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ON SOME 3-DIMENSIONAL COMPLETE RIEMANNIAN MANIFOLDS SATISFYING $R(X, Y)\cdot R = 0$

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1. Introduction. Let (M, g) be a Riemannian manifold. By R we denote the Riemannian curvature tensor. By $T_x(M)$ and Exp_x we denote the tangent space to M at x and the exponential mapping of (M, g) at x. For $X, Y \in T_x(M)$, $R(X, Y)$ operates on the tensor algebra as a derivation at each point $x \in M$. In a locally symmetric space ($\nabla R = 0$), we have

$$(*) \qquad R(X, Y)\cdot R = 0 \text{ for any point } x \in M \text{ and } X,\ Y \in T_x(M)\ .$$

We consider the converse under some additional conditions.

THEOREM A (S. Tanno [8]). *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold. If (M, g) satisfies $(*)$ and the scalar curvature S is positive and bounded away from 0 on M, then (M, g) is of constant curvature.*

Other results concerning this problem may be found in references. In this paper, we shall prove

THEOREM B. *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold satisfying $(*)$. If the volume of (M, g) is finite, then (M, g) is of constant curvature, and hence, $\nabla R = 0$.*

COROLLARY B. *Let (M, g) be a compact and irreducible 3-dimensional Riemannian manifold satisfying $(*)$. Then (M, g) is of constant curvature.*

It may be noticed that $(*)$ implies in particular

$$(**) \qquad R(X, Y)\cdot R_1 = 0\ ,$$

where R_1 denotes the Ricci tensor of (M, g).

In this paper, (M, g) is assumed to be connected, complete and of class C^∞ unless otherwise specified.

2. Preliminaries. Let (M, g) be a 3-dimensional Riemannian manifold. Assume $(*)$. $\dim M = 3$ implies

$$(2.1) \qquad R(X, Y) = R^1X \wedge Y + X \wedge R^1Y - (S/2)X \wedge Y\ ,$$

where

$$g(R^1X, Y) = R_1(X, Y) \quad \text{and} \quad (X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y .$$

Let (K_1, K_2, K_3) be eigenvalues of the Ricci transformation R^1 at a point x. Then $(*)$ is equivalent to

$$(K_i - K_j)(2(K_i + K_j) - S) = 0 . \tag{2.2}$$

Therefore we may have only three cases:

$$(K, K, K) , \quad (K, K, 0) , \quad (0, 0, 0) \quad \text{at each point.}$$

First, if (K, K, K), $K \neq 0$, holds at some point x, then it holds on some open neighborhood U of x. Hence U is an Einstein space, and K is constant on U and on M. Therefore (M, g) is of constant curvature (cf. H. Takagi and K. Sekigawa [6]). From now, we assume that rank $R^1 \leqq 2$ on M. Let $W = \{x \in M; \text{rank } R^1 = 2 \text{ at } x\}$. By W_0 we denote one component of W. On W_0, we have two C^∞-distributions T_1 and T_0 such that

$$T_1 = \{X; R^1X = KX\} ,$$
$$T_0 = \{Z; R^1Z = 0\} .$$

For X, $Y \in T_1$ and $Z \in T_0$, by (2.1), we have

$$R(X, Y) = KX \wedge Y , \quad R(X, Z) = 0 . \tag{2.3}$$

This shows that T_0 is the nullity distribution. Since the index of nullity at each point of M is 1 or 3, the nullity index of (M, g) is 1. Thus integral curves of T_0 are geodesics (and complete if (M, g) is complete) (cf. Y. H. Clifton and R. Maltz [2], etc.). Let $(E_1, E_2, E_3) = (E)$ be a local field of orthonormal frame such that $E_3 \in T_0$ (consequently, $E_1, E_2 \in T_1$) and

$$\nabla_{E_3}E_i = 0 , \qquad i = 1, 2, 3 .$$

We call this (E) an adapted frame field. If we put $\nabla_{E_i}E_j = \sum_{k=1}^3 B_{i\,jk}E_k$, then we get $B_{i\,jk} = -B_{i\,kj}$ and

$$B_{3\,ij} = 0 , \qquad i, j = 1, 2, 3 . \tag{2.4}$$

The second Bianchi identity and (2.3) give

$$E_3K + K(B_{1\,31} + B_{2\,32}) = 0 , \quad \text{or} \quad \operatorname{div} E_3 = -E_3K/K . \tag{2.5}$$

By (2.4) and $R(E_i, E_3)E_3 = \nabla_{E_i}\nabla_{E_3}E_3 - \nabla_{E_3}\nabla_{E_i}E_3 - \nabla_{[E_i, E_3]}E_3 = 0$, we get

$$
\begin{aligned}
&E_3B_{1\,31} + (B_{1\,31})^2 + B_{1\,32}B_{2\,31} = 0 \,, \\
&E_3B_{1\,32} + B_{1\,31}B_{1\,32} + B_{1\,32}B_{2\,32} = 0 \,, \\
&E_3B_{2\,31} + B_{2\,31}B_{1\,31} + B_{2\,32}B_{2\,31} = 0 \,, \\
&E_3B_{2\,32} + (B_{2\,32})^2 + B_{2\,31}B_{1\,32} = 0 \,.
\end{aligned} \tag{2.6}
$$

(2.5) and $(2.6)_2$, (2.5) and $(2.6)_3$, (2.5) and $(2.6)_{1,4}$ imply

$$B_{1\,32} = C_1(E)K \,, \qquad B_{2\,31} = C_2(E)K \,, \tag{2.7}$$

$$B_{1\,31} - B_{2\,32} = D(E)K \,, \tag{2.8}$$

where $C_1(E)$, $C_2(E)$ and $D(E)$ are functions defined on the same domain as (E) such that $E_3C_1(E) = E_3C_2(E) = E_3D(E) = 0$.

By (2.5) and (2.8), we get

$$2B_{1\,31} = D(E)K - E_3K/K \,. \tag{2.9}$$

Now, let $\gamma_x(s)$ be an integral curve of T_0 through $x = \gamma_x(0) \in W_0$ with arc-length parameter s, i.e., $\gamma_x(s) = \mathrm{Exp}_x s(E_3)_x$. Then $(2.6)_1$, (2.7) and (2.9) give

$$\frac{1}{2}\frac{d}{ds}\Big(\frac{1}{K}\frac{dK}{ds}\Big) = HK^2 + \frac{1}{4}\Big(\frac{1}{K}\frac{dK}{ds}\Big)^2 \,, \quad \text{along} \quad \gamma_x(s) \,, \tag{2.10}$$

where

$$H = H(E) = D(E)^2/4 + C_1(E)C_2(E) \,.$$

(2.10) implies that H is independent of the choice of the adapted frame fields (E). Solving (2.10), we get

$$K = \gamma \,, \quad (\text{for } H = 0) \,, \quad \text{or} \tag{2.11}$$

$$K = \pm 1/((\alpha s - \beta)^2 - H/\alpha^2) \,, \quad (\text{for } H \neq 0) \,, \tag{2.12}$$

where α, β and γ are constant along $\gamma_x(s)$, $\alpha \neq 0$.

With respect to our arguments, without loss of essentiality, we may assume that M is orientable. Let (E) be any adapted frame field which is compatible with the orientation. We call it an oriented adapted frame field. Then we see that $f = (C_1(E) - C_2(E))K$ is independent of the choice of oriented adapted frame fields, and hence f is a function of class C^∞ on W_0. $f = 0$ holds on an open set $U \subset W_0$, if and only if T_1 is integrable on U. This is a geometric meaning of f. In the sequel, we assume that the volume of (M, g) is finite. We can see that $H = H(E) = D(E)^2/4 + C_1(E)C_2(E)$ is a function of class C^∞ on W_0. Let $W(H) = \{x \in W_0; H \neq 0$ at $x\}$. We assume that $W(H) \neq \varnothing$. Let $W(H)_0$ be one component of $W(H)$. By (2.12) and completeness of (M, g), H must be negative on $W(H)_0$. For each point $x \in W(H)_0$, consider $\gamma_x(s)$. Then $\gamma_x(s) \in W(H)_0$, for all s. Let $x_0 = \gamma_x(\beta/\alpha)$. For $(E_1)_{x_0}, (E_2)_{x_0} \in T_1(x_0)$, there exists a 2-dimensional submanifold, $\{\varphi(u_1, u_2) \in W(H)_0;\ (u_1, u_2) \in (-\varepsilon, \varepsilon)^2, \varepsilon > 0\}$, such that $\varphi(0, 0) = x_0$ and $(\partial\varphi/\partial u_1)(0, 0) = (E_1)_{x_0}$, $(\partial\varphi/\partial u_2)(0, 0) = (E_2)_{x_0}$. Now, we define a mapping

$$\Phi: (-\varepsilon, \varepsilon)^2 \times (-\delta, \delta) \rightarrow W(H)_0 \quad \text{by}$$

$$\Phi(u_1, u_2, w_3) = \mathrm{Exp}_{\varphi(u_1,u_2)}\, w_3E_3, \text{ for some } \delta > 0 . \tag{2.13}$$

Then Φ is of class C^∞ and furthermore, for small ε, δ, $V(\varepsilon, \delta) = \{\Phi(u_1, u_2, w_3) \in W(H)_0;\ (u_1, u_2, w_3) \in (-\varepsilon, \varepsilon)^2 \times (-\delta, \delta)\}$ is a local coordinate neighborhood with origin at x_0. In $V(\varepsilon, \delta)$, by (2.12), we get

$$K = \pm 1/((Aw_3 - B)^2 - H/A^2) , \tag{2.14}$$

where A and B are functions of class C^∞ on $V(\varepsilon, \delta)$ such that $\partial A/\partial w_3 = \partial B/\partial w_3 = 0$ on $V(\varepsilon, \delta)$ and $A = \alpha$, $B = 0$ at x_0.

By continuity of A and B in (2.14), there is ε_0, $0 < \varepsilon_0 < \varepsilon$ such that $-\delta/4 < B/A < \delta/4$, for $(u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2$.

Now, we define a mapping $\psi: (-\varepsilon_0, \varepsilon_0)^2 \rightarrow V(\varepsilon, \delta)$ by

$$\psi(u_1, u_2) = \mathrm{Exp}_{\varphi(u_1, u_2)}\, (B(u_1, u_2)/A(u_1, u_2))E_3 . \tag{2.15}$$

And furthermore, we define a mapping $\Psi: (-\varepsilon_0, \varepsilon_0)^2 \times (-\delta_0, \delta_0) \rightarrow V(\varepsilon, \delta)$ by

$$\Psi(u_1, u_2, u_3) = \mathrm{Exp}_{\psi(u_1,u_2)}\, u_3E_3 , \qquad \delta_0 = \delta/4 . \tag{2.16}$$

Then Ψ is of class C^∞ and

$$U(\varepsilon_0, \delta_0) = \{\Psi(u_1, u_2, u_3) \in V(\varepsilon, \delta);\ (u_1, u_2, u_3) \in (-\varepsilon_0, \varepsilon_0)^2 \times (-\delta_0, \delta_0)\}$$

is a local coordinate neighborhood with origin at x_0.

Between w_3 in $V(\varepsilon, \delta)$ and u_3 in $U(\varepsilon_0, \delta_0)$, the following relation holds:

$$w_3 = u_3 + B/A , \qquad \text{in } U(\varepsilon_0, \delta_0) . \tag{2.17}$$

Thus (2.14) and (2.17) imply

$$K = \pm 1/((Au_3)^2 - H/A^2) , \qquad \text{on } U(\varepsilon_0, \delta_0) . \tag{2.18}$$

Let $\gamma(u_1, u_2)$ be the integral curve of T_0 starting from $\psi(u_1, u_2)$, $(u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2$, i.e., $\gamma(u_1, u_2)(s) = \mathrm{Exp}_{\psi(u_1,u_2)}\, sE_3$. Then, in $U(\varepsilon_0, \delta_0)$, u_3 can be considered as the arc-length parameter of $\gamma(u_1, u_2)$. We put $L(u_1, u_2) = \{\gamma(u_1, u_2)(s) \in M;\ -\infty < s < \infty\}$. Since $\dim T_0 = 1$, taking account of (2.12) and (2.18), we can see that $\gamma(u_1, u_2)(s_1) \neq \gamma(u_1, u_2)(s_2)$ for $s_1 \neq s_2$. From (2.12) and (2.18), $dK/ds = 0$ for $s = 0$ and otherwise $dK/ds \neq 0$ along $L(u_1, u_2)$, for any $(u_1, u_2) \in (-\varepsilon_0, \varepsilon_0)^2$. Thus, we can see that if $(u_1, u_2) \neq (v_1, v_2)$, (u_1, u_2), $(v_1, v_2) \in (-\varepsilon_0, \varepsilon_0)^2$, then $L(u_1, u_2) \cap L(v_1, v_2) = \emptyset$.

Now, we put

$$U(\varepsilon_0)=\{\hat{\Psi}(u_1, u_2, u_3)\in M; (u_1, u_2)\in(-\varepsilon_0, \varepsilon_0)^2, -\infty<u_3<\infty\},$$

where $\hat{\Psi}$ denotes an extension of Ψ defined by

$$\hat{\Psi}(u_1, u_2, u_3)=\mathrm{Exp}_{\Psi(u_1,u_2)}u_3E_3, \quad \text{on} \quad (-\varepsilon_0, \varepsilon_0)^2\times(-\infty, \infty).$$

Then, from the above arguments, we have the following

LEMMA 2.1. *$U(\varepsilon_0)$ is a local coordinate neighborhood with origin at x_0.*

For any $G>0$, we put

$$V_G=\{\hat{\Psi}(u_1, u_2, u_3)\in U(\varepsilon_0); (u_1, u_2)\in(-\varepsilon_0/2, \varepsilon_0/2)^2, 0<u_3<G\}.$$

Then $\bar{V}_G\subset U(\varepsilon_0)$. Let $\mathrm{vol}(M, g)$ and $\mathrm{vol}(V_G)$ denote the volumes of (M, g) and the open subspace V_G of (M, g), respectively. Then, by the assumption, we have

$$\mathrm{vol}(V_G)<\mathrm{vol}(M, g)<\infty, \quad \text{for any} \quad G>0. \tag{2.19}$$

On the other hand, since $E_3=\partial/\partial u_3$ on $U(\varepsilon_0)$, we have

$$\mathrm{div}\, E_3=(1/\sqrt{g_0})(\partial\sqrt{g_0}/\partial u_3) \quad \text{on} \quad U(\varepsilon_0),$$

where

$$g_0=\det(g_{ij}), \quad g_{ij}=g(\partial/\partial u_i, \partial/\partial u_j), \qquad i, j=1, 2, 3.$$

Thus, by (2.5), we get

$$(1/\sqrt{g_0})(\partial\sqrt{g_0}/\partial u_3)+(1/K)(\partial K/\partial u_3)=0 \quad \text{on} \quad U(\varepsilon_0). \tag{2.20}$$

Solving (2.20), we get

$$\sqrt{g_0}=C/K, \tag{2.21}$$

where $C=C(u_1, u_2)$ is a function of class C^∞ on $U(\varepsilon_0)$.

Thus, from (2.18) and (2.21), we get

$$\begin{aligned}\mathrm{vol}(V_G)&=\int_{V_G}dM=\int_{-\varepsilon_0/2}^{\varepsilon_0/2}\int_{-\varepsilon_0/2}^{\varepsilon_0/2}\int_0^G(C/K)du_1du_2du_3\\&\geqq a(\varepsilon_0)^2G, \quad \text{for any} \quad G>0,\end{aligned}$$

where

$$a=\min_{\substack{-\varepsilon_0/2\leqq u_1,\, u_2\leqq\varepsilon_0/2\\ u_3=0}} C/K>0.$$

But, this contradicts (2.19). Thus we have the following

LEMMA 2.2. *If $\mathrm{vol}(M, g)$ is finite, then, for each point $x\in W$, $S=2K$ is constant along $\gamma_x(s)$, $-\infty<s<\infty$.*

3. Proof of Theorem B. In the sequel, we shall assume that $\mathrm{vol}(M, g)$ is finite and rank R^1 is at most 2 on M and rank $R^1 = 2$ at some point of M. From Lemma 2.2, $H = 0$ on W_0. Let $V = \{x \in W_0; f(x) \neq 0\}$. Now, we assume that $V \neq \emptyset$. Let V_0 be one component of V. $H = H(E) = 0$ implies $D(E)^2 = -4C_1(E)C_2(E)$. Put $\cos 2\theta(E) = K(C_1(E) + C_2(E))/f$ and $\sin 2\theta(E) = KD(E)/f$. Define (E^*) by $E_3^* = E_3$ and

$$E_1^* = \cos\theta(E)E_1 - \sin\theta(E)E_2 \,,$$
$$E_2^* = \sin\theta(E)E_1 + \cos\theta(E)E_2 \,.$$

Then we have $D(E^*) = 0$. Furthermore, for (E) and (E'), we have $E_1^*(E) = \pm E_1^*(E')$ and $E_2^*(E) = \pm E_2^*(E')$. $H = 0$ and $D(E^*) = 0$ imply $C_1(E^*)C_2(E^*) = 0$. So we can assume that $C_2(E^*) = 0$ (otherwise, change $(E_1^*, E_2^*, E_3^*) \to (E_2^*, -E_1^*, E_3^*)$). Then we get

$$B^*_{1\,32} \neq 0 \,, \qquad B^*_{1\,31} = B^*_{2\,31} = B^*_{2\,32} = 0 \,, \tag{3.1}$$

where

$$\nabla_{E_i^*} E_j^* = \sum_{k=1}^{3} B^*_{i\,jk} E_k^* \,.$$

$R(E_1^*, E_3^*)E_2^* = 0$ implies

$$E_3^* B^*_{1\,21} = 0 \,. \tag{3.2}$$

$R(E_1^*, E_2^*)E_3^* = 0$ implies $B^*_{2\,21} = 0$ and

$$E_2^* B^*_{1\,32} + B^*_{1\,21} B^*_{1\,32} = 0 \,. \tag{3.3}$$

$R(E_1^*, E_2^*)E_1^* = -KE_2^*$ implies

$$E_2^* B^*_{1\,21} + (B^*_{1\,21})^2 = -K \,. \tag{3.4}$$

By $B^*_{2\,ij} = 0$, each trajectory of E_2^* is a geodesic. Put $h = B^*_{1\,21}$ and $F = (E_1^* f)^2$. Then F is a function of class C^∞ on V_0. From Lemma 2.2, and (3.3), we get

$$\begin{aligned} E_3^*(E_1^* f) &= E_1^*(E_3^* f) + [E_3^*, E_1^*]f \\ &= -B^*_{1\,32}(E_2^* f) = f^2 h \,, \qquad \text{i.e.}\,, \end{aligned}$$

$$d(E_1^* f)/ds = f^2 h \,, \quad \text{along} \quad \gamma_x(s) \,, \quad x \in V_0 \,. \tag{3.5}$$

From Lemma 2.2, for each point $x \in V_0$, $\gamma_x(s) \in V_0$, $-\infty < s < \infty$. Taking account of (3.2) and solving (3.5), we get

$$F = (f(x)^2 h(x) s + c)^2 \,, \quad \text{along} \quad \gamma_x(s) \,, \quad -\infty < s < \infty \,, \tag{3.6}$$

where c is constant along $\gamma_x(s)$.

Let $V^*=\{x\in V_0; h(x)\neq 0\}$. From (3.4), we see that $V^*\neq\emptyset$. Let V_0^* be one component of V^*. Then, by (3.2), we see that, for each point $x\in V_0^*$, $\gamma_x(s)\in V_0^*$, $-\infty<s<\infty$. For each point $x\in V_0^*$, consider $\gamma_x(s)$. Let $x_0=\gamma_x(-c/f(x)^2h(x))$ in (3.6).

Then we have

$$F=((f^2h)w_3+k)^2\ , \quad \text{on} \quad V(\varepsilon,\delta)\cap V_0^*\ , \tag{3.7}$$

where $k=k(u_1, u_2)$ is a function of class C^∞ on $V(\varepsilon,\delta)\cap V_0^*$ such that $k(0,0)=0$, and $V(\varepsilon,\delta)$ is a local coordinate neighborhood with origin at x_0 constructed by the similar fashion as in § 2. From (3.7), by applying the similar arguments as in the proof of Lemma 2.1, to the function F instead of K, we can construct a local coordinate neighborhood

$$U(\varepsilon^*)=\{\Psi^*(u_1, u_2, u_3)\in V_0^*; (u_1, u_2)\in(-\varepsilon^*, \varepsilon^*)^2, -\infty<u_3<\infty\}\ ,$$

with origin at x_0 such that $F=((f^2h)u_3)^2$ on $U(\varepsilon^*)$, where $\varepsilon^*>0$, and Ψ^* is a mapping of class C^∞ defined by the similar way as $\hat{\Psi}$ in § 2. For any $G>0$, let $V_G^*=\{\Psi^*(u_1, u_2, u_3)\in U(\varepsilon^*); (u_1, u_2)\in(-\varepsilon^*/2, \varepsilon^*/2)^2, 0<u_3<G\}$. From Lemma 2.2, and (2.5), we have $\operatorname{div} E_3^*=0$. Thus, we can see that if $G\to\infty$, then $\operatorname{vol}(V_G^*)\to\infty$. But, this is a contradiction. Thus, we can conclude that $f=0$ on W_0 and hence T_1 is integrable on W_0. Thus, T_1 and T_0 are parallel on W_0(cf. S. Tanno [7]). If W is dense in M, the restricted homogeneous holonomy group of (M, g) is reducible. If W is not dense in M, then the interior of the complement of W in M is flat. Hence, also in this case, the restricted homogeneous holonomy group of (M, g) is reducible. Lastly, if $\operatorname{rank} R^1=0$ on M, then (M, g) is flat. Therefore, this completes a proof of Theorem B.

From our arguments in this paper, we can also show the following

THEOREM C. *Let (M, g) be a complete and simply connected 3-dimensional Riemannian manifold satisfying $(*)$. If the volume of (M, g) is finite, then (M, g) is isometric to a 3-dimensional sphere.*

REFERENCES

[1] K. ABE, A characterization of totally geodesic submanifolds in S^N and CP^N by an inequality, Tôhoku Math. J., 23 (1971), 219–244.

[2] Y. H. CLIFTON AND R. MALTZ, The K-nullity spaces of the curvature operator, Michigan Math. J., 17 (1970), 85–89.

[3] S. KOBAYASHI AND K. NOMIZU, Foundations of differential geometry, Vol. I, Wiley-Interscience, New York, 1963.

[4] A. ROSENTHAL, Riemannian manifolds of constant nullity, Michigan Math. J., 14 (1967), 469–480.

[5] K. SEKIGAWA, On some 3-dimensional Riemannian manifolds, Hokkaido Math. J., 2(1973), 259–270.

[6] H. TAKAGI AND K. SEKIGAWA, On some 3-dimensional Riemannian manifolds satisfying a certain condition on the curvature tensor, Sci. Rep. Niigata Univ., 7 (1969), 23-27.
[7] S. TANNO, A theorem on totally geodesic foliations and its applications, Tensor N. S., 24 (1972), 116-122.
[8] S. TANNO, 3-dimensional Riemannian manifolds satisfying $R(X, Y)\cdot R = 0$, to appear.

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