(2) $f_1(s) = \alpha^{-1} \circ f \circ \alpha(s)$, $f_2(s) = \alpha^{-1} \circ \tilde{g} \circ \alpha(s)$ for all $s \in (0, 1)$.

The topological foliation on $S_2 \times [0, 1]$, obtained by using f_1 and f_2, has exceptional leaves. Image $(\Phi) \cong Z \oplus Z$.

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