ON SOME TYPES OF ISOPARAMETRIC HYPERSURFACES IN SPHERES I

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1. Introduction. We shall exhibit two series of non-homogeneous isoparametric hypersurfaces in spheres in this paper, and then give a classification of some types of isoparametric hypersurfaces in a forthcoming paper.

We begin with a few definitions and notations to explain our results more precisely. Let $M$ be a Riemannian manifold with metric $(, )$. The induced inner product on cotangent vectors is also denoted by $(, )$. A differentiable function $f$ defined on an open set $U$ in $M$ is called isoparametric if $df \wedge d(df, df) = 0$ and $df \wedge d(\Delta f) = 0$, where $\Delta$ denotes the Laplacian on $M$. A hypersurface $M$ (a submanifold of codim 1) in $M$ is called isoparametric if, for each point $p$ of $M$, there exist an open neighborhood $U$ of $p$ in $M$ and an isoparametric function $f$ defined on $U$ such that $U \cap M = \{ q \in U \mid f(q) = f(p) \}$.

Let $\mathcal{S} = \{ M_t \mid t \in I \}$ be a family of hypersurfaces in $\bar{M}$ parametrized by an open interval $I$. $\mathcal{S}$ is called a family of isoparametric hypersurfaces if there exist an open set $U$ in $\bar{M}$ and an isoparametric function $f$ on $U$ such that $M_t = f^{-1}(t)$ for each $t \in I$. Two families $\mathcal{S} = \{ M_t \mid t \in I \}$ and $\mathcal{S}' = \{ M'_t \mid t' \in I' \}$ of isoparametric hypersurfaces in $\bar{M}$ are identified if there exists a diffeomorphism $\varphi$ of $I$ onto $I'$ such that $M_t = M'_{\varphi(t)}$ for each $t \in I$. Also, if we have an imbedding $\varphi$ of $I$ into $I'$ such that $M_t \subset M'_{\varphi(t)}$ for each $t \in I$, then we write $\mathcal{S} \subset \mathcal{S}'$.

Now, let $\bar{M} = S^{N-1}$ be the unit sphere in an $N$-dimensional Euclidean space $\mathbb{R}^N$ centered at the origin, and $M$ a locally closed hypersurface in $\bar{M}$. $M$ is said to be homogeneous if a suitable subgroup of $O(N)$ acts transitively on $M$ where $O(N)$ denotes the real orthogonal group of $\mathbb{R}^N$. It is known that $M$ is isoparametric if and only if $M$ has locally constant principal curvatures (Cartan [2]). Thus, every homogeneous hypersurface in $S^{N-1}$ is isoparametric. Two hypersurfaces $M$ and $M'$ in $S^{N-1}$ are said to be equivalent if a suitable orthogonal transformation of $\mathbb{R}^N$ transforms $M$ onto $M'$. Similarly, two families of isoparametric hypersurfaces in
$S^{n-1}$ are equivalent if a suitable orthogonal transformation of $R^n$ transforms one to the other.

The following results are due to Münzner [5]. For every connected isoparametric hypersurface $M$ in $S^{n-1}$, there exists a unique maximal (relative to the above order $\subset$) family $\mathcal{M} = \{M_t | t \in I\}$ of isoparametric hypersurfaces in $S^{n-1}$ such that each $M_t$ is closed in $S^{n-1}$ and for some $t$ $M_t$ is an open submanifold of $M$. If $M$ and $M'$ are equivalent, then $\mathcal{M}$ and $\mathcal{M}'$ are equivalent in our sense. Further the classification problem of such maximal families is reduced to an algebraic one in the following way. Let $F$ be a homogeneous polynomial function of degree $g$ on $R^n$. For $g > 2$, let $m_1$ and $m_2$ be positive such that $m_1 + m_2 + m_1 + m_2 + \cdots = N - 2$, and let $m_1 = N - 2 > 0$ for $g = 1$. Assume $F$ satisfies

\[
(M) \quad \begin{cases} 
(dF, dF) = g^{n-2} \\
\Delta F = cr^{g-2}
\end{cases}
\]

where $c = (1/2)(m_1 - m_2)g^2$ for $g \geq 2$ and $c = 0$ for $g = 1$ and where $r$ is the radius function and $\Delta$ is the Laplacian on $R^n$. Then the restriction $f$ of $F$ to $S^{n-1}$ is isoparametric on $S^{n-1}$, and $\mathcal{J} = \{M_t = f^{-1}(t) | t \in (-1, 1)\}$ is a maximal family of isoparametric hypersurfaces in $S^{n-1}$ such that each $M_t$ is connected and closed. Conversely, any maximal family of isoparametric hypersurfaces in $S^{n-1}$ is given in the above way. Such two families $\mathcal{J}$ and $\mathcal{J}'$, are equivalent if and only if there exists an element $\sigma$ in $O(N)$ such that

\[F(\sigma^{-1}x) = \pm F'(X) \quad x \in R^n.\]

In this case, $F$ and $F''$ are said to be equivalent. Münzner also has shown that the above (M) has a solution only if $g = 1, 2, 3, 4$ or $6$ and that $m_1 = m_2$ if $g = 3$.

Geometrically, the above integers $g, m$, and $m_2$ are related to each isoparametric hypersurface $M_t$ as follows. Consider the unit normal vector field $X_t = \text{grad } f / (df, df)^{1/2}$ for each $M_t$. Let

$k_1(t) > \cdots > k_{g(t)}(t)$

be the distinct principal curvatures of $M_t$ relative to $X_t$ and $m_j(t)$ the multiplicity of $k_j(t)$ for each $j$. Then $g(t)$ and $m_j(t)$ are constant, and we have

\[
\begin{align*}
g(t) &= g(t), \\
m_1 &= m_i(t) = m_2(t) = \cdots, \\
m_2 &= m_3(t) = m_4(t) = \cdots, \\
k_j(t) &= \cot \left( \frac{1}{g} ((j - 1)\pi + \cos^{-1}(t)) \right)
\end{align*}
\]
for \( j = 1, 2, \ldots, g \).

We come to the problem of classifying equivalent classes of polynomials \( F \) satisfying the above condition (M). In the case where \( g = 1 \) or \( g = 2 \) it is easy. Cartan solved it in the case \( g = 3 \) ([3]) and proposed a problem: Is every closed isoparametric hypersurface in \( S^{n-1} \) homogeneous? Recently, Takagi [6] classified the case where \( g = 4 \) and \( m_1 \) or \( m_2 = 1 \), and his result still shows that the obtained ones are homogeneous.

In the present paper I, we shall investigate a homogeneous polynomial function \( F \) satisfying the differential equations (M) of Münzner in the case \( g = 4 \). To such an \( F \), we associate \( m_1 + 1 \) quadratic forms \( \{ p_a \} \) and \( m_1 + 1 \) cubic forms \( \{ q_a \} \) in \( m_1 + 2m_2 \) variables, and give a complete characterization of \( F \) in terms of \( \{ p_a \} \) and \( \{ q_a \} \) in Theorem 1. Using this, two series of non-homogeneous isoparametric hypersurfaces in spheres will be constructed in Theorem 2.

The polynomial functions \( F \) defining them are given explicitly as follows. We denote by \( F \) the real quaternion algebra \( H \) or the real Cayley algebra \( K \), and by \( u \rightarrow \bar{u} \) the canonical involution of \( F \). For the \( n \)-column vector space \( F^n \) over \( F \), the canonical inner product is denoted by \( (, ) \). For each positive integer \( r \), the space \( F^{2(r+1)} \) can be identified with \( R^N \) where \( N = 8(r + 1) \) or \( 16(r + 1) \). For a point \( x = u \times v \in F^{r+1} \times F^{r+1} = F^{2(r+1)} \), we set

\[
\begin{align*}
    u &= \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \\
    v &= \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}
\end{align*}
\]

where \( u_0, v_0 \in F, u_1, v_1 \in F^r \). Then we put

\[
F_s(u \times v) = 4(\| uv \|^2 - (u, v)^2) + (\| u_1 \|^2 - \| v_1 \|^2 + 2(u_0, v_0))^2
\]

where \( \| \| \) denotes the length of a vector, and

\[
F = r^4 - 2F_s.
\]

Then \( M_t = \{ x \in S^{n-1} \mid F(x) = t \} \) for each \( t \) in \((-1, 1)\) is isoparametric and its multiplicities \( m_1 \) and \( m_2 \) are given by

\[
m_1 = 3 \quad \text{and} \quad m_2 = 4r
\]

or

\[
m_1 = 7 \quad \text{and} \quad m_2 = 8r
\]

respectively according to \( F = H \) or \( K \).

The homogeneous isoparametric hypersurfaces in spheres have been classified by Hsiang-Lawson [4]. In Part II, we shall give an explicit form of \( F \) for each of them, and classify the polynomials \( F \) satisfying
the condition (M) in the case where \( g = 4 \) and \( m_1 \) or \( m_2 = 2 \). It will be shown that every closed isoparametric hypersurface in this case is homogeneous.

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2. Preliminaries. First we introduce a few notations for operations on polynomial functions and give some of their elementary properties. These notations and properties will be used consistently throughout our papers I and II.

Let \( R^n \) be an \( n \)-dimensional Euclidean space with inner product \((,\)\) and \( r \) the radius function of \( R^n \). The induced inner product on the dual space is also denoted by \((,\)\). For any polynomial functions \( f \) and \( g \) on \( R^n \), we denote by \( \langle f, g \rangle \) the polynomial function on \( R^n \) defined by

\[
\langle f, g \rangle(x) = ((df)_x, (dg)_x) \quad x \in R^n.
\]

The mapping \( (f, g) \rightarrow \langle f, g \rangle \) is bilinear and symmetric, and also satisfies

\[
\langle f, g_1 + g_2, h \rangle = \langle f, g_1, h \rangle + \langle f, g_2, h \rangle.
\]

Let \( \{x_1, \ldots, x_n\} \) be an orthonormal coordinate system for \( R^n \). Then \( \langle f, g \rangle \) is equivalently defined by

\[
\langle f, g \rangle = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i}.
\]

Especially, for a homogeneous polynomial \( f \) of degree \( k \) on \( R^n \), and for any positive integer \( l \) we have

\[
\langle r^{k+l}, f \rangle = 2klfr^{k+l-1}.
\]

We denote by \( \Delta \) the Laplacian on \( R^n \), that is,

\[
\Delta = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x_i)^2}.
\]

Then, for any positive integer \( k \), we have

\[
\Delta r^{2k} = 2l(n + 2k - 2) r^{2(k-1)}.
\]

Let \( V \) be a linear subspace of \( R^n \). We introduce the restriction forms of \( \langle , \rangle \) and \( \Delta \) as follows. Let \( W \) be the orthogonal complement of \( V \) so that we have \( R^n = V \oplus W \) (orthogonal decomposition). Choose orthonormal coordinate systems \( \{v_i\} \) and \( \{w_j\} \) for \( V \) and \( W \) respectively. Then any polynomial functions \( f \) and \( g \) on \( R^n \) can be expressed as polynomials in variables \( \{v_i\} \) and \( \{w_j\} \). We put
They are determined independently on the choices of coordinate systems, and sometimes they will be also denoted by $\langle f, g \rangle_{(v_i)}$ and $\Delta_{(v_i)}f$. From the definitions it follows that, for an arbitrary orthogonal decomposition $R^* = V \oplus W$, we have

\begin{equation}
\langle f, g \rangle = \langle f, g \rangle_v + \langle f, g \rangle_w \tag{2.9}
\end{equation}
and

\begin{equation}
\Delta f = \Delta_v f + \Delta_w f. \tag{2.10}
\end{equation}

Let $f$ be a polynomial function on $R^n$, and $V$ a linear subspace of $R^*$. $f$ is said to be homogeneous of degree $k$ on $V$ if $f$ is homogeneous of degree $k$ with respect to the variables $\{v_i\}$ in the expression of $f$ as a polynomial in $\{v_i\}$ and $\{w_i\}$.

Let $V$ be a linear subspace of $R^*$. Every polynomial function $f$ on $V$ can be considered also as a polynomial function on $R^*$ canonically through the orthogonal decomposition $R^* = V \oplus W$. By this identification, it follows that for polynomial functions $f$ and $g$ on $V$ we have

\begin{equation}
\langle f, g \rangle_v = \langle f, g \rangle \tag{2.11}
\end{equation}
and

\begin{equation}
\Delta_v f = \Delta f. \tag{2.12}
\end{equation}

Finally, for a quadratic form $f$ on $R^n$, we define a symmetric linear mapping $\eta(f)$ of $R^*$ by

\begin{equation}
\eta(f)(x), x' = f(x, x') \tag{2.13}
\end{equation}
where $f$ is considered in the usual way as a symmetric bilinear form on $R^n$. The correspondence $f \rightarrow \eta(f)$ is one to one from the set of quadratic forms on $R^n$ onto the set of symmetric linear mappings of $R^n$.

For quadratic forms $f$ and $g$ on $R^n$, we have

\begin{equation}
\eta(\langle f, g \rangle) = 2(\eta(f)\eta(g) + \eta(g)\eta(f)) \tag{2.14}
\end{equation}
and especially

\begin{equation}
\eta(\langle f, f \rangle) = 4(\eta(f))^2. \tag{2.15}
\end{equation}
Furthermore, we have
They can be verified easily.

Now, let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$ centered at the origin. We need the following preliminary lemmas.

**Lemma 1.** Let $F$ be a homogeneous polynomial function of degree $g$ on $\mathbb{R}^n$ satisfying

$$\langle F, F \rangle = g^2 r^{2g-2}.$$  

Then the restriction $f$ of $F$ to $S^{n-1}$ is singular at a point $x$ of $S^{n-1}$ if and only if

$$(dF)_x = \pm (dr^g)_x.$$  

**Proof.** By definition, $f$ is singular at $x$ if and only if $(df)_x = 0$. Note that a tangent vector $X$ in $T_x(\mathbb{R}^n)$ is contained in $T_x(S^{n-1})$ if and only if $(dr^g)_x(X) = 0$.

Thus, $(df)_x = 0$ if and only if

$$(dF)_x = c (dr^g)_x$$  

for some constant $c$. Since $(dF, dF) = \langle F, F \rangle = (dr^g, dr^g)$ from our assumption, we see that $(df)_x = 0$ if and only if

$$(dF)_x = \pm (dr^g)_x.$$  

q.e.d.

**Lemma 2.** Let $F$ be as in Lemma 1. Then the restriction $f$ of $F$ to $S^{n-1}$ ranges from $-1$ to $1$ unless it is constant, and $f$ is singular at a point $x$ of $S^{n-1}$ if and only if $F(x) = \pm 1$.

**Proof.** Let $x$ be a point of $S^{n-1}$ and choose an orthonormal coordinate system $\{u_1, \ldots, u_{N-1}, z\}$ such that $z(x) = 1$ and $u_i(x) = 0$ for $i = 1, 2, \ldots, N-1$. We expand $F$ as a polynomial in $z$ as

$$F = a_0 z^g + a_1 z^{g-1} + \cdots + a_g$$  

where $a_h$ is a homogeneous polynomial of degree $h$ in $u_1, \ldots, u_{N-1}$. We have

$$(dF)_x = \left( \frac{\partial F}{\partial z} \right)(x)(dz)_x + \sum_{i=1}^{N-1} \left( \frac{\partial F}{\partial u_i} \right)(x)(du_i)_x$$

$$= ga_0 (dz)_x + \sum_{i=1}^{N-1} \left( \frac{\partial F}{\partial u_i} \right)(x)(du_i)_x$$

and

$$\Delta f = 2 \text{Tr} \left( \gamma(f) \right).$$
(d\rho^*)_x = g(r^* \, r \, dr)_x = g(dz)_x.

First suppose that \( f \) is singular at \( x \). Then, by Lemma 1 we have 
\[(dF)_x = \pm (dr^*)_x,\] and hence \( a_0 = \pm 1 \). This shows \( F(x) = a_0 = \pm 1 \). Conversely, suppose \( F(x) = \pm 1 \), i.e., \( a_0 = \pm 1 \). We have

\[ \langle F, F \rangle(x) = \langle (dF)_x, (dF)_x \rangle = g^2 a_0^2 + \sum_{i=1}^{N-1} \left( \frac{\partial F}{\partial u_i}(x) \right)^2. \]

Since \( \langle F, F \rangle = g^2 r^* = \langle F, F \rangle(x) = g^2 \), and hence we have \( (\partial F/\partial u_i)(x) = 0 \) for \( i = 1, 2, \ldots, N-1 \). Thus, we have \( (dF)_x = \pm (dr^*)_x \), and hence \( f \) is singular at \( x \) by Lemma 1.

We have proved the latter assertion in Lemma 2. The former assertion follows from the latter since \( S^{N-1} \) is compact. q.e.d.

**Lemma 3.** Let \( F \) be as in Lemma 1, and put
\[ F = \sum_{i_1, \ldots, i_N} a_{i_1} \cdots x_{i_1}^i \cdots x_{i_N}^i \]
where \( \{x_1, \ldots, x_N\} \) is an orthonormal coordinate system for \( \mathbb{R}^N \). Assume that the degree \( g \) is even and \( F \) satisfies
\[ F \big|_{x_{i_1} = \cdots = x_{i_N} = 0} = \left( \sum_{i=1}^k x_i^i \right)^{g/2}. \]
Then we have
\[ a_{i_1 \cdots i_N} = 0 \]
whenever \( i_1 + \cdots + i_k = g - 1 \).

**Proof.** Put \( F = \sum F_h \) where \( F_h \) is the homogeneous part of degree \( h \) in the variables \( x_1, \ldots, x_k \):
\[ F_h = \sum_{i_1, \ldots, i_N} a_{i_1} \cdots x_{i_1}^i \cdots x_{i_N}^i. \]
The assumption says \( F_g = (\sum x_i^i)^{g/2} \). We shall show \( F_{g-1} = 0 \). Put
\[ G = F_{g-2} + \cdots + F_0, \]
so that we have
\[ F = F_g + F_{g-1} + G. \]
Now, we have
\[ \frac{\partial F}{\partial x_i} = g x_i \left( \sum_{i=1}^k x_i^i \right)^{(g/2) - 1} + \frac{\partial F_{g-1}}{\partial x_i} + \frac{\partial G}{\partial x_i}. \]
for $i = 1, \ldots, k$, and
\[
\frac{\partial F}{\partial x_i} = \frac{\partial F_{i-1}}{\partial x_i} + \frac{\partial G}{\partial x_i}
\]
for $j = k + 1, \ldots, N$, and hence
\[
\langle F, F \rangle = \sum_{i=1}^{k} \left( \frac{\partial F}{\partial x_i} \right)^2 + \sum_{j=k+1}^{N} \left( \frac{\partial F}{\partial x_j} \right)^2
\]
\[
= \sum_{i=1}^{k} \left( g^i x_i \left( \sum_{i=1}^{k} x_i^2 \right)^{g-2} + \left( \frac{\partial F_{i-1}}{\partial x_i} \right)^2 + \left( \frac{\partial G}{\partial x_i} \right)^2 \right)
\]
\[
+ 2gx_i \left( \sum_{i=1}^{k} x_i^2 \right)^{(g/2)-1} \left( \frac{\partial F_{i-1}}{\partial x_i} + \frac{\partial G}{\partial x_i} \right) + 2 \frac{\partial F_{i-1}}{\partial x_i} \frac{\partial G}{\partial x_i} \right)
\]
\[
+ \sum_{j=k+1}^{N} \left( \frac{\partial F_{j-1}}{\partial x_j} \right)^2 + \left( \frac{\partial G}{\partial x_j} \right)^2 + 2 \frac{\partial F_{j-1}}{\partial x_i} \frac{\partial G}{\partial x_i} \right).
\]
On the other hand, we have
\[
\langle F, F \rangle = g^2 x^2 g-2 = g^2 \left( \sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{N} x_j^2 \right)^{g-1} .
\]
Comparing the homogeneous terms of degree $2g - 2$ in the variables $x_1, \ldots, x_k$ in the above two equations, we get
\[
\sum_{j=k+1}^{N} \left( \frac{\partial F_{j-1}}{\partial x_j} \right)^2 = 0 ,
\]
and hence
\[
\frac{\partial F_{j-1}}{\partial x_j} = 0 \quad \text{for } j = k + 1, \ldots, N .
\]
Since $F_{j-1}$ is linear in $x_{k+1}, \ldots, x_N$, we have $F_{j-1} = 0$. This proves Lemma 3. q.e.d.

3. Reductions. From now on we shall concern with isoparametric hypersurfaces in $S^{N-1}$ with 4 distinct principal curvatures. So we investigate a homogeneous polynomial function $F$ of degree 4 on $R^n$ satisfying $\langle F, F \rangle = 16r^2$ and $\Delta F = 8(m_2 - m_1)r^2$. These two equations will be replaced by equivalent ones step by step, and in the latter part of this section two families $\{p_a\}$ and $\{q_a\}$ of polynomials will be associated to $F$ on a suitable coordinate system. Our first purpose is to give a complete characterization of such an $F$ in terms of $\{p_a\}$ and $\{q_a\}$ (Theorem 1 in §4).

Let $m_1$ and $m_2$ be two positive integers such that $N = 2(m_1 + m_2 + 1)$, and $F$ a homogeneous polynomial function of degree 4 on $R^n$. Consider
the following two conditions on $F$;

(3.1) $\langle F, F \rangle = 16r^6$,

(3.2) $\Delta F = 8(m_2 - m_1)r^4$.

As a first step of reductions, we choose a unit vector $e$ in $\mathbb{R}^n$ such that the restriction $f$ of $F$ to $S^{n-1}$ takes its maximum at the point $e$. Let $X$ be the orthogonal complement of the 1-dimensional subspace $Re$ so that we have

(3.3) $\mathbb{R}^n = X \oplus Re$.

Let $z$ be the coordinate function on $Re$ defined by $z(e) = 1$ and $\{x_1, \ldots, x_{n-1}\}$ an orthonormal coordinate system for $X$.

**Lemma 4.** Assume that $F$ satisfies (3.1) and (3.2). Then, $F$ can be written in the form

(3.4) $F = z^4 + Ax^2 + Bz + C$

where $A$, $B$ and $C$ are homogeneous polynomial functions on $X$ of degree 2, 3 and 4 respectively, and $A$, $B$ and $C$ satisfy the following equations (1-1)~(1-8) listed below. Conversely, assume that a homogeneous polynomial function $F$ of the above form (3.4) is given with $A$, $B$ and $C$ satisfying (1-1)~(1-8). Then $F$ satisfies (3.1) and (3.2).

(1-1) $\langle A, A \rangle + 16A = 48\left(\sum_{i=1}^{N-1} x_i^2\right)$

(1-2) $\langle A, B \rangle + 4B = 0$

(1-3) $\langle B, B \rangle + 2\langle A, C \rangle + 4A^2 = 48\left(\sum_{i=1}^{N-1} x_i^2\right)^3$

(1-4) $\langle B, C \rangle + 2AB = 0$

(1-5) $\langle C, C \rangle + B^2 = 16\left(\sum_{i=1}^{N-1} x_i^2\right)^3$

(1-6) $\Delta A + 12 = 8(m_2 - m_1)$

(1-7) $\Delta B = 0$

(1-8) $\Delta C + 2A = 8(m_2 - m_1)\left(\sum_{i=1}^{N-1} x_i^2\right)$.

**Proof.** Assume that $F$ satisfies (3.1) and (3.2). We first remark that the restriction $f$ of $F$ to $S^{n-1}$ is not a constant. In fact, suppose that $f$ is a constant $c$ on $S^{n-1}$. Then we have $F = cr^4$. Since $\langle F, F \rangle = 16r^6$, we have $c = \pm 1$. On the other hand,
\[ \Delta F = c \Delta r^4 = c(8 + 4N)r^2 = 8(m_2 - m_1)r^2. \]

Hence, \( \pm (8 + 4N) = 8(m_2 - m_1) \). It follows that \( m_1 = -1 \) or \( m_2 = -1 \). This is a contradiction.

By Lemma 2, we have \( F(e) = 1 \). By the choice of coordinates, we have

\[ F \mid_{x_1 = \cdots = x_{N-1} = 0} = (z^3)^2. \]

Applying Lemma 3, we see that \( F \) has the form

\[ F = z^4 + A x^2 + B z + C \]

where \( A, B \) and \( C \) are homogeneous polynomials in \( x_1, \cdots, x_{N-1} \) of degree 2, 3 and 4 respectively. We write (3.1) and (3.2) in terms of \( A, B \) and \( C \).

We have

\[ \langle F, F \rangle = \left( \frac{\partial F}{\partial z} \right)^2 + \langle F, F \rangle \]

\[ = 16z^4 + 4Az^2 + B^2 + 16Az^4 + 8Bz^3 + 4ABz + \langle F, F \rangle \]

\[ = 16z^4 + (16A + \langle A, A \rangle)z^4 + (8B + 2\langle A, B \rangle)z^3 \]

\[ + (4A^2 + \langle B, B \rangle + 2\langle A, C \rangle)z^2 + (4AB + 2\langle B, C \rangle)z \]

\[ + B^2 + \langle C, C \rangle, \]

and

\[ 16r^4 = 16(z^3 + \sum x_i)^3 \]

\[ = 16z^6 + 48(\sum x_i^2)z^4 + 48(\sum x_i^3)z^2 + 16(\sum x_i^3). \]

Comparing the coefficients of \( z^h \) for each \( h \), we see that (3.1) is equivalent to (1-1)\( \sim \)(1-5) as a whole.

Next, we have

\[ \Delta F = \Delta_{\{x_i\}} F + \Delta_x F \]

\[ = 12z^2 + 2A + (\Delta_x A)z^2 + (\Delta_x B)z + \Delta_x C, \]

and

\[ 8(m_2 - m_1)r^2 = 8(m_2 - m_1)(x^2 + \sum x_i^2). \]

Hence, (3.2) is equivalent to (1-6)\( \sim \)(1-8). Thus, we have the first assertion of Lemma 4.

The converse follows clearly from the above argument. q.e.d.

**Lemma 5.** Let \( A \) be a quadratic form on \( X \) satisfying (1-1) and (1-6). Then, \( X \) has a unique orthogonal decomposition

\[ X = Y \oplus W \]

(3.5)
with \( \dim W = m_1 + 1 \) such that \( A \) has the form

\[
A = 2\left( \sum_{j=1}^{n} y_j^2 \right) - 6\left( \sum_{a=0}^{n} w_a^2 \right)
\]

where \( \{y_j\} \) and \( \{w_a\} \) are orthonormal coordinate systems for \( Y \) and \( W \) respectively, and \( n = m_1 + 2m_2 \). Conversely, if \( A \) is of the above form with respect to an orthogonal decomposition \( X = Y \oplus W \) with \( \dim W = m_1 + 1 \), then \( A \) satisfies (1-1) and (1-6).

**Proof.** We denote by \( \bar{A} \) the symmetric mapping \( \eta(A) \) of \( X \) associated to \( A \). Then (1-1) and (1-6) are equivalent to

\[
(\bar{A})^2 + 4\bar{A} - 12 I_X = 0
\]

and

\[
\text{Tr}(\bar{A}) = 4(m_2 - m_1) - 6
\]

respectively, where \( I_X \) denotes the identity mapping of \( X \). Assume (1-1) and (1-6). (1-1)' shows that an eigenvalue of \( \bar{A} \) is 2 or \(-6\). Decompose \( X \) into the eigenspaces:

\[
X = Y \oplus W
\]

where \( Y \) and \( W \) are the eigenspaces for the eigenvalues 2 and \(-6\) respectively. This is an orthogonal decomposition since \( \bar{A} \) is symmetric. From (1-6)' it follows that \( \dim Y = m_1 + 2m_2 \) and \( \dim W = m_1 + 1 \). This shows our first assertion. The converse is easily seen. q.e.d.

**Lemma 6.** Assume (1-1) and (1-6) for \( A \). Then, \( B \) satisfies (1-2) if and only if \( B \) is homogeneous of degree 2 on \( Y \) and of degree 1 on \( W \).

**Proof.** Write

\[
B = \sum_{h=0}^{3} B_h
\]

where \( B_h \) is the homogeneous part of degree \( h \) on \( W \) and hence of degree \( 3 - h \) on \( Y \). Consider (1-2). Since \( A = 2(\sum y_j^2) - 6(\sum w_a^2) \) by Lemma 5, we have

\[
\langle A, B \rangle + 4B
\]

\[
= \langle A, B \rangle_Y + \langle A, B \rangle_W + 4B
\]

\[
= 2(\sum y_j^2, B)_Y - 6(\sum w_a^2, B)_W + 4B
\]

\[
= 2(2B_2 + 4B_1 + 6B_0, B)_W + 4B
\]

\[
= 2(2B_2 + 4B_1 + 6B_0, 2B + B_2 + 6B_0)
\]

\[
= -32B_2 - 16B_1 + 16B_0.
\]
Thus (1-2) is equivalent to $B_3 = 0$, $B_2 = 0$ and $B_0 = 0$. This shows Lemma 6.

Hereafter we assume (1-1), (1-6) together with (1-2). The orthogonal decomposition $X = Y \oplus W$ in Lemma 5 gives us the second reduction. Let \{\(y_\alpha\)\} and \{\(w_\alpha\)\} be orthonormal coordinate systems for $Y$ and $W$ respectively where $j$ runs from 1 to $n = m_1 + 2m_2$ and $\alpha$ runs from 0 to $m_1$. In view of Lemma 6, we can define $m_1 + 1$ quadratic forms $p_{\alpha_1}, \ldots, p_{\alpha_{m_1}}$ on $Y$ by

\[
B = 8 \sum_{\alpha=0}^{m_1} p_{\alpha} w_{\alpha} .
\]  

For $C$, we put

\[
C = \sum_{h=0}^{4} C_h
\]

where $C_h$ is the homogeneous part of degree $h$ on $W$ and hence of degree $4 - h$ on $Y$, and we define $m_1 + 1$ cubic forms $q_{\alpha_1}, \ldots, q_{\alpha_{m_1}}$ on $Y$ by

\[
C_1 = 8 \sum_{\alpha=0}^{m_1} q_{\alpha} w_{\alpha} .
\]

**Lemma 7.** The equation (1-3) holds if and only if we have

(i) $C_4 = (\sum w_\alpha^2)^2$,
(ii) $C_3 = 0$,
(iii) $C_2 = 2 \sum_{\alpha,\beta} \langle p_{\alpha}, p_{\beta} \rangle w_\alpha w_\beta - 6(\sum y_\alpha^2)(\sum w_\alpha^2)$,
(iv) $C_1 = (\sum y_\alpha^2)^2 - 2 \sum p_\alpha^2$.

**Proof.** Recall (1-3):

\[
\langle B, B \rangle + 2 \langle A, C \rangle + 4 A^2 = 48(\sum x_i^2)^2 .
\]

We have

\[
4 A^2 = 4(2(\sum y_\alpha^2) - 6(\sum w_\alpha^2))^2 ,
\]

\[
\langle B, B \rangle = \langle B, B \rangle_Y + \langle B, B \rangle_W = 64 \sum_{\alpha,\beta} \langle p_{\alpha}, p_{\beta} \rangle w_\alpha w_\beta + 64 \sum p_\alpha^2 ,
\]

\[
2 \langle A, C \rangle = 2 \langle A, C \rangle_Y + 2 \langle A, C \rangle_W = 4(\sum y_\alpha^2, \sum C_h) - 12(\sum w_\alpha^2, \sum C_h) = 8(C_2 + 2C_2 + 3C_1 + 4C_0) - 24(4C_4 + 3C_3 + 2C_2 + C_1) - 96C_4 - 64C_5 - 32C_2 + 32C_0 .
\]
and
\[ 48(\sum x_i^2) = 48(\sum w^2_i) + 96(\sum y_i)(\sum w_i) + 48(\sum y_j^2). \]
Summarizing their homogeneous terms, (1-3) is equivalent to
\[ 4 \cdot 36(\sum w^2_i) - 96C_4 = 48(\sum w^2_i), \]
\[ -64C_4 = 0, \]
\[ -96(\sum y_i)(\sum w_i) + 64 \sum \langle p_a, p_b \rangle w_aw_b - 32C_4 = 96(\sum y_i)(\sum w_i), \]
\[ 16(\sum y_j^2) + 64 \sum p_a^2 + 32C_0 = 48(\sum y_i^2). \]
Now Lemma 7 follows. q.e.d.

REMARK 1. By Lemmas 4, 5, 6 and 7, it follows that the polynomial function \( F \) can be constructed uniquely from \( \{p_a\} \) and \( \{q_a\} \).

Our \( \{p_a\} \) and \( \{q_a\} \) associated to \( F \) depend on the choice of \( e \) in \( S^{N-1} \)
such that \( F(e) = 1 \) and on the choice of an orthonormal coordinate system \( \{w_a\} \) for \( W \). Let \( F' \) be another homogeneous polynomial function of degree 4 on \( R^N \) satisfying (3.1) and (3.2). Choose \( e' \) in \( S^{N-1} \) and \( \{w'_a\} \) for \( W' \) in the same way, so that we have \( \{p'_a\} \) and \( \{q'_a\} \) on \( F' \) associated to \( F' \).

We say that \( F \) and \( F' \) are \( O(N) \)-equivalent if there exists an element \( \sigma \) in \( O(N) \) such that
\[ F'(x) = F(\sigma^{-1}x) \text{ for } x \in R^N. \]

Let \( V \) and \( V' \) be two finite-dimensional vector spaces over \( R \). For a linear isomorphism \( \tau \) of \( V \) onto \( V' \), and for a polynomial function \( f \) on \( V \), we denote by \( \tau f \) the polynomial function on \( V' \) obtained by
\[ (\tau f)(v') = f(\tau^{-1}v'). \]

With these notations, we state the following two remarks for a later use.

REMARK 2. Suppose that \( F \) and \( F' \) are \( O(N) \)-equivalent by an element \( \sigma \) in \( O(N) \) such that \( \sigma(e) = e' \). Then \( \sigma \) induces orthonormal transformations \( \sigma_w : W \rightarrow W' \) and \( \sigma_y : Y \rightarrow Y' \). By a suitable choice of \( \{w'_a\} \) for \( W' \), we have
\[ \sigma_y p_a = p'_a, \quad \sigma_y q_a = q'_a \]
for \( \alpha = 0, 1, \ldots, m \). Conversely, suppose that there exists an orthonormal transformation \( \tau \) of \( Y \) onto \( Y' \) such that
\[ \tau p_a = p'_a, \quad \tau q_a = q'_a \]
for \( \alpha = 0, 1, \ldots, m \). Then \( F \) and \( F' \) are \( O(N) \)-equivalent by an element \( \sigma \) in \( O(N) \) such that \( \sigma(e) = e' \).
Remark 3. Consider the case where the isoparametric hypersurface in $S^{n-1}$ defined by $F = c$ for some constant $c$ is homogeneous by a subgroup of $O(N)$. Then it follows that the singular submanifold

$$M_0 = \{ x \in S^{n-1}; F(x) = 1 \}$$

is also homogeneous by the $e$-component of the same group. Therefore $F$ and $F'$ are $O(N)$-equivalent if and only if there exist an orthogonal matrix $(\tau_{a\beta})$ of degree $m_i + 1$ and an orthonormal transformation $\sigma$ of $Y$ onto $Y'$ such that

$$p_\beta' = \sum_{a} \tau_{a\beta}(\sigma p_a),
q_\beta' = \sum_{a} \tau_{a\beta}(\sigma q_a)$$

for $\beta = 0, 1, \ldots, m_i$.

Remarks 2 and 3 are immediate consequences of the preceding lemmas.

4. A characterization by $\{p_a\}$ and $\{q_a\}$. We continue the argument of the preceding section under the assumptions (1-1), (1-2), (1-3) and (1-6). The equations (1-4), (1-5), (1-7) and (1-8) will be reformulated first in terms of $B$, $C_0$ and $C_i$, and then in terms of $\{p_a\}$ and $\{q_a\}$, using Lemmas 5, 6 and 7.

First we list the equations:

(2-1) \quad \langle B, C_z \rangle_Y = 8B(\sum w_n^z)
(2-2) \quad \langle B, C_0 \rangle_Y = 0
(2-3) \quad \langle B, C_z \rangle_w + \langle B, C_0 \rangle_Y + 4B(\sum y_j^z) = 0
(2-4) \quad \langle B, C_i \rangle_w = 0
(2-5) \quad \langle C_z, C_z \rangle_Y + 16C_4(\sum w_n^z) = 48(\sum y_j^z)^4(\sum w_n^z)^2
(2-6) \quad \langle C_z, C_i \rangle_Y + 4C_i(\sum w_n^z) = 0
(2-7) \quad \langle C_z, C_z \rangle_w + \langle C_z, C_i \rangle_Y + 2\langle C_z, C_0 \rangle_Y + B^2 = 48(\sum y_j^z)^4(\sum w_n^z)
(2-8) \quad \langle C_i, C_i \rangle_w + \langle C_i, C_0 \rangle_Y = 0
(2-9) \quad \langle C_i, C_i \rangle_w + \langle C_i, C_0 \rangle_Y = 16(\sum y_j^z)^3
(2-10) \quad \Delta_y B = 0
(2-11) \quad \Delta_y C_z = (8m_2 - 12m_i)(\sum w_n^z)
(2-12) \quad \Delta_y C_i = 0
(2-13) \quad \Delta_w C_z + \Delta_y C_0 = (8m_2 - 8m_1 - 4)(\sum y_j^z).
**Lemma 8.** The following implications hold:

(i) \((1-4) \Rightarrow (2-1), (2-2), (2-3) \text{ and } (2-4),\)

(ii) \((1-5) \Rightarrow (2-5), (2-6), (2-7), (2-8) \text{ and } (2-9),\)

(iii) \((1-7) \Rightarrow (2-10),\)

(iv) \((1-8) \Rightarrow (2-11), (2-12) \text{ and } (2-13).\)

**Proof.** In each of \((1-4), (1-5), (1-7) \text{ and } (1-8),\) we replace \(A\) by \(2(\sum y_j) - 6(\sum w_a^i),\) \(C\) by \(C_4 + C_5 + C_6 + C_7,\) and then \(C_4\) by \((\sum w_a^k)^t.\)

Decomposing the results into the homogeneous part with respect to the variables \(w_a's,\) we can conclude Lemma 8. We give here the proof of (i). The rest can be shown in a similar way.

Recall \((1-4):\)

\[\langle B, C \rangle + 2AB = 0.\]

We have

\[
\langle B, C \rangle = \langle B, C \rangle_r + \langle B, C \rangle_w
\]

\[= \langle B, C_4 \rangle_r + \langle B, C_5 \rangle_r + \langle B, C_6 \rangle_r + \langle B, C_7 \rangle_r
\]

\[+ \langle B, C_8 \rangle_w + \langle B, C_9 \rangle_w + \langle B, C_10 \rangle_w + \langle B, C_11 \rangle_w.\]

Note \(\langle B, C_4 \rangle_r = 0, \langle B, C_5 \rangle_w = 0,\) and \(\langle B, C_6 \rangle_w = \langle B, (\sum w_a^i)^t \rangle_w = 4B(\sum w_a^i).\)

Thus, we have

\[
\langle B, C \rangle + 2AB
\]

\[= \langle B, C_4 \rangle_r - 8B(\sum w_a^i)
\]

\[+ \langle B, C_5 \rangle_r
\]

\[+ \langle B, C_6 \rangle_r + \langle B, C_7 \rangle_w + 4B(\sum y_j^r)
\]

\[+ \langle B, C_8 \rangle_w,
\]

from which we can see easily \((1-4) \Rightarrow (2-1) \sim (2-4).\) \(\text{q.e.d.}\)

Now we reformulate the above equations \((2-1) \sim (2-13)\) in terms of \(\{p_a\} \text{ and } \{q_a\}\) as follows:

\[(3-1)\]

\[
\left\{ \begin{array}{c}
\langle \langle p_a, p_a \rangle, p_a \rangle = 16p_a, \\
da(\langle p_a, p_a \rangle) = 16m_a
\end{array} \right.
\]

for each \(\alpha;\)

\[(3-2)\]

\[2\langle \langle p_a, p_\beta \rangle, p_\beta \rangle + \langle \langle p_\beta, p_\beta \rangle, p_a \rangle = 16p_a\]

for distinct \(\alpha, \beta;\)

\[(3-3)\]

\[\langle \langle p_\alpha, p_\beta \rangle, p_\gamma \rangle + \langle \langle p_\beta, p_\gamma \rangle, p_\alpha \rangle + \langle \langle p_\gamma, p_\alpha \rangle, p_\beta \rangle = 0\]

for mutually distinct \(\alpha, \beta, \gamma;\)

\[(3-4)\]

\[\langle p_\alpha, q_\alpha \rangle = 0\]

for each \(\alpha;\)

\[(3-5)\]

\[\langle p_\alpha, q_\beta \rangle + \langle p_\beta, q_\alpha \rangle = 0\]

for distinct \(\alpha, \beta;\)
\begin{align}(3-6) \quad & \langle \langle p_\alpha, p_{\beta} \rangle, q_\gamma \rangle + \langle \langle p_\beta, p_\gamma \rangle, q_\alpha \rangle + \langle \langle p_\gamma, p_\alpha \rangle, q_\beta \rangle = 0 \\
& \text{for mutually distinct } \alpha, \beta, \gamma; \\
&(3-7) \quad \sum_{\alpha=0}^{m_1} p_\alpha q_\alpha = 0; \\
&(3-8) \quad 16 \left( \sum_{\alpha=0}^{m_1} q_\alpha^2 \right) = 16 G(\sum y_j) - \langle G, G \rangle; \\
&(3-9) \quad 8 \langle q_\alpha, q_\alpha \rangle = 8(\langle p_\alpha, p_\alpha \rangle(\sum y_j^2) - p_\alpha^2) + \langle \langle p_\alpha, p_\alpha \rangle, G \rangle \\
& \quad - 24G - 2 \sum_{j=0}^{m_1} \langle p_\alpha, p_\gamma \rangle^2 \quad \text{for each } \alpha; \\
&(3-10) \quad 8 \langle q_\alpha, q_\beta \rangle = 8(\langle p_\alpha, p_\beta \rangle(\sum y_j^2 - p_\alpha p_\beta) + \langle \langle p_\alpha, p_\beta \rangle, G \rangle \\
& \quad - 2 \sum_{j=0}^{m_1} \langle p_\alpha, p_\gamma \rangle \langle p_\beta, p_\gamma \rangle \quad \text{for distinct } \alpha, \beta; \\
\end{align}

where \( G = \sum_{\alpha=0}^{m_1} p_\alpha^2 \) and the indices \( \alpha, \beta, \gamma \) run from 0 to \( m_1 \).

**Lemma 9.** The following implications hold:

(i) \( (2-1), (2-10), (2-11) \implies (3-1), (3-2), (3-3) \)

(ii) \( (3-1), (3-2), (3-3) \implies (2-1), (2-10) \),

(iii) \( (2-6) \implies (3-6); \)

(iv) \( (2-4) \implies (3-7); \)

(v) \( (2-9) \implies (3-8); \)

(vi) \( (2-7) \implies (3-9), (3-10). \)

We give here the proofs of (i) and (iii). The rest can be proved similarly.

**Proof of (i).** Recall (2-10): \( \Delta \gamma B = 0. \) This is equivalent to \( \Delta p_\alpha = 0. \)

Consider (2-11):

\[
\Delta \gamma C_2 = (8m_2 - 12m_3)(\sum w_\gamma^2). 
\]

Using \( C_2 = 2 \sum \langle p_\alpha, p_\beta \rangle w_\alpha w_\beta - 6(\sum y_j^2)(\sum w_\alpha^2), \) we get

\[
\Delta \gamma C_2 = 2 \sum \Delta \gamma \langle \langle p_\alpha, p_\beta \rangle \rangle w_\alpha w_\beta - 12(m_1 + 2m_2)(\sum w_\alpha^2). 
\]

Thus, (2-11) can be written as

\[
2 \sum \Delta \gamma \langle \langle p_\alpha, p_\beta \rangle \rangle w_\alpha w_\beta = \{12(m_1 + 2m_2) + 8m_2 - 12m_3\}(\sum w_\alpha^2) = 32m_3(\sum w_\gamma^2). 
\]

And hence we see that (2-11) is equivalent to

\begin{align}(2-11-1) \quad & \Delta \langle \langle p_\alpha, p_\alpha \rangle \rangle = 16m_2 \quad \text{for each } \alpha, \\
& \text{and} \end{align}
Now consider (2-1): \( \langle B, C_2 \rangle_Y = 8B(\sum w_i^2) \).

We have

\[
\langle B, C_2 \rangle_Y - 8B(\sum w_i^2) = 2\langle B, \sum \langle p_a, p_b \rangle w_a w_b \rangle_Y - 6\langle B, (\sum y_i^2)(\sum w_i^2) \rangle_Y - 8B(\sum w_i^2) = 16\sum p_a w_a, \sum \langle p_a, p_b \rangle w_a w_b \rangle_Y - 32B(\sum w_i^2) = 16\left\{ \sum \left\langle \langle p_a, p_b \rangle, p_\gamma \right\rangle w_a w_\beta w_\gamma - 16 \sum p_a w_a w_\beta \right\}.
\]

Now we have the implication (2-1), (2-10), (2-11) \( \Rightarrow \) (3-1), (3-2), (3-3).

From the above argument, we also have the implication (3-1), (3-2), (3-3) \( \Rightarrow \) (2-1), (2-10).

**Proof of (iii).** Recall (2-6): \( \langle C_2, C_i \rangle_Y + 4C_i(\sum w_i^2) = 0. \) By Lemma 7, \( C_2 = 2 \sum \langle p_a, p_b \rangle w_a w_b - 6(\sum y_i^2)(\sum w_i^2) \). We have

\[
\langle C_2, C_i \rangle_Y + 4C_i(\sum w_i^2) = 16\sum \langle p_a, p_b \rangle w_a w_b, \sum q_\tau w_\tau \rangle_Y - 6(\sum w_i^2)(\sum y_i^2), C_i \rangle_Y + 4C_i(\sum w_i^2) = 16 \sum \left\langle \langle p_a, p_b \rangle, q_\tau \rangle w_a w_\beta w_\gamma - 32C_i(\sum w_i^2) = 16\left\{ \sum \left\langle \langle p_a, p_b \rangle, q_\gamma \right\rangle w_a w_\beta w_\gamma - 16 \sum q_\alpha w_\alpha w_\beta \right\}.
\]

Thus, we see that (2-6) is equivalent to the following three conditions as a whole:

(2-6-1) \( \langle \langle p_a, p_a \rangle, q_\alpha \rangle = 16q_\alpha \) for each \( \alpha \);

(2-6-2) \( 2\langle \langle p_a, p_b \rangle, q_\alpha \rangle + \langle \langle p_a, p_a \rangle, q_\beta \rangle = 16q_\beta \) for distinct \( \alpha, \beta \);

(2-6-3) \( \langle \langle p_a, p_b \rangle, q_\gamma \rangle + \langle \langle p_\beta, p_\gamma \rangle, q_\alpha \rangle + \langle \langle p_\gamma, p_\alpha \rangle, q_\beta \rangle = 0 \) for distinct \( \alpha, \beta, \gamma \).

Thus we have (2-6) \( \Rightarrow \) (3-6) = (2-6-3).

q.e.d.

Lemma 9 shows the first assertion of the following Theorem 1.

**Theorem 1.** Let \( m_1 \) and \( m_2 \) be positive integers such that \( N = 2(m_1 + m_2 + 1) \), and put \( n = m_1 + 2m_2 \).

Assume that a homogeneous polynomial function \( F \) of degree 4 on \( \mathbb{R}^n \) satisfies \( \langle F, F \rangle = 16r^4 \) and \( \Delta F = 8(m_2 - m_1)r^4 \). Then two families \{\( p_a \)\} and \{\( q_a \)\} of polynomials associated to \( F \) in § 3 satisfy the equations (3-1) \( \sim \) (3-10).

Conversely, assume that there are given \( m_1 + 1 \) quadratic forms \( p_0, \ldots, p_{m_1} \) and \( m_1 + 1 \) cubic forms \( q_0, \ldots, q_{m_1} \) both on \( \mathbb{R}^n \) such that they
satisfy the equations (3-1)~(3-10). Then the polynomial function $F$ on $\mathbb{R}^n$ constructed from $\{p_a\}$ and $\{q_a\}$ as in § 3 satisfies $\langle F, F \rangle = 16r^8$ and $\Delta F = 8(m - m_*)_r^8$.

To prove "the converse" in Theorem 1, it suffices, in view of Lemma 9, to show that (2-3), (2-5), (2-6), (2-8), (2-11), (2-12) and (2-13) follow from (3-1)~(3-10). We first show (2-3), (2-8) and (2-13) below, and then reformulate the rest in terms of $\{p_a\}$ and $\{q_a\}$. They will be proved in § 5.

**Lemma 10.** (2-3), (2-8) and (2-13) follow from (3-1)~(3-10).

**Proof.** Recall (2-3): $\langle B, C_2 \rangle_w + \langle B, C_0 \rangle_Y + 4B(\sum y_j^2) = 0$. We have

$$
\begin{align*}
\langle B, C_0 \rangle_Y &= \langle B, 2 \sum (p_a, p_b)w_aw_bw \rangle_w - \langle B, 6(\sum y_j^2)(\sum w_a^2) \rangle_w \\
&= 32 \sum p_a<p_a, p_b> w_a w_b - 96(\sum p_a w_a)(\sum y_j^2)
\end{align*}
$$

and

$$
\begin{align*}
\langle B, C_0 \rangle_Y &= \langle B, (\sum y_j^2)^2 \rangle_Y - \langle B, 2G \rangle_Y \\
&= 8B(\sum y_j^2) - 16 \sum <p_a, G> w_a.
\end{align*}
$$

Thus, we have

$$
\begin{align*}
\langle B, C_2 \rangle_w + \langle B, C_0 \rangle_Y + 4B(\sum y_j^2)
&= 32 \sum <p_a, p_b> p_aw_a - 16 \sum <p_a, G> w_a \\
&= 16(\sum w_a(2 \sum <p_a, p_b> p_b - <p_a, G>)).
\end{align*}
$$

Since $G = \sum p_j^2$, we have $\langle p_a, G \rangle = 2\sum b <p_a, p_b> p_b$, and hence we have (2-3).

Next recall (2-8): $\langle C_2, C_1 \rangle_w + \langle C_1, C_0 \rangle_Y = 0$. We have

$$
\begin{align*}
\langle C_2, C_1 \rangle_w &= \langle 2 \sum (p_a, p_b)w_aw_bw, 8 \sum q_aw_a \rangle_w \\
&- \langle 6(\sum y_j^2)(\sum w_a^2), 8 \sum q_aw_a \rangle_w \\
&= 32 \sum <p_a, p_b> q_aw_b - 96(\sum y_j^2)(\sum q_aw_a),
\end{align*}
$$

and

$$
\begin{align*}
\langle C_1, C_0 \rangle_Y &= \langle C_1, (\sum y_j^2)^2 \rangle_Y - 2\langle C_1, G \rangle_Y \\
&= 12C(\sum y_j^2) - 2\langle C_1, G \rangle_Y \\
&= 96(\sum y_j^2) \sum q_aw_a - 16 \sum <q_a, G> w_a.
\end{align*}
$$

Hence we have

$$
\begin{align*}
\langle C_2, C_1 \rangle_w + \langle C_1, C_0 \rangle_Y &
= 16(2 \sum <p_a, p_b> q_aw_a - \sum <q_a, G> w_a).
\end{align*}
$$
Now we see that (2-8) is equivalent to
\[ 2 \sum_{\beta} \langle p_{\alpha}, p_{\beta} \rangle q_{\beta} = \langle q_{\alpha}, G \rangle \]
for each \( \alpha \).

By definition, \( \langle q_{\alpha}, G \rangle = \langle q_{\alpha}, \sum p_{\beta}^2 \rangle = 2 \sum_{\beta} \langle q_{\alpha}, p_{\beta} \rangle p_{\beta} \). Using (3-4) and (3-5), we have
\[ \langle q_{\alpha}, G \rangle = -2 \sum_{\beta} \langle p_{\alpha}, q_{\beta} \rangle p_{\beta} \]
Consider (3-7): \( \sum p_{\beta} q_{\beta} = 0 \). We have
\[ 0 = \langle p_{\alpha}, \sum p_{\beta} q_{\beta} \rangle = \sum_{\beta} \langle p_{\alpha}, p_{\beta} \rangle q_{\beta} + \sum_{\beta} \langle p_{\alpha}, q_{\beta} \rangle p_{\beta} \]
This proves the required equation.

Finally recall (2-13): \( \Delta w C_{2} + \Delta r C_{0} = (8(m_{2} - m_{4}) - 4)(\sum y_{i}^{2}) \). We have
\[
\begin{align*}
\Delta w C_{2} &= \Delta w \{ 2 \sum \langle p_{\alpha}, p_{\beta} \rangle w_{\alpha} w_{\beta} - 6(\sum y_{i}^{2})(\sum w_{\alpha}^{2}) \} \\
&= 4 \sum \langle p_{\alpha}, p_{\alpha} \rangle - 12(m_{i} + 1)(\sum y_{i}^{2})
\end{align*}
\]
and
\[
\begin{align*}
\Delta r C_{0} &= \Delta r \{(\sum y_{i}^{2})^{4} - 2G \} \\
&= (8 + 4n)(\sum y_{i}^{2}) - 2 \sum \Delta r p_{\alpha}^{2} \\
&= (8 + 4n)(\sum y_{i}^{2}) - 2 \sum (2p_{\alpha} \Delta p_{\alpha} + 2\langle p_{\alpha}, p_{\alpha} \rangle)
\end{align*}
\]
Since \( \Delta p_{\alpha} = 0 \) by (3-1), we have
\[
\Delta w C_{2} + \Delta r C_{0} = ((8 + 4n) - 12(m_{i} + 1))(\sum y_{i}^{2})
\]
Now
\[
8 + 4n - 12(m_{4} + 1) = 4(2m_{2} + m_{1}) - 12m_{4} - 4 \\
= 8(m_{2} - m_{1}) - 4
\]
and hence we have (2-13). q.e.d.

**Lemma 11.** (2-5) and (2-12) can be written as:

(2-5)'
\[
\sum_{\alpha, \beta, T, \delta} \langle \langle p_{\alpha}, p_{\beta} \rangle, \langle p_{T}, p_{\delta} \rangle \rangle w_{\alpha} w_{\beta} w_{T} w_{\delta} = 16 \sum_{\alpha, \beta, T} \langle p_{\alpha}, p_{\beta} \rangle w_{\alpha} w_{\beta} w_{T}^{2} ;
\]

(2-12)'
\[
\Delta q_{\alpha} = 0 \quad \text{for each } \alpha
\]
respectively.

**Proof.** Recall (2-5): \( \langle C_{2}, C_{2} \rangle_{Y} + 16C_{4}(\sum w_{\alpha}^{2}) = 48(\sum w_{\alpha}^{2})(\sum y_{i}^{2}) \), and \( C_{2} = 2 \sum \langle p_{\alpha}, p_{\beta} \rangle w_{\alpha} w_{\beta} - 6(\sum y_{i}^{2})(\sum w_{\alpha}^{2}) \).

We have
\[ \langle C, C \rangle_Y = 4 \sum_{a, \beta, r, \delta} \langle p_a, p_\beta \rangle w_a w_\beta \sum_{i} w_i^2 - 96 \sum_{a, \beta} \langle p_a, p_\beta \rangle w_a w_\beta (\sum w_i^2) + 4 \cdot 36 (\sum w_i^2)^4 (\sum y_j^2), \]

and

\[ 16 C_d (\sum w_i^2) = 32 \sum_{a, \beta} \langle p_a, p_\beta \rangle w_a w_\beta w_i^2 - 96 (\sum y_j^2) (\sum w_i^2)^4. \]

They show that (2-5) is equivalent to (2-5)'.

Recall (2-12): \( \Delta C_i = 0 \). Since \( C_i = \sum q_a w_a \), clearly (2-12) is equivalent to (2-12)'.

Note that (2-6) and (2-11) have been reformulated in the proof of Lemma 9.

5. The third decomposition of \( R^\gamma \). In this section, first the family \( \{ p_a \} \) of quadratic forms on \( Y \) will be characterized in matricial forms. Then we shall give a further decomposition of the space \( Y \). The proof of Theorem 1 will be completed.

For each quadratic form \( p_a \) on \( Y \), we define the symmetric linear mapping \( P_a \) of \( Y \) as in §2 by

\[ P_a = \eta(p_a). \]

We have

**Lemma 12.** The conditions (3-1), (3-2) and (3-3) on \( \{ p_a \} \) are equivalent to the following conditions (i), (ii) and (iii) respectively:

(i) For each \( \alpha \), we have

\[ (4-1)_a \]

\[ P_a^3 = P_a, \quad \text{Tr } P_a = 0, \quad \text{rank } P_a = 2m; \]

(ii) For each distinct \( \alpha, \beta \), we have

\[ (4-2)_{a, \beta} \]

\[ P_a = P_\beta P_a + P_\alpha P_\beta + P_\beta P_\alpha P_\beta; \]

(iii) For each mutually distinct \( \alpha, \beta, \gamma \) we have

\[ (4-2)_{a, \beta, \gamma} \]

\[ \Theta(P_a P_\beta P_\gamma) = 0, \]

where \( \Theta \) denotes the sum of terms obtained by interchanging the indices over all permutations.

Note \( \dim Y = n = m + 2m \). Lemma 12 follows by direct verifications, using (2.14), (2.15) and (2.16).

**Lemma 13.** (2-5) follows from (3-1), (3-2) and (3-3).

**Proof.** Recall, by Lemma 11, (2-5) \( \iff \) (2-5)'.

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\[
\sum_{a,\beta,\gamma,\delta} \langle \langle p_a, p_\beta \rangle, \langle p_\gamma, p_\delta \rangle \rangle w_a w_\beta w_\gamma w_\delta = 16 \sum_{a,\beta} \langle p_a, p_\beta \rangle w_a w_\beta w_\gamma^2.
\]

The monomials of \(w_a\)'s appearing in (2-5)' are classified in the following types:

\[w_\alpha^\alpha, w_\beta^\beta w_\beta, w_\alpha^\alpha w_\beta^\beta, w_\alpha^\alpha w_\gamma w_\delta, w_\alpha w_\beta w_\gamma w_\delta\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are all distinct. Now (2-5)' decomposes into the following five equations:

(2-5-1) \[\langle \langle p_\alpha, p_\alpha \rangle, \langle p_\alpha, p_\alpha \rangle \rangle = 16 \langle p_\alpha, p_\alpha \rangle ,\]
(2-5-2) \[\langle \langle p_\beta, p_\alpha \rangle, \langle p_\beta, p_\alpha \rangle \rangle = 8 \langle p_\beta, p_\alpha \rangle ,\]
(2-5-3) \[\langle \langle p_\gamma, p_\beta \rangle, \langle p_\gamma, p_\beta \rangle \rangle + 2 \langle \langle p_\alpha, p_\beta \rangle, \langle p_\gamma, p_\beta \rangle \rangle = 8 \langle p_\gamma, p_\beta \rangle ,\]
(2-5-4) \[\langle \langle p_\beta, p_\gamma \rangle, \langle p_\beta, p_\gamma \rangle \rangle + 2 \langle \langle p_\alpha, p_\gamma \rangle, \langle p_\beta, p_\gamma \rangle \rangle = 8 \langle p_\beta, p_\gamma \rangle ,\]
(2-5-5) \[\langle \langle p_\alpha, p_\gamma \rangle, \langle p_\beta, p_\gamma \rangle \rangle = 0 ,\]

where \(\alpha, \beta, \gamma, \delta\) are all distinct.

We give here a proof of (2-5-4). In the following verification, \(P_\alpha, P_\beta, \ldots\) are denoted simply by \(\alpha, \beta, \ldots\), and the notation \(\langle , \rangle\) is also used for mappings, i.e., \(\langle \alpha, \beta \rangle = 2(\alpha \beta + \beta \alpha)\).

To prove (2-5-4), it suffices to show

\[\langle \langle \alpha, \alpha \rangle, \langle \beta, \gamma \rangle \rangle + 2 \langle \langle \alpha, \beta \rangle, \langle \alpha, \gamma \rangle \rangle = 8 \langle \beta, \gamma \rangle .\]

The left hand side

\[= 8(\langle \alpha^4, (\beta \gamma + \gamma \beta) \rangle + \langle (\alpha \beta + \beta \alpha), (\alpha \gamma + \gamma \alpha) \rangle)\]

\[= 16(\alpha^2 \beta \gamma + \alpha^2 \gamma \beta + \beta \gamma \alpha^2 + \gamma \beta \alpha^2 + \alpha \beta \alpha \gamma + \alpha \beta \gamma \alpha + \beta \alpha \gamma + \beta \alpha \gamma + \alpha \gamma \alpha \beta + \gamma \alpha \beta + \gamma \alpha \beta) .\]

The right hand side

\[= 16(\beta \gamma + \gamma \beta) .\]

From (4-2)\(_{\gamma, a}\): \(\gamma = \alpha^2 \gamma + \gamma \alpha^2 + \alpha \gamma \alpha\), we have
\[\gamma \beta = \alpha^2 \gamma \beta + \gamma \alpha^2 \beta + \alpha \gamma \alpha \beta .\]

From (4-2)\(_{\beta, a}\): \(\beta = \alpha^2 \beta + \beta \alpha^2 + \alpha \beta \alpha\), we have
\[\beta \gamma = \alpha^2 \beta \gamma + \beta \alpha^2 \gamma + \alpha \beta \alpha \gamma .\]
Substituting them, we see that it suffices to show
\[ \beta_1 \alpha^2 + \gamma_1 \beta_\gamma + \alpha_\beta \gamma \alpha + \beta_\alpha \gamma \alpha + \alpha_\gamma \beta \alpha + \gamma_1 \alpha \beta \alpha = 0. \]

Now the left hand side of this equation coincides with \( \Theta(\alpha \beta \gamma) \alpha \), which is 0 by (4-3)\( _{a,b,\gamma} \).

The rest of equations can be proved in a similar way. q.e.d.

From now on in this section we assume (3-1) and (3-2). We choose an arbitrary index \( \alpha \), say \( \alpha = 0 \).

By virtue of (4-1)\( _{\alpha} \), each \( P_\alpha \) has the eigenvalues 1, -1 and 0. We decompose the space \( Y \) into the eigenspaces of \( P_\alpha \);

\[ Y = U \bigoplus V \bigoplus Z \]

where \( U, V \) and \( Z \) are the eigenspaces of \( P_\alpha \) for the eigenvalues 1, -1 and 0 respectively. Note that the decomposition (5.2) is orthogonal since \( P_\alpha \) is symmetric and that, by (4-1)\( _{\alpha} \), we have

\[
\begin{align*}
\dim U &= \dim V = m_2, \\
\dim Z &= m_1.
\end{align*}
\]

Now, with respect to orthonormal bases of \( U, V \) and \( W \), \( P_\alpha \) is represented by the matrix;

\[
P_\alpha \sim \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

where 1 denotes the identity matrix of degree \( m_2 \). Similarly, we have

**Lemma 14.** For each \( \alpha > 0 \), \( P_\alpha \) is represented by the following matrix;

\[
P_\alpha \sim \begin{pmatrix}
0 & a_\alpha & b_\alpha \\
a_\alpha' & 0 & c_\alpha \\
b_\alpha' & c_\alpha & 0
\end{pmatrix}
\]

where \( a_\alpha \) is \( m_2 \times m_3 \), \( b_\alpha \) and \( c_\alpha \) are \( m_2 \times m_2 \) and ' indicates the transpose. Further they satisfy

\[
\begin{align*}
(a_\alpha a_\alpha' + 2b_\alpha b_\alpha' = 1, \\
b_\alpha b_\alpha = c_\alpha c_\alpha);
\end{align*}
\]

\[
\begin{align*}
(b_\alpha c_\alpha a_\alpha' + a_\alpha c_\alpha b_\alpha' = 0, \\
c_\alpha b_\alpha a_\alpha + a_\alpha' b_\alpha c_\alpha = 0, \\
c_\alpha' a_\alpha b_\alpha + b_\alpha' a_\alpha c_\alpha = 0.
\end{align*}
\]

Conversely, assume that a matrix of the above form is given and satisfies (5.4), (5.5). Then it satisfies (4-1)\( _{\alpha} \), (4-2)\( _{\alpha} \), and (4-2)\( _{\beta,\alpha} \).
PROOF. Consider \((4-2)_{\alpha,0}\): \[ P_{\alpha} = P_{\alpha}^0P_{\alpha} + P_{\alpha}P_{\alpha}^0 + P_{\alpha}P_0P_0. \]
This gives the required form for \(P_{\alpha}\). Similarly, \((4-2)_{0,\alpha}\):
\[ P_0 = P_0^0P_0 + P_0^0P_0 + P_0P_\alphaP_\alpha \]
gives (5.4). If we assume \((4-2)_{\alpha,0}, (4-2)_{0,\alpha}\), then \((4-1)_{\alpha}\) is equivalent to (5.5). Note that the condition: \(\text{rank } P_{\alpha} = 2m_\alpha\) follows from (5.4) and (5.5). q.e.d.

COROLLARY 1. \((2-11-2)\) holds, i.e., we have \(\Delta\langle p_{\alpha}, p_\beta\rangle = 0\) for each distinct \(\alpha, \beta\).

PROOF. Without loss of generality, we may assume \(\beta = 0\). We have \(\Delta\langle p_{\alpha}, p_\alpha\rangle = 4 \text{Tr} (P_{\alpha}P_{\alpha} + P_{\alpha}P_0)\).
It can be easily verified that \(\text{Tr} (P_{\alpha}P_{\alpha}) = 0\) and \(\text{Tr} (P_{\alpha}P_0) = 0\) for \(\alpha > 0\) using Lemma 14. q.e.d.

Let \(\{u_i\}, \{v_i\}\) and \(\{z_k\}\) be orthonormal coordinate systems for \(U, V\) and \(Z\) respectively. We consider the homogeneous degree with respect to the variables \(z_i, \ldots, z_{m_1}\) for polynomial functions on \(Y\). Let
\[ p_{\alpha} = \sum_{h} p_{\alpha,h}, \quad q_{\alpha} = \sum_{h} q_{\alpha,h} \]
be the decompositions into homogeneous parts with respect to \(z_i, \ldots, z_{m_1}\), where \(h\) indicates the total degree on \(\{z_i\}\).

COROLLARY 2. For each \(\alpha > 0\), we have
(i) \(p_{\alpha,2} = 0\),
(ii) \(\langle p_{\alpha}, p_{\alpha,0}\rangle = 0\).

One can verify them using matricial forms given in Lemma 14.

LEMMA 15. We have, from (3-8) and (3-4),
(i) \(q_{\alpha,3} = 0\) for each \(\alpha\),
(ii) \(q_0\) is homogeneous of degree 1 on \(U, V\) and \(W\).

PROOF. (i) Recall (3-8):
\[ 16\left(\sum_{\alpha} q_{\alpha}^2\right) = 16(\sum y_i^2)G - \langle G, G\rangle \]
where \(G = \sum_{\alpha} p_{\alpha}^2\) and \(\sum y_i^2 = \sum u_i^2 + \sum v_i^2 + \sum z_i^2\). In the equation (3-8),
consider the homogeneous parts of degree 6 with respect to $z_1, \ldots, z_m$.

Since $p_{\alpha,\beta} = 0$, the total degree of $G$ with respect to $z_k$'s is less than 4.

Similarly, the total degree of $\langle G, G \rangle$ with respect to $z_k$'s is less than 6, since $\langle G, G \rangle = 4 \sum \langle p_\alpha, p_\beta \rangle p_\alpha p_\beta$. Thus, we have $\sum q_{\alpha,\beta}^2 = 0$, and hence $q_{\alpha,\beta} = 0$ for each $\alpha$.

(ii) For $\alpha = 0$, (3-4) gives

$$\langle p_0, q_0 \rangle = 0.$$  

Now we have $p_0 = \sum u_i^2 - \sum v_i^2$, and hence

$$\langle p_0, q_0 \rangle = 2 \sum u_i \frac{\partial q_0}{\partial u_i} - 2 \sum v_i \frac{\partial q_0}{\partial v_i}.$$  

If $S$ is homogeneous of degree $k$ and $l$ with respect to $\{u_i\}$ and $\{v_i\}$ respectively, then we have

$$\langle p_0, S \rangle = 2(k - l)S.$$  

Thus, $\langle p_0, q_0 \rangle = 0$ implies that each non zero term of $q_0$ consists of monomials with the same degree on $\{u_i\}$ and $\{v_i\}$. Since $q_0$ is cubic and $q_{0,3} = 0$ by (i), we have (ii).

**Corollary.** (2-12) and (2-6-1) follow from (3-1)~(3-10).

**Proof.** Recall (2-12) $\iff$ (2-12)': $\Delta q_\alpha = 0$ for each $\alpha$. Without loss of generality, we may assume $\alpha = 0$. Then $\Delta q_0 = 0$ follows from (ii) of Lemma 15.

Next, recall (2-6-1): $\langle \langle p_\alpha, p_\beta \rangle, q_\alpha \rangle = 16q_\beta$ for each $\alpha$. Again we may assume $\alpha = 0$ without loss of generality. Since $p_0 = \sum u_i^2 - \sum v_i^2$, we have

$$\langle p_0, p_0 \rangle = 4(\sum u_i^2 + \sum v_i^2).$$  

By (ii) of Lemma 15, $q_0 = q_{0,1}$. Now we have

$$\langle \langle p_0, p_0 \rangle, q_0 \rangle = \langle \langle p_0, p_0 \rangle, q_{0,1} \rangle = 16q_{0,1} = 16q_0.$$  

This proves our corollary.

**Lemma 16.** (2-6-2) follows from (3-1)~(3-10).

**Proof.** Recall (2-6-2): $2\langle \langle p_\alpha, p_\beta \rangle, q_\alpha \rangle + \langle \langle p_\alpha, p_\alpha \rangle, q_\beta \rangle = 16q_\beta$ for each distinct $\alpha, \beta$. Interchanging the indices, it suffices to show

$$2\langle \langle p_0, p_0 \rangle, q_0 \rangle + \langle \langle p_0, p_0 \rangle, q_\alpha \rangle = 16q_\alpha$$  

for $\alpha > 0$. From $\langle p_0, p_0 \rangle = 4(\sum u_i^2 + \sum v_i^2)$, we have

$$\langle \langle p_0, p_0 \rangle, q_{0,1} \rangle = 8(3 - h)q_{0,1}.$$
for any \( h \). Since \( q_{a,3} = 0 \) by (i) of Lemma 15, it suffices now to show
\[
\langle\langle p_0, p_a\rangle, q_0\rangle = 4q_{a,2} - 4q_{a,0}.
\]
By Corollary 2 of Lemma 14, it suffices to show
\[
(\ast)\quad \langle\langle p_0, p_{a,1}\rangle, q_0\rangle = 4q_{a,2} - 4q_{a,0}.
\]
Now we consider the total degree on the variables \( u_1, \ldots, u_{m_2} \). Let
\[
p_{a,1} = s_1 + s_0,
q_{a,0} = f_3 + f_2 + f_1 + f_0,
q_{a,1} = g_2 + g_1 + g_0,
q_{a,2} = h_1 + h_0
\]
be the decompositions into homogeneous parts, where each suffix indicates the total degree on \( u_1, \ldots, u_{m_2} \). Let
\[
p_{a,1} = s_{11} + s_{10},
q_{a,0} = f_{11} + f_{10} + f_{01} + f_{00},
q_{a,1} = g_{11} + g_{10} + g_{01} + g_{00},
q_{a,2} = h_{11} + h_{10} + h_{01} + h_{00}
\]
be the decompositions into homogeneous parts, where each suffix indicates the total degree on \( u_1, \ldots, u_{m_2} \). Recall (3-5). We have
\[
\langle p_0, q_{a}\rangle + \langle p_a, q_0\rangle = 0,
\]
and hence
\[
\langle p_0, q_{a,0}\rangle + \langle p_0, q_{a,1}\rangle + \langle p_0, q_{a,2}\rangle
+ \langle p_{a,0}, q_{0,1}\rangle + \langle p_{a,1}, q_{0,1}\rangle = 0.
\]
Equivalently, we have
\[
\{\langle p_0, q_{a,2}\rangle + \langle p_{a,1}, q_0\rangle_{|u_i, v_j}\}
+ \{\langle p_0, q_{a,1}\rangle + \langle p_{a,0}, q_{0,1}\rangle\}
+ \{\langle p_0, q_{a,0}\rangle + \langle p_{a,1}, q_0\rangle_{|z}\} = 0.
\]
Observing the degree with respect to \( z_1, \ldots, z_{m_1} \) of each term in the above equation, we obtain:
\[
(1)\quad \langle p_0, q_{a,2}\rangle + \langle p_{a,1}, q_0\rangle_{|u_i, v_j} = 0,
(2)\quad \langle p_0, q_{a,1}\rangle + \langle p_{a,0}, q_0\rangle = 0,
(3)\quad \langle p_0, q_{a,0}\rangle + \langle p_{a,1}, q_0\rangle_{|z} = 0.
\]
From \( p_0 = \sum u_i^2 - \sum v_i^2 \), we obtain:
\[
(4)\quad \langle p_0, q_{a,2}\rangle = 2h_1 - 2h_0,
(5)\quad \langle p_0, q_{a,1}\rangle = 4g_1 - 4g_0,
(6)\quad \langle p_0, q_{a,0}\rangle = 2(3f_3 + f_2 - f_1 - 3f_0).
\]
On the other hand, we have
\[
\langle p_{a,1}, q_0\rangle_{|u_i, v_j} = \langle s_0, q_0\rangle_{|u_i, v_j} + \langle s_1, q_0\rangle_{|u_i, v_j}
= \langle s_0, q_0\rangle_{V} + \langle s_1, q_0\rangle_{V}.
\]
Substituting this and (4) into (1), we get

\[
\begin{align*}
2h_i + \langle s_i, q_0 \rangle_v &= 0, \\
2h_0 - \langle s_i, q_0 \rangle_v &= 0.
\end{align*}
\] (7)

Similarly, substituting \( \langle p_{a1}, q_0 \rangle_z = \langle s_i, q_0 \rangle_z + \langle s_0, q_0 \rangle_z \) and (6) into (3), we get

\[
\begin{align*}
\langle f_i = f_0 = 0, \\
2f_z + \langle s_i, q_0 \rangle_z &= 0, \\
2f_1 - \langle s_0, q_0 \rangle_z &= 0.
\end{align*}
\] (8)

Since \( \langle p_0, p_{a1} \rangle = \langle p_0, s_i \rangle + \langle p_0, s_i \rangle = -2s_0 + 2s_i \), (7) and (8) give the required equation (*). q.e.d.

Note that we have completed the proof of Theorem 1.

6. A further characterization. In this section we give a further characterization of \( \{p_a\} \) and \( \{q_a\} \) under an additional condition (A) for a later use. Let \( \{p_a\} \) be \( m_i + 1 \) quadratic forms on \( Y \) satisfying (3-1) and (3-2). With the notations in § 5, we state

**Lemma 17.** The following three conditions are mutually equivalent:

(i) \( \langle p_{\alpha}, p_{\beta} \rangle = 0 \) for distinct \( \alpha, \beta \);

(ii) \( \langle p_{\alpha}, p_{\alpha} \rangle = \langle p_{\beta}, p_{\beta} \rangle \) for distinct \( \alpha, \beta \);

(iii) \( p_{a1} = 0 \) for each \( a \).

**Proof.** As one can see easily, to prove Lemma 17, it suffices to show that, for each \( \alpha > 0 \), the following three conditions are mutually equivalent:

(i)' \( \langle p_0, p_{\alpha} \rangle = 0 \);

(ii)' \( \langle p_{\alpha}, p_{\alpha} \rangle = \langle p_{\alpha}, p_{\alpha} \rangle \);

(iii)' \( p_{a1} = 0 \).

Using Lemma 14, we give matricial representations for \( \langle p_0, p_{\alpha} \rangle, \langle p_{\alpha}, p_{\alpha} \rangle \) and \( p_{a1} \). In the following, the indices for submatrices are omitted. We have

\[
\begin{align*}
\langle p_0, p_{\alpha} \rangle &\sim 2 \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & -c \\ b' & -c' & 0 \end{pmatrix}, \\
\langle p_{\alpha}, p_{\alpha} \rangle &\sim 4 \begin{pmatrix} aa' + bb' & bc' & ac \\ cb' & a' + cc' & a'b' \\ c'a' & b'a & b'b + c'c' \end{pmatrix}.
\end{align*}
\]
Thus, (i)′ ⇒ (iii)′ and (iii)′ ⇒ (ii)′ are clear. Suppose (ii)′. Then \( aa' + bb' = 1 \). Since \( aa' + 2bb' = 1 \) by Lemma 14, we see \( bb' = 0 \), and hence \( b = 0 \). Similarly we have \( c = 0 \). This proves (ii)′ ⇒ (iii)′. q.e.d.

From now on we denote by (A) one of the three conditions in Lemma 17. Now assume that \( \{p_a\} \) satisfy the condition (A) together with (3-1) and (3-2). We remark here that the image and the kernel of \( P_a \) are independent on \( a \) and that the condition (3-3) follows automatically. We put, for each \( a \),

\[
R_a = P_a \mid_{U \oplus V}.
\]

We see that \( R_a \) is a symmetric mapping of \( U \oplus V \) into itself and for \( \alpha = 0 \), \( R_0 \mid_U = 1_U \), \( R_0 \mid_V = -1_V \). Furthermore it is easily seen that the family \( \{R_a\} \) satisfies the following two conditions:

\[
\begin{align*}
(5-1) & \quad R_a^2 = 1_{U \oplus V}, \quad \text{Tr } R_a = 0 \quad \text{for each } \alpha; \\
(5-2) & \quad R_\alpha R_\beta + R_\beta R_\alpha = 0 \quad \text{for distinct } \alpha, \beta.
\end{align*}
\]

Conversely, we have

**Lemma 18.** Let \( \{R_a\} \) be \( m_1 + 1 \) symmetric mappings of \( U \oplus V \) into itself satisfying (5-1) and (5-2). Then we can associate \( m_1 + 1 \) quadratic forms \( \{p_a\} \) on \( Y \) satisfying (3-1), (3-2) and the condition (A) with the relation (6.1) for each \( \alpha \).

**Proof.** For each \( R_a \), we define \( P_a \) by

\[
P_a = \begin{cases} 
R_a & \text{on } U \oplus V \\
0 & \text{on } Z.
\end{cases}
\]

Then \( P_a \) is a symmetric mapping of \( Y = U \oplus V \oplus Z \). Now (5-1) implies (4-1) for each \( \alpha \). From the construction of \( P_a \), it follows that (4-2) is a consequence of (5-2). Let \( p_a \) be the quadratic form on \( Y \) corresponding to \( P_a \). \( \{p_a\} \) satisfy the required conditions. q.e.d.

**Lemma 19.** Assume that \( \{p_a\} \) satisfy (3-1), (3-2) and the condition (A). Let \( \{q_a\} \) be \( m_1 + 1 \) cubic forms on \( Y \). Then (3-3) and (3-6) follow immediately. The conditions (3-8), (3-9) and (3-10) can be written equivalently as

\[
(5-8) \quad \sum q_i^2 = G(\sum z_i^2),
\]
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\[(5-9) \quad \langle q_\alpha, q_\alpha \rangle = G - p_\alpha^2 + 4(\sum u_i^2 + \sum v_i^2)(\sum z_i^2) \quad \text{for each } \alpha ,\]

\[(5-10) \quad \langle q_\alpha, q_\beta \rangle = -p_\alpha p_\beta \quad \text{for distinct } \alpha, \beta\]

respectively.

**Proof.** By Lemma 17, we see that (3-3) and (3-6) follow immediately from (A). For \( G = \sum p_\alpha^2 \), consider \( \langle G, G \rangle \). We have

\[
\langle G, G \rangle = \sum_{\alpha, \beta} \langle p_\alpha, p_\beta \rangle = 4 \sum_{\alpha, \beta} p_\alpha p_\beta \langle p_\alpha, p_\beta \rangle
\]

\[= 4 \sum \alpha p_\alpha^2 \langle p_\alpha, p_\alpha \rangle = 4 \left( \sum p_\alpha^2 \right) \langle p_\alpha, p_\beta \rangle
\]

\[= 16G(\sum u_i^2 + \sum v_i^2) .
\]

This gives (3-8) \( \Rightarrow \) (5-8). Since each \( p_\beta \) is a quadratic form on \( U \oplus V \), we have

\[
\langle \langle p_\alpha, p_\alpha \rangle, p_\beta \rangle = \langle \langle p_\alpha, p_\alpha \rangle, p_\beta \rangle
\]

\[= \langle \langle p_\alpha, p_\alpha \rangle, p_\beta \rangle_{U \oplus V} = 16p_\beta .
\]

Thus, we have

\[
\langle \langle p_\alpha, p_\alpha \rangle, G \rangle = \sum_{\beta} \langle \langle p_\alpha, p_\alpha \rangle, p_\beta \rangle
\]

\[= \sum_{\beta} 2p_\beta \langle \langle p_\alpha, p_\alpha \rangle, p_\beta \rangle = 32G .
\]

This and Lemma 17 give (3-9) \( \Rightarrow \) (5-9). Lemma 17 gives also (3-10) \( \Rightarrow \) (5-10).

By Lemmas 18 and 19, it follows that for a given \( \{R_\alpha\} \) satisfying (5-1) and (5-2), the required conditions for \( \{q_\alpha\} \) in Theorem 1 are now (3-4), (3-5), (3-7), (5-8), (5-9) and (5-10).

For a later use, we give the following lemma.

**Lemma 20.** Let \( \{p_\alpha\} \) be \( m_1 + 1 \) quadratic forms on \( Y \) satisfying (3-1), (3-2) and (A). Then \( p_\alpha, \ldots, p_{m_1} \), are algebraically independent over \( R \).

**Proof.** First we prove that \( p_\alpha, \ldots, p_{m_1} \) are linearly independent over \( R \). Suppose \( \sum a_\alpha p_\alpha = 0, a_\alpha \in R \). We have for any \( \beta \),

\[
\langle p_\beta, \sum a_\alpha p_\alpha \rangle = a_\beta \langle p_\beta, p_\beta \rangle ,
\]

and hence \( a_\beta = 0 \). Next suppose

\[
\sum a_{i_0\ldots i_m} p_{i_0}^{i_0} \ldots p_{i_m}^{i_m} = 0 .
\]

Since each \( p_\alpha \) is a quadratic form, we have
for each \( l \). We shall show \( a_{i_0 \ldots i_{m_1}} = 0 \) for all \( i_0, \ldots, i_{m_1} \). This will be shown by the induction on \( l = i_0 + \cdots + i_{m_1} \). The case \( l = 1 \) has been proved. For each \( \beta \), we have
\[
\langle p_{\beta}, \sum a_{i_0 \ldots i_{m_1}} p_{i_0}^0 \cdots p_{i_{m_1}}^1 \rangle = \sum i_{\beta} a_{i_0 \ldots i_{m_1}} p_{i_0}^0 \cdots p_{i_{m_1}}^1 = 0.
\]
Using this, one can complete easily the proof. q.e.d.

7. Representations of a Clifford algebra. In this section we prove certain lemmas concerning representations of a Clifford algebra for a later use.

Let \( F \) be an associative division algebra over \( \mathbb{R} \), i.e., \( F = \mathbb{R}, \mathbb{C} \) or the real quaternion algebra \( \mathbb{H} \). We denote by \( M_m(F) \) the algebra of all \( m \times m \) matrices with coefficients in \( F \), and by \( 1_m \) the unit matrix in \( M_m(F) \). \( M_m(F) \) is called the total matrix algebra over \( F \) of degree \( m \).

For each non-negative integer \( \kappa \), we denote by \( C_\kappa \) the Clifford algebra over \( \mathbb{R} \) associated to the negative definite quadratic form \( -<,> \) on \( \mathbb{R}^\kappa \), where \( <,> \) is the usual inner product on \( \mathbb{R}^\kappa \). Let \( \{e_1, \ldots, e_\kappa\} \) be an orthonormal base for \( \mathbb{R}^\kappa \) with respect to \( <,> \). Then \( C_\kappa \) is the associative algebra over \( \mathbb{R} \) with the unit 1 generated by \( e_1, \ldots, e_\kappa \) with the relations:
\[
\begin{align*}
e_k^2 &= -1 & \text{ for each } k, \\
e_k e_l + e_l e_k &= 0 & \text{ for each distinct } k, l,
\end{align*}
\]
and \( \{1, e_1 \cdots e_\kappa; k_1 < \cdots < k_r, 1 \leq r \leq \kappa\} \) forms a basis of the underlying vector space of \( C_\kappa \), and hence \( \dim C_\kappa = 2^\kappa \). We denote by \( x \rightarrow x^* \) the canonical involution of \( C_\kappa \), that is, the anti-automorphism of \( C_\kappa \) satisfying \( e_k = -e_k \) for each \( k \). A homomorphism
\[
\rho: C_\kappa \rightarrow M_m(\mathbb{R}) \text{ with } \rho(1) = 1_m
\]
is called a representation of \( C_\kappa \) of degree \( m \). Two representations \( \rho, \tilde{\rho} \) of \( C_\kappa \) of degree \( m \) are said to be equivalent if there exists \( A \in GL(m, \mathbb{R}) \) such that \( \tilde{\rho}(x) = A \rho(x) A^{-1} \) for each \( x \in C_\kappa \). The set of equivalence classes of representations of \( C_\kappa \) of degree \( m \) will be denoted by \( \mathcal{R}_m(C_\kappa) \).

We consider a representation \( \rho \) of \( C_\kappa \) of degree \( m \) satisfying
\[
(7.1) \quad \rho(x^*) = \rho(x)', \quad \text{ for each } x \in C_\kappa,
\]
where ' indicates the transpose of a matrix. Two representations \( \rho, \tilde{\rho} \) of \( C_\kappa \) satisfying (7.1) are said to be orthogonally equivalent if there exists
\[ \sigma \in O(m) \text{ such that } \tilde{\sigma}(x) = \sigma \rho(x) \sigma^{-1} \text{ for each } x \in C_\kappa. \]

The set of orthogonal equivalence classes of representations of \( C_\kappa \) of degree \( m \) satisfying (7.1) will be denoted by \( \mathcal{R}_m(\kappa, \ast) \).

**Lemma 21.** The natural map:
\[ \mathcal{R}_m(\kappa, \ast) \rightarrow \mathcal{R}_m(\kappa) \]
is a bijection.

**Proof.** The bracket operation \([x, y] = xy - yx\) on \( C_\kappa \) defines a Lie algebra over \( \mathbb{R} \), which is denoted by \( g \). Since \( C_\kappa \) is a semi-simple algebra over \( \mathbb{R} \), it is the direct sum of a finite number of total matrix algebras. It follows that \( g \) has a natural structure of reductive algebraic Lie algebra over \( \mathbb{R} \).

Now the canonical involution \( x \rightarrow x^* \) of \( C_\kappa \) is a positive involution in the sense that the symmetric bilinear form \( Tr(L_{xy}) \) on \( C_\kappa \) is positive definite, \( L_x \) being the left regular representation of \( C_\kappa \): \( L_x y = xy \).

In fact, for \( x_0 = e_{i_1} \cdots e_{i_r} \), \( y_0 = e_{j_1} \cdots e_{j_s} \), we have
\[ x_0 y_0^* = \begin{cases} 1 & r = s, \{i_1, \ldots, i_r\} = \{j_1, \ldots, j_s\} \\ \pm e_{k_1} \cdots e_{k_t}, t > 0 & \text{otherwise} \end{cases} \]
where
\[ \{k_1, \ldots, k_t\} = \{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_s\} - \{i_1, \ldots, i_r\} \cap \{j_1, \ldots, j_s\} \]
Thus we have
\[ Tr(L_{x_0 y_0^*}) = \begin{cases} \dim C_\kappa = 2^r > 0 & r = s, \{i_1, \ldots, i_r\} = \{j_1, \ldots, j_s\} \\ 0 & \text{otherwise} \end{cases} \]
and hence \( Tr(L_{xy^*}) \) is positive definite on \( C_\kappa \). Thus, by a theorem of Weil [8], the map \( \theta \) of \( g \) defined by \( x \rightarrow x^* \) is a Cartan involution of \( g \).

We shall show first the surjectivity. Let \( \rho \) be a representation of \( C_\kappa \) of degree \( m \). Then the representation
\[ \rho: g \rightarrow gl(m, \mathbb{R}) \]
is completely reducible. Hence there exists a Cartan involution \( \theta_0 \) of \( gl(m, \mathbb{R}) \) such that
\[ \theta_0(\rho(x)) = \rho(\theta(x)) \quad \text{for each } x \in g. \]
\( \theta_0 \) can be expressed as
\[ \theta_0(X) = -P^{-1}X'P \quad \text{for } X \in gl(m, \mathbb{R}) \]

* The proof of surjectivity is due to I. Satake.
by a positive definite symmetric matrix $P \in M_m(R)$. Thus we have

$$\rho(x^*) = P^{-1}\rho(x)'P$$

for $x \in C_\varepsilon$.

Put $A = P^{1/2}$ and

$$\tilde{\rho}(x) = A\rho(x)A^{-1}$$

for $x \in C_\varepsilon$.

Then we have for each $x \in C_\varepsilon$

$$\tilde{\rho}(x^*) = A\rho(x^*)A^{-1} = AP^{-1}\rho(x)'PA^{-1}$$

$$= A'^{-1}\rho(x)'A' = \tilde{\rho}(x)' ,$$

and hence $\tilde{\rho}$ satisfies (7.1). This proves the surjectivity of the map.

To prove the injectivity, let $\rho$ and $\tilde{\rho}$ be mutually equivalent representations of $C_\varepsilon$ satisfying (7.1). Let $A \in GL(m, R)$ such that

(7.2) $$\tilde{\rho}(x) = A\rho(x)A^{-1}$$

for $x \in C_\varepsilon$.

Then we have $\tilde{\rho}(x^*) = A\rho(x^*)A^{-1}$ for each $x \in C_\varepsilon$. From the condition (7.1) we have $\tilde{\rho}(x)' = A\rho(x)'A^{-1}$ and hence

(7.3) $$\tilde{\rho}(x) = A'^{-1}\rho(x)A'$$

for $x \in C_\varepsilon$.

(7.2) and (7.3) imply that the symmetric matrix $A'A$ commutes with each $\rho(x)$. Now write $A$ as the product: $A = \sigma P$ of $\sigma \in O(m)$ and a positive definite symmetric matrix $P$. Then $A'A = P^2$ commutes with each $\rho(e_i)$. From the condition (7.1), $\tau_i = \exp t\rho(e_i)$ is in $O(m)$ for each $t \in R$, and hence $\tau, P\tau^{-1}$ is also a positive definite symmetric matrix. It follows from $\tau_iP^{\tau_i^{-1}} = (\tau_iP\tau_i^{-1})^t = P^2$ that each $\tau_i$ commutes with $P$ and hence each $\rho(e_i)$ commutes with $P$. Since $C_\varepsilon$ is generated by $e_1, \cdots, e_\varepsilon$, we have

$$\tilde{\rho}(x) = \sigma\rho(x)\sigma^{-1}$$

for $x \in C_\varepsilon$.

Thus $\rho$ and $\tilde{\rho}$ are orthogonally equivalent.

q.e.d.

The subspace of $C_\varepsilon$ spanned by $e_1, \cdots, e_\varepsilon$ is identified with $R^\varepsilon$ in a natural way, and any orthogonal transformation $\sigma$ of $R^\varepsilon(\sigma \in O(\varepsilon))$ is extended uniquely to an automorphism $\sigma$ of $C_\varepsilon$. For a representation $\rho$ of $C_\varepsilon$ of degree $m$, we define another representation $\sigma\rho$ by

$$(\sigma\rho)(x) = \rho(\sigma^{-1}x)$$

for $x \in C_\varepsilon$.

If $\rho$ satisfies (7.1), then $\sigma\rho$ also satisfies (7.1), since the automorphism $\sigma$ of $C_\varepsilon$ commutes with the canonical involution $x \rightarrow x^*$. The correspondence $(\sigma, \rho) \rightarrow \sigma\rho$ gives an action of $O(\varepsilon)$ on $R_m(C_\varepsilon)$ and on $R_m(C_\varepsilon, *)$. Let $O(\varepsilon)\backslash R_m(C_\varepsilon)$ and $O(\varepsilon)\backslash R_m(C_\varepsilon, *)$ denote the spaces of $O(\varepsilon)$-orbits respectively. Since the natural map $R_m(C_\varepsilon, *) \rightarrow R_m(C_\varepsilon)$ is $O(\varepsilon)$-equivariant, Lemma 21 gives us the natural bijection
We cite Atiyah-Bott-Shapiro [1]: We have an isomorphism

\[ C_{\kappa+8} \cong C_{\kappa} \otimes M_{\alpha}(R), \]

and the Clifford algebras \( C_{\kappa}'s \) for \( \kappa \leq 8 \) are given by the following table:

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>( C_{\kappa} )</th>
<th>( d(\kappa) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( C )</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>( H )</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>( H \oplus H )</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>( M_{4}(H) )</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>( M_{4}(C) )</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>( M_{8}(R) )</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>( M_{8}(R) \oplus M_{8}(R) )</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>( M_{16}(R) )</td>
<td>16</td>
</tr>
</tbody>
</table>

where \( d(\kappa) \) denotes the degree of irreducible representations of \( C_{\kappa} \). We have

\[ d(\kappa + 8) = 16d(\kappa) \]
in virtue of the isomorphism (7.4).

**Lemma 22.** For \( \kappa \geq 1 \), \( O(\kappa) \backslash R_{\kappa+1}(C_{\kappa}, *) \) is not empty if and only if \( \kappa = 1, 3 \) or 7. For \( \kappa = 1, 3 \) or 7, \( O(\kappa) \backslash R_{\kappa+1}(C_{\kappa}, *) \) consists of exactly one element, represented by an irreducible representation of \( C_{\kappa} \).

**Proof.** By Lemma 21, it suffices to show the above for the set \( O(\kappa) \backslash R_{\kappa+1}(C_{\kappa}) \). From (7.5) we have

\[
d(\kappa + 8) - (\kappa + 8) = 16d(\kappa) - \kappa - 8
= (15d(\kappa) - 8) + (d(\kappa) - \kappa) > d(\kappa) - \kappa.
\]

It follows that if \( R_{\kappa+1}(C_{\kappa}) \) is not empty, then \( \kappa \leq 8 \) and \( R_{\kappa+1}(C_{\kappa}) \) consists of equivalent classes of irreducible representations. From the table cited above we get the first assertion of Lemma 22.

In case \( \kappa = 1 \), \( C_{1} = C \) and \( R_{4}(C_{1}) \) consists of just one class. In case \( \kappa = 3 \), \( C_{3} = H \oplus H \) and \( R_{8}(C_{3}) \) consists of two classes. Putting \( z = e_{1}e_{2}e_{3} \) in \( C_{8} \), we define \( f_{+}, f_{-} \in C_{8} \) by

\[
f_{+} = \frac{1}{2}(1 + z), \quad f_{-} = \frac{1}{2}(1 - z).
\]
Then they are primitive idempotents of $C_{\gamma}$ defining the decomposition $C_{\gamma} = H \oplus H$. Since $-1_{\gamma} \in O(3)$ transforms $f_{\gamma}$ into $f_{-\gamma}$, $O(3) \setminus \mathcal{K}(C_{\gamma})$ consists exactly one element. In case $\gamma = 7$, we see similarly that $O(7) \setminus \mathcal{K}(C_{\gamma})$ consists exactly one element, making use of the element $z = e_{\gamma}e_{2} \cdots e_{7} \in C_{\gamma}$.

For $\gamma = 1, 3, 7$, we have $C_{\gamma-1} \cong R, H, M_{9}(R)$ respectively. Hence we have

**Lemma 23.** For $\gamma = 1, 3, 7$, the set $\mathcal{K}(C_{\gamma-1}, \ast)$ is not empty if and only if $m$ is a multiple of $1, 4, 8$ respectively. In these cases, $\mathcal{K}(C_{\gamma-1}, \ast)$ consists exactly one class.

Now, let $\kappa, m$ be positive integers. Consider a family $\{a_{k}\}_{1 \leq k \leq \kappa}$ of $\kappa$ matrices in $M_{m}(R)$ satisfying the following condition:

\begin{align}
& \left\{ \begin{array}{l}
  a'_{k}a_{k} = 1_{m} \\
  a'_{k}a_{l} + a'_{l}a_{k} = 0
\end{array} \right. \quad \text{for each } k, l \; \text{distinct,}
\end{align}

Two such families $\{a_{k}\}, \{\bar{a}_{k}\}$ are said to be equivalent and denoted by $\{a_{k}\} \sim \{\bar{a}_{k}\}$ if there exist $\sigma, \tau \in O(m)$ such that

$$
\bar{a}_{k} = \sigma a_{k} \tau^{-1}
$$

for each $k$. They are classified in terms of representations of Clifford algebras as follows.

**Lemma 24.** The set of equivalence classes of families $\{a_{k}\}$ of $\kappa$ matrices in $M_{m}(R)$ satisfying the condition (7.6) is in a bijective correspondence with the set $\mathcal{K}(C_{\gamma-1}, \ast)$.

**Proof.** Let $\rho$ be a representation of $C_{\gamma-1}$ of degree $m$ satisfying (7.1). We define $\kappa$ matrices $a_{1}, \ldots, a_{\kappa}$ by

$$
\left\{ \begin{array}{l}
  a_{k} = \rho(e_{k}) \quad 1 \leq k \leq \kappa - 1, \\
  a_{\kappa} = 1_{m}.
\end{array} \right.
$$

Since we have

$$
\left\{ \begin{array}{l}
  a'_{k} = -a_{k}, a'_{\kappa} = -1_{m} \quad \text{for each } k, 1 \leq k \leq \kappa - 1, \\
  a_{k}a_{l} + a_{l}a_{k} = 0 \quad \text{for distinct } k, l, 1 \leq k, l \leq \kappa - 1,
\end{array} \right.
$$

the family $\{a_{k}\}$ satisfies the condition (7.6). The correspondence $\rho \mapsto \{a_{k}\}$ induces a map of $\mathcal{K}(C_{\gamma-1}, \ast)$ into the set of equivalence classes of families $\{a_{k}\}$ satisfying (7.6). One can see easily that it is bijective. \text{q.e.d.}

Next, consider a family $\{A_{k}\}_{1 \leq k \leq \kappa}$ of $\kappa$ matrices in $M_{m}(R)$ satisfying the following condition:
\[ \begin{align*}
A_k^2 &= -A_k, \quad A_k^2 = -1_m \quad \text{for each } k, \\
A_k A_l + A_l A_k &= 0 \quad \text{for distinct } k, l.
\end{align*} \]

Note that the condition (7.7) implies the condition (7.6). Two such families \{A_k\}, \{\tilde{A}_k\} are said to be equivalent and denoted by \{A_k\} \approx \{\tilde{A}_k\} if there exist \(\sigma \in O(m)\) and \(\tau = (\tau_k) \in O(k)\) such that

\[ \tilde{A}_k = \sum_{i=1}^k \tau_{ki}(\sigma A_i \sigma^{-1}) \quad \text{for each } k. \]

They are also classified in terms of representations of Clifford algebras as follows.

**Lemma 25.** The set of equivalence classes of families \{A_k\} of \(\kappa\) matrices in \(M_\kappa(R)\) satisfying the condition (7.7) is in a bijective correspondence with the set \(O(\kappa) \backslash R_m(C_\kappa, *)\).

**Proof.** For each representation \(\rho\) of \(C_\kappa\) of degree \(m\) satisfying (7.1), we define \(\kappa\) matrices \(A_1, \ldots, A_k\) by

\[ A_k = \rho(e_k) \quad \text{for each } k. \]

Then the family \{A_k\} satisfies the condition (7.7). The correspondence \(\rho \rightarrow \{A_k\}\) induces a bijection required in our lemma. q.e.d.

From Lemmas 22 ~ 25, we have

**Lemma 26.** There exists a family \{A_k\} of \(\kappa\) matrices in \(M_{\kappa_1}(R)\) satisfying the condition (7.7) if and only if \(\kappa = 1, 3, 7\). For \(\kappa = 1, 3, 7\), there exists a family \{a_k\} of \(\kappa\) matrices in \(M_\kappa(R)\) satisfying the condition (7.6) if and only if \(m\) is a multiple of 1, 4, 8 respectively. In these cases, both of equivalence classes of \{A_k\} and \{a_k\} are unique.

8. Examples of non-homogeneous isoparametric hypersurfaces. Now we come back to families of quadratic forms \{p_\alpha\} and cubic forms \{q_\alpha\} on \(Y = R^*\). In this section we shall classify polynomials \{p_\alpha\}, \{q_\alpha\} under certain conditions and construct two series of non-homogeneous isoparametric hypersurfaces.

As in §5, let

\[ Y = U \oplus V \oplus Z \]

be the eigenspace decomposition of the symmetric mapping \(P_\alpha\) corresponding to \(p_\alpha\), where \(U, V\) and \(Z\) are the eigenspaces for the eigenvalues 1, \(-1\) and 0 respectively. Recall \(\dim U = \dim V = m_2\) and \(\dim Z = m_1\). We choose orthonormal coordinate systems \{u_i\}, \{v_i\} and \{z_k\} for \(U, V\) and \(Z\) respectively. Each symmetric mapping \(P_k\) corresponding to \(p_k\) for \(k \geq 1\) will be represented by a matrix with respect to these coordinates.
LEMMA 27. Assume that $P_0$ is represented in the above way. Then the family $\{p_a\}$ satisfies (3-1), (3-2) and the condition (A) if and only if (1) each $P_k (1 \leq k \leq m_1)$ is represented by a matrix of the form

\[
\begin{pmatrix}
0 & a_k & 0 \\
\end{pmatrix}
\]

with $a_k \in M_{m_2}(R)$ and (2) the family $\{a_k\}$ satisfies the condition (7.6) for $\kappa = m_1$ and $m = m_2$.

PROOF. First suppose $\{p_a\}$ satisfies (3-1), (3-2) and (A). Then the family $\{R_a\}$ of symmetric mappings of $U \oplus V$ associated to $\{p_a\}$ in §6 satisfies (5-1) and (5-2). The condition (5-2) for $a = 0$ and $\beta = k$ implies that $R_k$ is represented by a matrix of the form

\[
\begin{pmatrix}
0 & a_k \\
a_k' & 0 \\
0 & 0
\end{pmatrix}
\]

with $a_k \in M_{m_2}(R)$. Now (5-1) gives

(i) $a_k a'_k = a'_k a_k = 1_{m_2}$ for each $k$,

and also (5-2) gives

(ii) \[
\begin{cases}
\begin{array}{c}
a_k a'_i + a'_i a_k = 0 \\
a_k' a_l + a'_l a_k = 0
\end{array}
\end{cases}
\]

for distinct $k, l$ where $1 \leq k, l \leq m_1$. (i) and (ii) together are equivalent to the condition (7.6), thereby obtaining (1) and (2) of Lemma 27.

The converse follows from the above argument and Lemma 18.

q.e.d.

Now let $\{p_a\}$ be a family of quadratic forms on $Y$ satisfying (3-1), (3-2) and (A), and let $\{q_a\}$ be a family of cubic forms on $Y$. We assume the following additional condition:

(B) For each $\alpha$, $q_\alpha$ is expressed as

\[
q_\alpha = \sum_\beta \lambda_{\alpha \beta} p_\beta
\]

where $\lambda_{\alpha \beta}$'s are linear forms on $Z$.

First note that the above expression of $q_\alpha$ is unique by virtue of Lemma 20. We put

\[
\lambda_{\alpha \beta} = \sum_{k=1}^{m_1} a_{\alpha \beta k} z_k
\]
for each \( \alpha, \beta \), and define \( m \) matrices \( A_1, \ldots, A_m \) in \( M_{m+1}(R) \) by

\[
A_k = (a_{\alpha \beta k})_{0 \leq \alpha, \beta \leq m}
\]

for each \( k, 1 \leq k \leq m \).

**Lemma 28.** As in the above, suppose that \( \{p_a\} \) and \( \{q_a\} \) satisfy (3-1) and (3-2) together with (A) and (B). Then, \( \{p_a\} \) and \( \{q_a\} \) satisfy the conditions (3-4), (3-5), (3-7), (5-8), (5-9) and (5-10) if and only if the family \( \{A_k\} \) of \( m \) matrices in \( M_{m+1}(R) \) satisfies the condition (7.7) and the following condition:

\[
\frac{1}{2} \sum_k \left( a_{\alpha \beta k}a_{\beta \gamma k} + a_{\alpha \gamma k}a_{\beta \delta k} \right) = \delta_{\alpha \delta} \delta_{\gamma \delta}
\]

for each \( \alpha, \beta, \gamma, \delta \) with \( \{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset \).

**Proof.** Note that the above condition (8.3) is equivalent to the following two conditions:

\[
\begin{align*}
A &+ A' = 0 \quad \text{for each } k; \\
A_k' A_k + A_k A_k' &+ 0 \quad \text{for distinct } k, l.
\end{align*}
\]

First we show the following implications: (3-7) \( \iff \) (7.7.1); (7.7.1) \( \iff \) (3-4) and (3-5); and then (5-8) \( \iff \) (7.7.2).

Recall (3-7): \( \sum p_a q_a = 0 \). We have

\[
\sum_a p_a q_a = \sum_{\alpha, \beta} \lambda_{\alpha \beta} p_a p_{\beta} = \frac{1}{2} \sum_k \left\{ \sum_{\alpha, \beta} (a_{\alpha \beta k} + a_{\alpha \beta k}) p_a p_{\beta} \right\} z_k.
\]

By Lemma 20, we see (3-7) \( \iff \) (7.7.1). Since each \( \lambda_{\beta \gamma} \) is a linear form on \( Z \), we have \( \langle p_a, \lambda_{\beta \gamma} \rangle = 0 \). Thus, we have

\[
\langle p_a, q_\beta \rangle = \sum_\gamma \lambda_{\beta \gamma} \langle p_a, p_\gamma \rangle = \lambda_{\beta \gamma} \langle p_a, p_\gamma \rangle.
\]

using Lemma 17. Therefore we can write

\[
\langle p_a, q_\beta \rangle + \langle p_\beta, q_a \rangle = (\lambda_{\alpha \beta} + \lambda_{\alpha \beta}) \langle p_a, p_\gamma \rangle.
\]

This shows (7.7.1) \( \iff \) (3-4) and (3-5). Recall (5-8): \( \sum q^*_a = G(\sum x^*_i) \). We have
\[ \sum_{\alpha} q_{\alpha}^2 = \sum_{\alpha} \left( \sum_{\beta} \lambda_{\alpha \beta} p_{\beta} \right)^2 = \sum_{\alpha, \beta, \gamma} \lambda_{\alpha \beta} \lambda_{\alpha \gamma} p_{\beta} p_{\gamma} \]

\[ = \frac{1}{2} \sum_{\alpha, \beta, \gamma, k} (a_{\alpha \beta} a_{\alpha \gamma} + a_{\alpha \beta} (a_{\alpha \gamma}) p_{\beta} p_{\gamma} x_k z_i , \]

and

\[ G(\sum_{k} z_k^2) = \left( \sum_{k} z_k^2 \right) \left( \sum_{\alpha} p_{\alpha}^2 \right) . \]

Now (5-8) is equivalent to

\[ \begin{align*}
\sum_{\alpha, \beta, \gamma, k} a_{\alpha \beta} a_{\alpha \gamma} p_{\beta} p_{\gamma} &= \sum_{\beta} p_{\beta}^2 \quad \text{for each } k , \\
\sum_{\alpha, \beta, \gamma, k} (a_{\alpha \beta} a_{\alpha \gamma} + a_{\alpha \beta} (a_{\alpha \gamma}) p_{\beta} p_{\gamma} &= 0 \quad \text{for distinct } k, l ,
\end{align*} \]

which is, by Lemma 20, equivalent to

\[ \begin{align*}
\sum_{\alpha} a_{\alpha \beta} a_{\alpha \gamma} &= \delta_{\beta \gamma} \quad \text{for each } \beta, \gamma, k , \\
\sum_{\alpha} (a_{\alpha \beta} a_{\alpha \gamma} + a_{\alpha \beta} (a_{\alpha \gamma}) &= 0 \quad \text{for each } \beta, \gamma \text{ and distinct } k, l .
\end{align*} \]

This is nothing but (7.7.2), thereby obtaining the implications described first.

Henceforth we assume the condition (7.7). Consider the condition (5-9). We have

\[ \langle q_{\alpha}, q_{\alpha} \rangle = \left( \sum_{\beta} \lambda_{\alpha \beta} p_{\beta} \right) \left( \sum_{\gamma} \lambda_{\alpha \gamma} p_{\gamma} \right) \]

\[ = \sum_{\beta, \gamma} \lambda_{\alpha \beta} \lambda_{\alpha \gamma} p_{\beta} p_{\gamma} + \sum_{\beta, \gamma} \lambda_{\alpha \beta} \lambda_{\alpha \gamma} \langle p_{\beta}, p_{\gamma} \rangle \]

\[ = \sum_{\beta, \gamma, k} a_{\alpha \beta} a_{\alpha \gamma} p_{\beta} p_{\gamma} + 4(\sum u_i^2 + \sum v_i^2) \sum_{\beta, \gamma, k} a_{\alpha \beta} a_{\alpha \gamma} x_k z_i , \]

and

\[ G - p_{\alpha}^2 + 4(\sum u_i^2 + \sum v_i^2)(\sum z_i^2) \]

\[ = \sum_{\alpha, \beta, \gamma} p_{\beta}^2 + 4(\sum u_i^2 + \sum v_i^2)(\sum z_i^2) . \]

Again by Lemma 20, we see that (5.9) is equivalent to the following three conditions:

(i) \[ \sum_{k} a_{\alpha \beta} a_{\gamma k} = \delta_{\beta \gamma} \quad \text{for each } \alpha, \beta, \gamma \text{ with } \beta \neq \alpha, \gamma \neq \alpha ; \]

(ii) \[ \sum_{k} a_{\alpha \beta} a_{\alpha k} = 0 \quad \text{for each } \alpha ; \]

(iii) \[ \sum_{\beta} a_{\alpha \beta} a_{\beta \alpha} = \delta_{\alpha \alpha} \quad \text{for each } \alpha, \beta, \gamma . \]

Since (ii) and (iii) follow from (7.7), we have (5-9) \( \Rightarrow \) (i) = (8.3.1). By a similar computation, we can see (5-10) \( \Rightarrow \) (8.3.2) and (8.3) \( \Rightarrow \) (5-10).

q.e.d.
Now we recall some properties of inner products on division algebras over $\mathbb{R}$. Let $F$ be a (not necessarily associative) division algebra over $\mathbb{R}$, i.e., $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or the real Cayley algebra $\mathbb{K}$. Let $c_0 = 1, c_1, \ldots, c_{d-1}$ be the standard units of $F$ with $d = \dim F$. $u \rightarrow \bar{u}$ denotes the canonical involution of $F$. We put $\mathfrak{F} = \{u \in F | \bar{u} = -u \}$. Then $\mathfrak{F}$ is a $(d-1)$-dimensional subspace of $F$ spanned by $c_1, \ldots, c_{d-1}$. The subspace $\mathfrak{K} = \{u \in F | \bar{u} = u \}$ will be identified with $\mathbb{R}$ in a natural way. On $F$,

$$(u, v) = \frac{1}{2}(uv + vu)$$

defines an inner product with the following properties:

$$(\bar{u}, \bar{v}) = (u, v),$$

$$(uv, w) = (v, \bar{u}w) = (u, \bar{w}v),$$

$$\bar{u}(vw) + v(\bar{u}w) = (wv)\bar{v} + (w\bar{v})u = 2(u, v)w.$$ 

$\{c_0, c_1, \ldots, c_{d-1}\}$ forms an orthonormal basis of $F$ with respect to the above inner product. The dual base $\{u_0, u_1, \ldots, u_{d-1}\}$ of $\{c_0, c_1, \ldots, c_{d-1}\}$ forms an orthonormal coordinate system for $F$, which we call standard. $(, )$ is extended to the $m$-column vector space $F^m$ by

$$(u, v) = \frac{1}{2}(u'\bar{v} + v'\bar{u})$$

for $u, v \in F^m$, where $'$ denotes the transpose. The standard orthonormal coordinate system for $F^m$ consists of $\{u^{(i)} | 0 \leq i \leq d - 1, 1 \leq \lambda \leq m \}$ where $\{u^{(i)} | 0 \leq i \leq d - 1 \}$ denotes the standard orthonormal coordinates for the $\lambda$-th component $u^{(i)}$ of $u \in F^m$. We write also $\|u\|$ for the norm $(u, u)^{1/2}$ of a vector $u$.

**Theorem 2.** Let $m_1$ and $m_2$ be positive integers such that $N = 2(m_1 + m_2 + 1)$, and set $n = m_1 + 2m_2$.

(i) There exist $m_1 + 1$ quadratic forms $\{p_a\}$ and $m_1 + 1$ cubic forms $\{q_a\}$ on $Y = \mathbb{R}^n$ satisfying the equations (3-1) $\sim$ (3-10) together with the conditions (A) and (B) if and only if the pair $(m_1, m_2)$ is one of the following three types: $(1, r), (3, 4r), (7, 8r)$ for some positive integer $r$. In these cases, the polynomial $F$ associated to such $\{p_a, q_a\}$ is unique up to (ON)-equivalence.

(ii) The polynomial $F$ on $\mathbb{R}^n$ associated to such $\{p_a, q_a\}$ is given explicitly as follows:

(a) $(m_1, m_2) = (1, r)$; We define a polynomial $F_0$ on $\mathbb{R}^{(r+2)} = C^{r+2}$ by
and set $F = r^4 - 2F_0$.

(b) $(m_1, m_2) = (3, 4r)$ or $(7, 8r)$; $F$ denotes $H$ or $K$ according to $m_1 = 3$ or 7. We define a polynomial $F_0$ on $R^N = F^{2(r+1)} = F^{r+1} \times F^{r+1}$ by

$$F_0(u \times v) = 4(||u'v||^2 - (u, v)^2) + (||v||^2 - ||v||^2 + 2(u_0, v_0))^2$$

for

$$u = (u_0, u_1), \quad v = (v_0, v_1), \quad u_0, v_0 \in F, \quad u_1, v_1 \in F^r,$$

and set $F = r^4 - 2F_0$.

In each case, $F$ satisfies the differential equations (M) of Münzner.

**Remark.** Takagi-Takahashi [7] gave the multiplicities of principal curvatures for homogeneous isoparametric hypersurfaces in spheres. Our pairs $(m_1, m_2)$ of multiplicities in the case (b) do not appear in their table except $(m_1, m_2) = (3, 4)$. Hence our isoparametric hypersurfaces given in the above case (b) are not homogeneous, possibly except the case where $(m_1, m_2) = (3, 4)$. However, in Part II it will be shown that our isoparametric hypersurfaces for $(m_1, m_2) = (3, 4)$ are also non-homogeneous.

**Proof of (i).** The “only if” part follows immediately from Lemmas 26, 27, 28. Conversely, assume that $(m_1, m_2)$ is $(1, r), (3, 4r)$ or $(7, 8r)$. Let $F = C, H$ or $K$ respectively, so that $\dim F = m_1 + 1$. In the following, indices $k, l, \cdots$ and $\alpha, \beta, \cdots$ run through $1, 2, \cdots, m_1$ and $0, 1, \cdots, m_1$ respectively. For $u, v \in F$ we have

$$(c_k u, v) = (u, \bar{c}_k v) = -(c_k v, u) \quad \text{for each } k$$

$$c_k(c_k u) = -\bar{c}_k(c_k u) = -(c_k, c_k)u = -u \quad \text{for each } k$$

$$c_k(c_k u) + c_l(c_k u) = -\bar{c}_k(c_k u) - \bar{c}_l(c_k u) = -2(c_k, c_l)u = 0$$

for distinct $k, l$.

We define $A_1, \cdots, A_{m_1} \in M_{m_1+1}(R)$ by

$$A_k = (a_{\alpha\beta})_{0 \leq \alpha, \beta \leq m_1} \quad \text{with} \quad a_{\alpha\beta} = (c_k c_{\alpha}, c_\beta)$$

for each $k$. Then $\{A_k\}$ satisfy (7.7) as is easily seen from the above properties. Consider (8.3). For each $\alpha, \beta, \gamma, \delta$ with $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$, we have
\[
\sum_k (a_{\alpha\gamma}a_{\beta\delta} + a_{\alpha\delta}a_{\beta\gamma}) \\
= \sum_i (c_{\alpha}c_{\beta}, c_{\gamma}c_{\delta}) + \sum_i (c_{\alpha}c_{\gamma}, c_{\beta}c_{\delta}) \\
= \sum_i (c_{\alpha}c_{\beta}, c_{\gamma}c_{\delta}) + \sum_i (c_{\alpha}c_{\gamma}, c_{\beta}c_{\delta}) \\
= \sum_i (c_{\alpha}c_{\beta}, c_{\gamma}c_{\delta}) + \sum_i (c_{\alpha}c_{\gamma}, c_{\beta}c_{\delta}) \\
= (c_{\alpha}c_{\beta}, c_{\gamma}c_{\delta}) + (c_{\alpha}c_{\gamma}, c_{\beta}c_{\delta}) \\
= 2(c_{\alpha}, c_{\beta})(c_{\gamma}, c_{\delta}),
\]
and hence we have (8.3) for \( \{A_k\} \).

Next, we define \( m_1 \) matrices \( \{a_k\} \) in \( M_{m_1}(R) \) as follows: for \( m_1 = 1 \)
\[
a_k = 1_r,
\]
and for \( m_1 = 3 \) or 7
\[
a_k = \begin{pmatrix} A_k & 0 \\ \vdots & \vdots & \vdots \\ 0 & A_k \end{pmatrix}
\]
where \( A_k \) appears \( r \)-times in the diagonal. One sees easily that \( \{a_k\} \) satisfy (7.6).

Now by Lemma 27 we can associate to the matrices \( \{a_k\} \) \( m_1 + 1 \) quadratic forms \( \{p_a\} \) on \( Y \), satisfying (3-1), (3-2) and (A). From the matrices \( \{A_k\} \), using (8.1) we can define \( m_1 + 1 \) cubic forms on \( Y \), satisfying (B). Our polynomials \( \{p_a\} \), \( \{q_a\} \) satisfy, in virtue of Lemma 28, (3-4), (3-5), (3-7), (5-8), (5-9), (5-10), and hence the equations (3-1) \( \sim \) (3-10) by Lemma 19, which proves the "if" part of (i).

It remains to prove the uniqueness. Let \( \{p_a, q_a\} \) and \( \{\tilde{p}_a, \tilde{q}_a\} \) be two families of polynomials on \( Y \) satisfying the conditions in (i), and let \( F \) and \( \tilde{F} \) be the associated polynomials on \( R^v \) respectively. Let
\[
(1) \quad Y = U \oplus V \oplus Z, \\
(2) \quad Y = \tilde{U} \oplus \tilde{V} \oplus \tilde{Z}
\]
be the eigenspace decompositions of symmetric mappings \( P_0, \tilde{P}_0 \) corresponding to \( p_0, \tilde{p}_0 \) respectively. We take orthonormal coordinate systems \( \{u_i\}, \{v_i\}, \{z_i\} \) for \( U, V, W \) respectively. Linear mappings of \( Y \) will be represented by matrices with respect to these coordinates.

Choosing \( \sigma_1 \in O(n) \) such that \( \sigma_1 U = \tilde{U}, \sigma_1 V = \tilde{V} \) and \( \sigma_1 Z = \tilde{Z} \), we put
Then the polynomials \( \{p_\alpha^{(1)}, q_\alpha^{(1)}\} \) also satisfy the conditions in (i) and the eigenspace decomposition of \( P_0^{(1)} \) corresponding to \( p_\alpha^{(1)} \) is the same as (1). The condition (B) for \( \{p_\alpha, q_\alpha\} \) and \( \{p_\alpha^{(1)}, q_\alpha^{(1)}\} \) gives \( \{A_k\} \) and \( \{A_k^{(1)}\} \) in \( M_{m_3+1}(\mathbb{R}) \) respectively, which satisfy (7.7) by Lemma 28. It follows from Lemma 26 that \( \{A_k\} \cong \{A_k^{(1)}\} \), that is, there exist \( \varphi = (\varphi_{kl}) \in O(m_3) \) and \( \tau = (\tau_{\alpha \beta}) \in O(m_3 + 1) \) such that

\[
A_k^{(1)} = \sum_i \varphi_{kl} (\tau A_l \tau^{-1})
\]

for each \( k \).

We put

\[
p_\alpha^{(2)} = \sum_\beta \tau_{\alpha \beta} p_\beta.
\]

Then the quadratic forms \( \{p_\alpha^{(2)}\} \) also satisfy (3-1), (3-2), (A). Let

\[
Y = U^{(2)} \oplus V^{(2)} \oplus Z
\]

be the eigenspace decomposition of \( P_0^{(2)} \) corresponding to \( p_\alpha^{(2)} \). Choosing \( \sigma_3 \in O(n) \) such that \( \sigma_3 U^{(2)} = U, \sigma_3 V^{(2)} = V, \sigma_3 Z = \text{identity} \), we put

\[
p_\alpha^{(3)} = \sigma_3 p_\alpha^{(2)}.
\]

Then \( \{p_\alpha^{(3)}\} \) also satisfy (3-1), (3-2), (A), and the eigenspace decomposition of \( P_0^{(3)} \) corresponding to \( p_\alpha^{(3)} \) is the same as (1). It follows from Lemma 27 that \( \{p_\alpha^{(1)}\} \) and \( \{p_\alpha^{(3)}\} \) define \( \{a_k^{(1)}\} \) and \( \{a_k^{(3)}\} \) in \( M_{m_3}(\mathbb{R}) \) respectively, satisfying (7.6). By Lemma 26, we have \( \{a_k^{(1)}\} \sim \{a_k^{(3)}\} \), that is, we can find \( \sigma, \sigma_3 \in O(m_3) \) such that

\[
\sigma_3 a_k^{(3)} \sigma^{-1} = a_k^{(1)}
\]

for each \( k \).

Putting together \( \sigma, \sigma_3 \) and \( \varphi^{-1} \), we get an element \( \sigma_3 \times \sigma_3 \times \varphi^{-1} \in O(m_3) \times O(m_3) \times O(n) \). Put \( \sigma = \sigma_3 \times \sigma_4 \times \varphi^{-1} \sigma_3 \in O(n) \). Then we have

\[
\tilde{p}_\alpha = \sum_\beta \tau_{\alpha \beta} (\sigma p_\beta), \quad \tilde{q}_\alpha = \sum_\beta \tau_{\alpha \beta} (\sigma q_\beta)
\]

for each \( \alpha \),

which gives the required uniqueness. In fact,

\[
\sum_\beta \tau_{\alpha \beta} (\sigma p_\beta) = \sigma p_\alpha^{(3)} = \sigma_4 (\sigma_3 \times \sigma_4 \times \varphi^{-1}) p_\alpha^{(3)} = \sigma_4 p_\alpha^{(1)} = \tilde{p}_\alpha.
\]

Denoting by \( a_{\alpha \beta k}, a_{\alpha \beta k}^{(1)} \) the \((\alpha, \beta)\)-elements of \( A_k, A_k^{(1)} \) respectively, we have

\[
\sigma_2^{-1} \left( \sum_\beta \tau_{\alpha \beta} (\sigma p_\beta) \right) = (\sigma_3 \times \sigma_4 \times \varphi^{-1}) \sigma_2 \left( \sum_\beta \tau_{\alpha \beta} \sigma_{\beta \gamma} z_{\gamma} p_\gamma \right)
\]

\[
= \sum_\beta \tau_{\alpha \beta} \sigma_{\beta \gamma} (\varphi^{-1} z_{\gamma}) (\sigma_3 \times \sigma_4 \times \varphi^{-1}) \sigma_2 \left( \sum_\gamma \tau_{\gamma \delta} p_\delta^{(2)} \right)
\]

\[
= \sum_\beta \tau_{\alpha \beta} \sigma_{\beta \gamma} \varphi_{\beta \gamma} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\gamma \tau_{\gamma \delta} p_\delta^{(2)} \right)
\]

\[
= \sum_\beta \tau_{\alpha \beta} \sigma_{\beta \gamma} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \tau_{\alpha \beta} \sigma_{\beta \gamma} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \sigma_{\alpha \beta k} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \sigma_{\alpha \beta k} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \sigma_{\alpha \beta k} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \sigma_{\alpha \beta k} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \sigma_{\alpha \beta k} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]

\[
= \sum_\delta \sigma_{\alpha \beta k} \sigma_3 \sigma_4 \sigma_2 \left( \sum_\delta \tau_{\delta \gamma} p_\delta^{(1)} \right)
\]
and hence
\[\sum_{\alpha} \tau_{\alpha\beta} (\sigma q) = \tilde{q}_{\alpha}.\]

It follows that \(F\) and \(\hat{F}\) are \(O(N)\)-equivalent.

**Proof of (ii).** (b) \(m_1 = 3\) or 7. Let \(F = H\) or \(K\) respectively. Let
\[U = F^r, \quad V = F^r, \quad \hat{Z} = F, \quad W = F, \quad Z = \mathfrak{Z} F \subset \hat{Z},\]
and let
\[R^N = U \oplus V \oplus \hat{Z} \oplus W,\]
\[Y = U \oplus V \oplus Z\]
be the orthogonal direct sums. Elements of \(U, V, Z, W\) will be denoted by \(u, v, z, w\) respectively. The standard orthonormal coordinate systems for \(U, V, \hat{Z}, W\) are denoted by \(\{u_{1}^{\alpha}\}, \{v_{1}^{\alpha}\}, \{z_{a}\}, \{w_{a}\}\) respectively, and they as a whole form an orthonormal coordinate system for \(R^N\). As a base point \(e\) in \(R^N\), we take the unit \(c_{0}\) in \(Z\) so that we have \(z = z_{0}\) in the notation of §3. We compute polynomials \(p_{\alpha}, q_{\alpha}\) on \(Y\) corresponding to matrices \(\{a_{\alpha}\}, \{A_{\alpha}\}\) given in the proof of (i), with respect to the above orthonormal coordinate system. We have
\[p_{0} = \sum_{0 \leq i \leq m_1} ((u_{0}^{(i)})^2 - (v_{0}^{(i)})^2) = ||u||^2 - ||v||^2,\]
\[p_{\alpha} = 2 \sum_{0 \leq i \leq m_1} (c_{\alpha} c_{i}, c_{\alpha}) u_{\alpha}^{(i)} v_{j}^{(i)} = 2 \sum_{0 \leq i \leq m_1} (c_{\alpha} v_{\alpha}^{(i)}, u_{\alpha}^{(i)}) = 2(c_{\alpha}, u' v),\]
\[q_{\alpha} = \sum_{F, \alpha} (c_{\alpha} c_{i}, c_{\alpha}) z_{k} p_{i}\]
\[= \sum_{k} \left\{ (c_{\alpha} c_{0}, c_{\alpha}) p_{0} + \sum_{i} (c_{\alpha} c_{i}, c_{\alpha}) p_{i} \right\} z_{k}\]
\[= \sum_{k} \left\{ (c_{\alpha} c_{0}, c_{\alpha})(||u||^2 - ||v||^2) + 2 \sum_{i} (c_{\alpha} c_{i}, c_{\alpha})(c_{i}, u' v) \right\} z_{k}\]
\[= (c_{\alpha}, \tilde{c}_{\alpha})(||u||^2 - ||v||^2) + 2 \sum_{i} (c_{i}, \tilde{c}_{\alpha})(c_{i}, u' v),\]
where we have
\[\begin{align*}
(c_{\alpha}, \tilde{c}_{\alpha}) = (z, c_{\alpha}),
\sum_{i} (c_{i}, \tilde{c}_{\alpha})(c_{i}, u' v) = (\tilde{c}_{\alpha}, u' v) - (c_{\alpha}, \tilde{c}_{\alpha})(c_{\alpha}, u' v)
&= (\tilde{c}_{\alpha}, u' v) - (z, c_{\alpha})(u, v).
\end{align*}\]

Hence we have
\[q_{\alpha} = (z, c_{\alpha})(||u||^2 - ||v||^2 - 2(u, v)) + 2(\tilde{c}_{\alpha}, u' v).\]
In particular, \(q_{0} = 2(\tilde{z}, u' v).\) Now we have
\[ \sum_a p_a w_a = (|| u ||^2 - || v ||^2)w_0 + 2 \sum_k (c_k, u'\overline{v})w_k \]
\[ = (|| u ||^2 - || v ||^2)w_0 + 2(w, u'\overline{v}) - 2(c_0, u'\overline{v})w_0 \]
\[ = (|| u ||^2 - || v ||^2 - 2(u, v))w_0 + 2(w, u'\overline{v}) , \]
\[ \sum_a q_aw_a = (z, w)(|| u ||^2 - || v ||^2 - 2(u, v)) + 2(\overline{z}w, u'\overline{v}) , \]
\[ \sum_a p^2_a = (|| u ||^2 - || v ||^2)^2 + 4 \sum_k (c_k, u'\overline{v})^2 \]
\[ = (|| u ||^2 - || v ||^2)^2 + 4 || u'\overline{v} ||^2 - 4(u, v)^2 . \]

Furthermore we have
\[ \langle p_\alpha, p_\beta \rangle = 4(|| u ||^2 + || v ||^2)\delta_{\alpha, \beta} \] for each \( \alpha, \beta \).

Recall Lemmas 4, 5, 6, 7. The polynomial \( F \) on \( R^n \) associated to \( \{ p_\alpha \}, \{ q_a \} \)

is given by
\[ F = z_0 + 8z_0(|| u ||^2 + || v ||^2 + || z ||^2) - 6 || w ||^2 \]
\[ + 8z_0(|| u ||^2 - || v ||^2 - 2(u, v))w_0 + 2w, u'\overline{v}) \]
\[ + (|| u ||^2 + || v ||^2)^2 - 2(|| u ||^2 - || v ||^2)^2 + 4 || u'\overline{v} ||^2 - 4(u, v)^2 \]
\[ + 8(z, w)(|| u ||^2 - || v ||^2 - 2(u, v)) + 2(\overline{z}w, u'\overline{v}) \]
\[ + 8(|| u ||^2 + || v ||^2) || w ||^2 - 6(|| u ||^2 + || v ||^2 + || z ||^2) || w ||^2 + || w ||^4 \]
\[ = z_0^2 + 2z_0(|| u ||^2 + || v ||^2 + || z ||^2) + (|| u ||^2 + || v ||^2 + || z ||^2)^2 \]
\[ - 6z_0^2 || w ||^2 - 6(|| u ||^2 + || v ||^2 + || z ||^2) || w ||^2 \]
\[ + 8z_0w_0(|| u ||^2 - || v ||^2 - 2(u, v)) + 8(z, w)(|| u ||^2 - || v ||^2 - 2(u, v)) \]
\[ + 16(z, w, u'\overline{v}) + 16(\overline{z}w, u'\overline{v}) \]
\[ - 2(|| u ||^2 - || v ||^2)^2 - 8 || u'\overline{v} ||^2 + 8(u, v)^2 \]
\[ + 8(|| u ||^2 + || v ||^2) || w ||^2 + || w ||^4 . \]

Putting \( \zeta = z_0 \epsilon_0 + z \in \tilde{Z} \) (\( z \in Z \)), we have
\[ F = (|| u ||^2 + || v ||^2 + || \zeta ||^2)^2 - 6(|| u ||^2 + || v ||^2 + || \zeta ||^2) || w ||^2 \]
\[ + 8(\zeta, w)(|| u ||^2 - || v ||^2 - 2(u, v)) + 16(\overline{z}w, u'\overline{v}) \]
\[ - 2(|| u ||^2 - || v ||^2)^2 - 8 || u'\overline{v} ||^2 + 8(u, v)^2 \]
\[ + 8(|| u ||^2 + || v ||^2) || w ||^2 + || w ||^4 \]
\[ = (|| u ||^2 + || v ||^2 + || \zeta ||^2 + || w ||^2)^2 - 8 || \zeta ||^2 || w ||^2 \]
\[ + 8(\zeta, w)(|| u ||^2 - || v ||^2 - 2(u, v)) + 16(\overline{z}w, u'\overline{v}) \]
\[ - 2(|| u ||^2 - || v ||^2)^2 - 8 || u'\overline{v} ||^2 + 8(u, v)^2 . \]

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we get
\[ F = r^4 - 2F'_0 \]
where
\[ F'_0 = 4(||u'\bar{v} - \zeta w||^2 - ((u, v) - (\zeta, w))^2) + (||u||^2 - ||v||^2 - 2(\zeta, w))^2. \]
We put \( u_0 = \xi, \quad v_0 = -\bar{w}, \quad \text{and} \]
\[ u_1 = \begin{pmatrix} u_0 \\ u \end{pmatrix}, \quad v_1 = \begin{pmatrix} v_0 \\ v \end{pmatrix} \in F^{r+1}. \]
Then we have
\[ F_0 = 4(||u'\bar{v} - \zeta w||^2 - ((u, v) - (\zeta, w))^2) + (||u||^2 - ||v||^2 + 2(u_0, v_0))^2, \]
which shows the case (b) of (ii).
(a) \( m_1 = 1. \) Let
\[ U = R^r, \quad V = R^r, \quad \hat{Z} = C, \quad W = C, \quad Z = \mathbb{C} \subseteq \hat{Z} \]
and let
\[ R^{2(r+3)} = U \oplus V \oplus \hat{Z} \oplus W, \]
\[ Y = U \oplus V \oplus Z \]
be the orthogonal direct sums. In the same way as (b), we get
\[ F = r^4 - 2F'_0 \]
where
\[ F'_0 = 4((u, v) - z_0w_1 + z_1w_0)^2 + (||u||^2 - ||v||^2 - 2(\zeta, w))^2. \]
We put
\[ \xi_\lambda = u_0^{(2)} + \sqrt{-1} v_0^{(2)} \quad \text{for} \quad \lambda = 1, \ldots, r, \]
\[ \xi_{r+1} = \frac{1}{\sqrt{2}}((z_1 - w_1) + \sqrt{-1}(z_0 + w_0)), \]
\[ \xi_{r+2} = \frac{1}{\sqrt{2}}((-z_0 + w_0) + \sqrt{-1}(z_1 + w_1)). \]
Then we have
\[ \sum_{i=1}^{r+2} \xi_i = (||u||^2 - ||v||^2 - 2(\zeta, w)) + 2\sqrt{-1}((u, v) - z_0w_1 + z_1w_0). \]
Thus we have
\[ F_0 = \left\| \sum_{i=1}^{r+1} \xi_i \right\|^2, \]
which shows (a) of (ii).

q.e.d.

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