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TRANSVERSALITY THEOREM FOR HOMOLOGY MANIFOLDS AND REPRESENTATION OF HOMOLOGY CLASSES OF HOMOLOGY MANIFOLDS

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1. Introduction. Martin showed the transversality theorem for homology manifolds. ([4])

THEOREM. ([4]) *Let M^m, $N^n \subset P^p$ be properly contained homology manifolds, where M is PL. Suppose that $\partial N \perp F/L$ for a normal bundle F over a cell subdivision L of ∂M. Then, if $r \geqq 2m+1$, we can make $N \times I^r$ transverse to M in $P \times I^r$, such that making transverse is relative to ∂M in that $\partial N \times I^r \perp F \oplus \theta^r$.*

In this theorem, he assumes that one submanifold M is a PL-manifold. We will show an improvement of this.

THEOREM. *Let M^m, $N^n \subset P$ be properly contained homology manifolds. Suppose that $\partial N \perp F/L$ for a normal bundle F over a cell subdivision L of ∂M. Then, if $r \geqq 2m+1$, we can make $N \times I^r$ transverse to M in $P \times I^r$ such that the making transverse is relative to ∂M in that $\partial N \times I^r \perp F \oplus \theta^r$.*

In Section 3, we will apply this for representation of homology classes. Similar results have been obtained by Adachi. ([3])

2. Transversality theorem. An $ND(r)$-space is a polyhedron X such that Link (σ^k, X), for every $k \leqq r$, is $(r-k-2)$-connected. X is an $ND(r)$-manifold if X is a homology manifold and an $ND(r)$-space, and if ∂X is an $ND(r-1)$-space.

If $f: X \rightarrow Y$ is a non-degenerated map, $S_2(f)$ is the closure of the set of points, $x \in X$, such that $f^{-1}(f(x))$ contains more than one. Then we have the next proposition.

PROPOSITION. (See [4], [8]) *Let Y be an $ND(r)$-space, let $X \subset P^p$ be polyhedra with $p \leqq r$ and let $f: P \rightarrow Y$ be a map such that $f \mid X$ is non-degenerated with* $\dim (S_2(f \mid x)) \leqq 2p - r$. *Then there exists an arbitrary close non-degenerated apploximation g to f such that $g \mid X = f \mid X$ and* $\dim (S_2 \mid g) \leqq 2p - r$.

If M is a closed homology n-manifold, and if $H_*(M; Z) \cong H_*(S^n; Z)$, we call M a homological homology n-sphere. A homological homology n-disk is an orientable homology n-manifold M with boundary, such that $\tilde{H}_*(M; Z) = 0$. An $ND(r)$-manifold M is called an $ND(r)$-homology sphere or disk if M is a homological homology sphere or disk.

We call a cone of homological homology sphere $v \cdot M$ a homology cell.

A simplicial complex K is called a homology cell complex if it is a union of homology cells, such that:

(1) each simplex is the interior of exactly one cell;

(2) if $a \cdot M$ is a cell, then $a \cdot (\partial M)$ and M are union of cells;

(3) there exists a total ordering of the vertices of K such that if $a \cdot M$ is a cell and $b \in M$ then $b < a$.

Let K be a homology cell complex. A polyhedron E is called a space over K if E is the union of compact polyhedra $E(C)$, for each cell C of K, such that:

(1) if $C \subset D$ are cells of K, then $E(C)$ is a PL-subspace of $E(D)$;

(2) for cells C, D in K, $E(C) \cap E(D) = \cup E(B)$, where B run over all cells in $C \cap D$.

DEFINITION. (See [5]) A space E over K is an S^n-homology or homotopy cobordism bundle, if

(1) for m-cell C in K, $E(C)$ is an $(m + n)$-homology manifold with boundary $E(\partial C)$, where $E(\partial C) = \cup E(D)$, for all $D \subset C$, $D \neq C$;

(2) for each cell C, there is a space G over K such that $G(C)$ is an H-cobordism or h-cobordism between $E(C)$ and $C \times S^n$.

Similarly, we define a D^n-homology or homotopy bundle over K,

(1) for m-cell of K, $E(C)$ is an $(m + n)$-manifold, whose boundary contains $E(\partial C)$ as a submanifold of codimension 0;

(2) there is a space G over K such that, for each m-cell C of K, $G(K)$ is an H-cobordism or h-cobordism of triads between $(E(C); E(\partial C), \bar{E}(C))$ and $(C \times D^n, \partial C \times D^n, C \times S^{n-1})$ where $\bar{E}(B) = \text{cl.}\,(\partial E(B) - E(\partial B))$.

We can define an oriented bundle too.

Let $BHML(k)$ be the classifying space of D^k-homology bundles and $BSHML(k)$ be a classifying space of oriented D^k-homology bundles. $EHML(k)$ is the total space of the universal bundle $\gamma(k)$ over $BHML(k)$ and $ESHML(k)$ is the total space of the universal bundle over $BSHML(k)$.

Maunder proved the next theorem. ([6])

THEOREM. *Let M be a homology m-manifold. Then there exist a homology m-manifold N, which is an $ND(m-3)$-space and has $\partial M = \partial N$, and a homotopy resolution $f: N \to M$ such that $f \mid \partial N$ is the* *identity.*

We prove the next lemma from this theorem.

LEMMA. *Let E^k and E'^k be S^k-homology or homotopy bundle over a homology manifold M^m and G be an isomorphism between E and E'. If $E^k(\sigma^i)$ and $E'^k(\sigma^i)$ are $ND(r-m+i)$-manifolds for all $i \leqq m$ where $r \leqq m+k-2$, there exists an isomorphism G' between E and E' such that $G'(\sigma^i)$ is an $ND(r-m+i+1)$-manifold for all $i \leqq m$.*

This lemma is induced by next lemmas.

LEMMA$_{(j)}$ ($j = 0, \cdots, m$). *Let E^k and E'^k be S^k-homology or homotopy bundles over a homology manifold M^m and let G be an isomorphism between E and E'. If $E^k(\sigma^i)$ and $E'^k(\sigma^i)$ are $ND(r-m+i)$-manifolds for all $i \leqq j$ where $r \leqq m+k-2$ and $j \leqq m$, there exists an isomorphism G_j between E and E' such that $G_j(\sigma^i)$ is an $ND(r-m+i+1)$-manifold for all $i \leqq j$ and there exists a space J_j over M such that $G_j(\sigma^i) \subset J_j(\sigma^i)$ is homotopy equivalent and the next conditions are satisfied;*

1) $E(\alpha) \cap J_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} E(\tau)$

   $E'(\alpha) \cap J_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} E'(\tau)$

2) $G(\alpha) \cap G_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} (E(\tau) \cup E'(\tau))$

3) $G(\alpha) \cap J_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} G(\tau)$

4) $G_j(\alpha) \cap J_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} G_j(\tau)$

5) $G_j(\alpha) \cap G_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} G_j(\tau)$

6) $J_j(\alpha) \cap J_j(\beta) = \bigcup_{\tau \in \alpha \cap \beta} J_j(\tau)$.

We will prove the lemma$_{(0)}$ by the induction. We divide the proof into two step.

1-st step. $G(\sigma^0)$ is k-dimensional homology manifold with boundary $E(\sigma^0) \cup E'(\sigma^0)$ which is an $ND(r-m)$-manifold. By Maunder's theorem and $r - m \leqq k-2$, there exists a homotopy resolution

$$f\colon G_0'(\sigma^0) \to G(\sigma^0)$$

such that $G_0'(\sigma^0)$ is $ND(r-m+1)$-manifold and the restriction of f to the boundary is an identity map. We define $J_0(\sigma^0)$ by the mapping cylinder of f. We put

$$G_0(\sigma^0) = \partial J_0(\sigma^0) - \text{Int. } G(\sigma^0)\,.$$

G_0 is an isomorphism between E and E' over a 0-skelton M^0 of M and J_0 is a space over M^0 such that $G_0(\sigma^0) \subset J_0(\sigma^0)$ is homotopy equivalent for $\sigma \in M^0$. For any cells $\alpha \in M$ and β, $\gamma \in M^0$, the next conditions are satisfied,

1) $E(\alpha) \cap J_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} E(\tau)$;

$E'(\alpha) \cap J_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} E'(\tau)$;

2) $G(\alpha) \cap G_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} (E(\tau) \cup E'(\tau))$;

3) $G(\alpha) \cap J_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} G(\tau)$;

4) $G_0(\beta) \cap J_0(\gamma) = \bigcup_{\tau \in \beta \cap \gamma} G_0(\tau)$;

5) $G_0(\beta) \cap G_0(\gamma) = \bigcup_{\tau \in \beta \cap \gamma} G_0(\tau)$;

6) $J_0(\beta) \cap J_0(\gamma) = \bigcup_{\tau \in \beta \cap \gamma} J_0(\tau)$.

2-nd step. We assume G_0 is an isomorphism between E and E' over s-skelton M^s of M and J_0 is a space over M^s such that $G_0(\sigma) \subset J_0(\sigma)$ is homotopy equivalent for any $\sigma \in M^s$. And we assume that next conditions are satisfied for any cells $\alpha \in M$ and β, $\gamma \in M^s$,

1) $E(\alpha) \cap J_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} E(\tau)$;

$E'(\alpha) \cap J_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} E'(\tau)$;

2) $G(\alpha) \cap G_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} (E(\tau) \cup E'(\tau))$;

3) $G(\alpha) \cap J_0(\beta) = \bigcup_{\tau \in \alpha \cap \beta} G(\tau)$;

4) $G_0(\beta) \cap J_0(\gamma) = \bigcup_{\tau \in \beta \cap \gamma} G_0(\tau)$;

5) $G_0(\beta) \cap G_0(\gamma) = \bigcup_{\tau \in \beta \cap \gamma} G_0(\tau)$;

6) $J_0(\beta) \cap J_0(\gamma) = \bigcup_{\tau \in \beta \cap \gamma} J_0(\tau)$.

We put $J_0(C) = \bigcup_{\sigma \in C} J_0(\sigma)$ and $G_0(C) = \bigcup_{\sigma \in C} G_0(\sigma)$ for $(s+1)$-cell $v \cdot C$. We define

$$J_0(v \cdot C) = (G(v \cdot C) \bigcup_{G(C)} J_0(C)) \times I$$

$$G_0(v \cdot C) = (G_0(C) \cup E(v \cdot C) \cup E'(v \cdot C)) \times I$$
$$\cup J_0(C) \times \{1\} \cup G(v \cdot C) \times \{1\} \subset J_0(v \cdot C) .$$

We identify $J_0(C)$ to $J_0(C) \times \{0\}$ and we put $J_0(\sigma) \cap J_0(\delta) = \bigcup_{\tau \in \sigma \cap \delta} J_0(\tau)$ for any $(s+1)$-cells σ and δ.

Let α, $u \cdot C$, $v \cdot D$ be cells.

1) If $\dim u \cdot C = s + 1$, we have

$$E(\alpha) \cap J_0(u \cdot C) = E(\alpha) \times \{0\} \cap (G(u \cdot C) \cup J_0(C)) \times I$$
$$= (E(\alpha) \cap (G(u \cdot C) \cup J_0(C))) \times \{0\}$$
$$= \bigcup_{\tau \in \alpha \cap u \cdot C} E(\tau) \bigcup_{\tau \in \alpha \cap C} E(\tau) = \bigcup_{\tau \in \alpha \cap u \cdot C} E(\tau) .$$

Similarly we have

$$E'(\alpha) \cap J_0(u\cdot C) = \bigcup_{\tau\in\alpha\cap u\cdot C} E'(\alpha)\ .$$

2) If $\dim u\cdot C = s+1$, we have

$$\begin{aligned} G(\alpha) \cap G_0(u\cdot C) &= G(\alpha) \times \{0\} \cap \{(G_0(C) \cup E(u\cdot C) \cup E'(u\cdot C)) \times I \\ &\qquad \cup J_0(C) \times \{1\} \cup G(u\cdot C) \times \{1\}\} \\ &= \{G(\alpha) \cap (G_0(C) \cup E(u\cdot C) \cup E'(u\cdot C))\} \times \{0\} \\ &= \bigcup_{\tau\in\alpha\cap C} E(\tau) \bigcup_{\tau\in\alpha\cap C} E'(\tau) \bigcup_{\tau\in\alpha\cap u\cdot C} (E(\tau) \cup E'(\tau)) \\ &= \bigcup_{\tau\in\alpha\cap u\cdot C} (E(\tau) \cup E'(\tau))\ . \end{aligned}$$

3) If $\dim u\cdot C = s+1$, we have

$$\begin{aligned} G(\alpha) \cap J_0(u\cdot C) &= G(\alpha) \times \{0\} \cap (G(u\cdot C) \cup J_0(C)) \times I \\ &= \{G(\alpha) \cap (G(u\cdot C) \cup J_0(C))\} \times \{0\} \\ &= \bigcup_{\tau\in\alpha\cap u\cdot C} G(\tau) \bigcup_{\tau\in\alpha\cap C} G(\tau) \\ &= \bigcup_{\tau\in\alpha\cap u\cdot C} G(\tau)\ . \end{aligned}$$

4) If $\dim u\cdot C = s+1$ and $\dim v\cdot D \leqq s$, we have

$$\begin{aligned} & G_0(u\cdot C) \cap J_0(v\cdot D) \\ &= ((G_0(C) \cup E(u\cdot C) \cup E'(u\cdot C)) \times I \cup J_0(C) \times \{1\} \cup G(v\cdot C) \times \{1\}) \\ &\quad \cap J_0(v\cdot D) \times \{0\} \\ &= \{(G_0(C) \cup E(u\cdot C) \cup E'(u\cdot C)) \cup J_0(v\cdot D)\} \times \{0\} \\ &= \bigcup_{\tau\in C\cap v\cdot D} G_0(\tau) \bigcup_{\tau\in u\cdot C\cap v\cdot D} (E(\tau) \cup E'(\tau)) \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\ . \end{aligned}$$

If $\dim u\cdot C \leqq s$, $\dim v\cdot C = s+1$, we have

$$\begin{aligned} & G_0(u\cdot C) \cap J_0(v\cdot D) \\ &= G_0(u\cdot C) \times \{0\} \cap (G(v\cdot D) \cup J_0(D)) \times I \\ &= \{G_0(u\cdot C) \cap (G(v\cdot D) \cup J_0(D))\} \times \{0\} \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} (E(\tau) \cup E'(\tau)) \bigcup_{\tau\in u\cdot C\cap D} G(\tau) = \bigcup_{\tau\in u\cdot C\cap v\cdot D} G(\tau)\ . \end{aligned}$$

If $\dim u\cdot C = \dim v\cdot D = s+1$, we have

$$\begin{aligned} G_0(u\cdot C) \cap J_0(v\cdot D) &= G_0(u\cdot C) \cap J_0(u\cdot C) \cap J_0(v\cdot D) \\ &= G_0(u\cdot C) \cap \bigcup_{\tau\in u\cdot C\cap v\cdot D} J_0(\tau) \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\ . \end{aligned}$$

5) If $\dim u\cdot C \leqq s$ and $\dim v\cdot D = s+1$, we have

$$\begin{aligned} G_0(u\cdot C)\cap G_0(v\cdot D) &= G_0(u\cdot C)\cap J_0(u\cdot C)\cap G_0(v\cdot D) \\ &= G_0(u\cdot C)\cap \Big(\bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\Big) \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\,. \end{aligned}$$

If $\dim u\cdot C = \dim v\cdot D = s+1$, we have

$$\begin{aligned} G_0(u\cdot C)\cap G_0(v\cdot D) &= J_0(u\cdot C)\cap G_0(u\cdot C)\cap J_0(v\cdot D)\cap G_0(v\cdot D) \\ &= (G_0(u\cdot C)\cap J_0(v\cdot D))\cap (G_0(v\cdot D)\cap J_0(u\cdot C)) \\ &= \Big(\bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\Big)\cap\Big(\bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\Big) \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} G_0(\tau)\,. \end{aligned}$$

6) If $\dim u\cdot C \leqq s$ and $\dim v\cdot D = s+1$, we have

$$\begin{aligned} J_0(u\cdot C)\cap J_0(v\cdot D) &= J_0(u\cdot C)\times\{0\}\cap (G(v\cdot D)\cup J_0(D))\times I \\ &= (J_0(u\cdot C)\cap (G(v\cdot D)\cup J_0(D)))\times\{0\} \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} G(\tau)\bigcup_{\tau\in u\cdot C\cap D} J_0(\tau) \\ &= \bigcup_{\tau\in u\cdot C\cap v\cdot D} J_0(\tau)\,. \end{aligned}$$

We have homology or homotopy equivalent maps

$$\begin{aligned} E(v\cdot C) &= E(v\cdot C)\times\{0\} \\ &\subset E(v\cdot C)\times\{0\}\cup G_0(C)\times\{0\} \\ &\subset (E(v\cdot C)\cup G_0(C))\times I \\ &\subset (E(v\cdot C)\cup G_0(C))\times I\cup J_0(C)\times\{1\} \\ &\subset (E(v\cdot C)\cup G_0(C))\times I\cup J_0(C)\times\{1\}\cup G(v\cdot C)\times\{1\} \\ &\subset (E(v\cdot C)\cup G_0(C)\cup E'(v\cdot C))\times I\cup J_0(C)\times\{1\}\cup G(v\cdot C)\times\{1\} \\ &= G_0(v\cdot C)\,. \end{aligned}$$

Then $E(v\cdot C)\subset G_0(v\cdot C)$ is homology or homotopy equivalent.

Similarly $E'(v\cdot C)\subset G_0(v\cdot C)$ is homology or homotopy equivalent. Then G_0 is extended to an isomorphism E and E' over M^{s+1}. We have homotopy equivalent maps

$$\begin{aligned} G(v\cdot C) &= G(v\cdot C)\times\{0\} \\ &\subset (G(v\cdot C)\cup J_0(C))\times\{0\} \\ &\subset (G(v\cdot C)\cup J_0(C))\times I = J_0(v\cdot C)\,. \end{aligned}$$

Then $G(v\cdot C)\subset J_0(v\cdot C)$ is homotopy equivalent.

Similarly $G(v\cdot C)\times\{1\}\subset J_0(v\cdot C)$ is homotopy equivalent. We have homotopy equivalent maps

$$\begin{aligned}&G(v\cdot C)\times\{1\}\subset\{G(v\cdot C)\cup J_0(C)\}\times\{1\}\\&\quad\subset(G(v\cdot C)\cup J_0(C))\times\{1\}\cup(G_0(C)\cup E(v\cdot C)\cup E'(v\cdot C))\times I\\&\quad=G_0(v\cdot C)\ .\end{aligned}$$

We have $G(v\cdot C)\times\{1\}\subset G_0(v\cdot C)\subset J_0(v\cdot C)$. Then $G_0(v\cdot C)\subset J_0(v\cdot C)$ is homotopy equivalent. By the induction of s, we have an isomorphism $G_0=\bigcup_\alpha G_0(\alpha)$ between E and E' and we have a space J_0 over M. Thus we have proved lemma$_{(0)}$.

We prove that lemma$_{(j)}$ implies lemma$_{(j+1)}$.

1-st step. We put $G_{j+1}(\sigma^i)=G_j(\sigma^i)$ and $J_{j+1}(\sigma^i)=J_j(\sigma^i)$ for $i\leqq j$. $G_j(v\cdot C^j)$ is a homology manifold with boundary $E(v\cdot C^j)\cup E'(v\cdot C^j)\cup G_j(C^j)$ which is an $ND(r-m+j+1)$-manifold. By Maunder's theorem and $r-m\leqq k-2$, there exists a homotopy resolution

$$f\colon G'_{j+1}(v\cdot C^j)\longrightarrow G_j(v\cdot C^j)$$

such that $G'_{j+1}(v\cdot C^j)$ is an $ND(r-m+j+2)$-manifold and the restriction of f to the boundary is an identity map. We define $J_{j+1}(v\cdot C^j)$ by the mapping cylinder of f. We put $G_{j+1}(v\cdot C^j)=\partial J_{j+1}(v\cdot C^j)-\text{Int }G_j(v\cdot C^j)$. Similar conditions of the first step of lemma$_{(0)}$ are satisfied.

2-nd step. G_{j+1} and J_{j+1} are constructed by similar way of 2-nd step of lemma$_{(0)}$. Then lemma$_{(j+1)}$ is induced. By the induction of j, we have an isomorphism $G_m=\bigcup_\sigma G_m(\sigma)$ between E and E' such that $G_m(\sigma^i)$ is an $ND(r-m+i+1)$-manifold for $i\leqq m$. Thus we have proved the lemma.

DEFINITION. (See [4].) Let P^p be a homology manifold and let M^m, and N^n be homology manifolds properly PL-embedded in P^p. Let E be a normal homology bundle for M in P over a homology cell subdivision K of M such that $E(\partial M)=E\cap\partial P$. Then we say that N is block transverse to the bundle E/K if $M\cap N$ is a cell subcomplex of K and $N\cap E=E(M\cap N)$. We write $N\perp E/K$.

DEFINITION. (See [4].) Let M and N be proper homology submanifolds of P. We say that we can make N transverse to M if there exists a triple of h-cobordism $(W; M\times I, V)$ between $(P; M, N)$ and $(P'; M, N')$, where M and N' are proper homology submanifolds of a homology manifold P', $N'\perp E/K$ for some normal bundle E of M in P' over a homology cell subdivision K of M, and $M\times I$ and V are proper submanifolds of W. If we already know that $\partial N\perp F/L$ for a normal bundle F of M in P over L ($|L|=\partial M$), we say that we can make N transverse to M relative to the boundary if $(W; M\times I, V)$ restricts to the product h-cobordism $(\partial P\times I; \partial M\times I, \partial N\times I)$ on the boundary and if we can choose K to extend L such that $E|L=F$. In this case we write

$$(W; M\times I, V)_{\mathrm{rel}\partial}: (P; M, L)_{(F,L)} \xrightarrow{t} (P'; M, N', E, K) .$$

We have next propositions. We show that Proposition $1_{(p-1)}$ implies Proposition $2_{(p)}$ and that Proposition $2_{(p)}$ implies Proposition $1_{(p)}$ by a similar way to Martin. ([4])

PROPOSITION $1_{(p)}$. *Let M^m and N^n be proper submanifolds of an $ND(r)$-manifold P^p, $p-m\geqq r\geqq 2m+1$, and $\partial N\perp F/L$ for a normal D^{p-m}-homotopy bundle F over ∂M in ∂P. Suppose that* cl. $(\partial P-F)$ *is as $ND(r-1)$-manifold. Then there exists*

$$(W; M\times I, V)_{\mathrm{rel}\partial}: (P; M, N)_{(F,L)} \xrightarrow{t} (P'; M, N'; E, K)$$

such that W is an $ND(r+1)$-manifold, E is a D^{p-m}-homotopy bundle over K.

PROPOSITION $2_{(p)}$. *Let Σ^p be a 1-connected $ND(r)$-homology sphere. Let Σ^m and Σ^n be homological homology spheres PL-embedded in Σ^p, with $p-m\geqq r\geqq 2m+3$. Suppose that $\Sigma^n\perp F/L$ for a normal homology bundle F over a homology cell subdivision L of Σ^m. Then Σ^p spans a 1-connected $ND(r+1)$-homology ball B^{p+1} which contains a homology ball D^{m+1} define by a cone of Σ^m and a contractible homology manifold C^{n+1} spanning Σ^n, both properly PL-embedded in B^{p+1} such that $C^{n+1}\perp E/K$ for a normal homotopy bundle E over a homology cell subdivision K of D^{m+1} extending L and with $E|L=F$.*

Since $P^p\times I^r$ is $ND(r+2)$-manifold, and M^m and $N^n\times I^r$ are proper submanifolds of $P^p\times I^r$ with $r\geqq 2m+1$, Proposition 1 implies the next transversality theorem of homology manifolds.

THEOREM. *Let M^m, $N^n\subset P^p$ be properly contained homology manifolds. Suppose that $\partial N\perp F/L$ for a normal bundle F over a cell subdivision L of ∂M. Then, if $r\geqq 2m+1$, we can make $N\times I^r$ transverse to M in $P\times I^r$ such that the making transverse is relative to ∂M in that $\partial N\times I^r\perp F+\theta^r$.*

We now prove that Proposition $2_{(1)},\cdots,2_{(p-1)}$ imply Proposition $1_{(p)}$.

Let J be a simplicial complex such that $|J|=P$ and let K be a subcomplex such that $|K|=M$. Let $\bar{K}$ be a dual cell complex of K. Let E_0 be a normal bundle over K induced by the dual cell complex $\bar{J}$ of J.

Then $\partial E_0(\sigma^i) \cap N = \emptyset$ or $\partial E_0(\sigma^i) \cap N$ is an $(n-m+i-1)$-dimensional homological homology sphere. We have $\partial E_0(\sigma^0) \cap M = \emptyset$. We assume that $(P; M, N)$ and $(P_k; M, N_k)$ are h-cobordant and E_k is normal bundle over M in P_k such that $\partial E_k(\sigma^i) \cap N_k = \emptyset$ or $\partial E_k(\sigma^i) \cap N$ is an $(n-m+i-1)$-dimensional homological homology sphere for any $i \leqq m$, and $E_k(\sigma^i \cap N_k) = E_k(\sigma^i) \cap N_k$ for $i \leqq k$.

If $\sigma \neq E_k(\tau)$ for $\tau \in k$, then we define

$$W(\sigma) = \sigma \times I\,, \quad P_{k+1}(\sigma) = \sigma \times \{1\}\,, \quad N_{k+1}(\sigma \cap N_k) = (\sigma \cap N_k) \times \{1\}$$

and $V(\sigma \cap N_k) = (\sigma \cap N_k) \times I$. We identify σ to $\sigma \times \{0\}$. If $\sigma \in K^k \cup \partial M$, we define

$$W(E_k(\sigma)) = E_k(\sigma) \times I\,, \quad P_{k+1}(E_k(\sigma)) = E_k(\sigma) \times \{1\}$$
$$N_{k+1}(\sigma \cap N_k) = (\sigma \cap N_k) \times \{1\} \quad \text{and} \quad V(\sigma \cap N_k) = (\sigma \cap N_k) \times I\,.$$

We identify σ to $\sigma \times \{0\}$. If $v \cdot C \in K^{k+1} - K^k$, there exists a contractible disk triple $(A; v \cdot C, \tau)$ with boundary $(\partial E_k(v \cdot C); C, D)$ where $D = \partial E_k(v \cdot C) \cap N_k$ and there exists a normal bundle E_{k+1} over $v \cdot C$ in A extending $E_k \mid C$ such that $E_{k+1}(v \cdot C) \cap \tau = E_{k+1}(v \cdot C \cap \tau)$, by Proposition $2_{(p-m-k-1)}$. We put

$$P_{k+1}(E(v \cdot C)) = A\,, \quad N_{k+1}(E_k(v \cdot C) \cap N_k) = \tau\,,$$
$$V(E_k(v \cdot C) \cap N) = \text{cone of } ((E_k(v \cdot C) \cap N_k) \cup V(\partial E_k(v \cdot C) \cap N) \cup \tau)$$

and

$$W(E_k(v \cdot C)) = \text{cone of } (E_k(v \cdot C) \cup W(\partial E_k(v \cdot C)) \cup A)\,.$$

Now suppose that we have $W(E_k(\sigma))$, $V(E_k(\sigma) \cap N_k)$, $P_{k+1}(E_k(\sigma))$ and $N_{k+1}(E_k(\sigma) \cap N_k)$ for each $\sigma \in K$ and $\dim \sigma < t$ $(t > k)$ such that $(W(E_k(\sigma));$ $\sigma \times I, V(E_k(\sigma) \cap N_k))$ is an h-cobordism $(E_k(\sigma); \sigma, E_k(\sigma) \cap N_k)$ and $(P_{k+1}(E_k(\sigma)); \sigma, N_{k+1}(E_k(\sigma) \cap N_k))$ and that for $\tau \in P_k$, $E_k(\sigma) \neq \tau$ we have $P_{k+1}(\tau)$ and $W(\tau)$. If $\dim \sigma = t$ we define

$$P_{k+1}(E_k(\sigma)) = \text{cone of } P_{k+1}(\partial E_k(\sigma))\,,$$
$$W(E_k(\sigma)) = \text{cone of } (E_k(\sigma) \cup W(\partial E_k(\sigma)) \cup P_{k+1}(E_k(\sigma)))$$
$$N_{k+1}(E_k(\sigma) \cap N_k) = \text{cone of } N_{k+1}(\partial E_k(\sigma) \cap N_k)$$

and

$$V(E_k(\sigma) \cap N_k) = \text{cone of } ((E_k(\sigma) \cap N_k) \cup V(\partial E_k(\sigma) \cap N_k) \cup N_{k+1}(E_k(\sigma) \cap N_k))\,.$$

By the induction of t we have an h-cobordism $(W; M \times I, V)$ between $(P_k; M, N_k)$ and $(P_{k+1}; M, N_{k+1})$ such that $E_{k+1}(\sigma^i \cap N_{k+1}) = E_{k+1}(\sigma^i) \cap N_{k+1}$ for $i \leqq k+1$. By the induction of k, we can make N transverse to M in P.

Now we prove that Proposition $1_{(p-1)}$ implies Proposition $2_{(p)}$.

Let Σ^p be a 1-connected $ND(r)$-homology sphere. Let Σ^m and Σ^n be homological homology spheres embedded in Σ^p with $p - m \geqq r \geqq 2m + 3$. Σ^n is transverse to E/Σ^m which is normal bundle over Σ^m in Σ^p. By the lemma, there exists an $ND(r + 1) - h$-cobordism G between E and $\Sigma^m \times D^{p-m}$ which is a trivialization of E. We can properly embed $\Sigma^m \times [1, 2]$ in G such that $\Sigma^m \times \{1\}$ is embedded in E and $\Sigma^m \times \{2\}$ is embedded in $\Sigma^m \times D^{p-m}$. We have $\Sigma^p \times [0, 1] \bigcup_f G$, $f\colon E \subset \Sigma^p = \Sigma^p \times \{1\}$ and $\Sigma^n \times [0, 1] \bigcup_g G \,|\, (\Sigma^m \cap \Sigma^n)$, $g\colon E \,|\, (\Sigma^m \cap \Sigma^n) \to \Sigma^n \times \{1\}$. We put

$$\bar{\Sigma}^n = \partial(\Sigma^n \times [0, 1] \bigcup_g G \,|\, (\Sigma^m \cap \Sigma^n)) - \Sigma^n \times \{0\}\,,$$

and

$$\bar{\Sigma}^p = \partial(\Sigma^p \times [0, 1] \bigcup_f G) - \Sigma^p \times \{0\}\,.$$

We define the normal bundle of $\Sigma^m \times \{2\}$ in $\bar{\Sigma}^p$ by E'. Then $\bar{\Sigma}^n$ is transversal to $E'/\Sigma^m \times \{2\}$. We can embed $\Sigma^m \times [2, 3]$ in $\bar{\Sigma}^p$ such that $\bar{\Sigma}^n$ is transversal to $\Sigma^m \times [2, 3]$ in $\bar{\Sigma}^p$ and that $\Sigma^m \times \{3\}$ is embedded in $\partial E'$. We put $X = \bar{\Sigma}^p - \operatorname{Int} E'$ and $Y = \bar{\Sigma}^n - \operatorname{Int} E'$. Since X is $(p - m - 2)$-connected and $ND(r + 1)$-manifold, a cone of $\Sigma^m \times \{3\}$ can be embedded in X. ∂Y is transversal to $\Sigma^m \times \{3\}$ in ∂X. By Proposition $1_{(p-1)}$, there exists an $ND(r + 1) - h$-cobordism $(A; B, C)$ we can make transverse $\bar{\Sigma}^n$ to cone of $\Sigma^m \times \{3\}$ in ∂X by $ND(r + 1) - h$-cobordism $(A; B, C)$. We have the triple $(\Sigma^p \times [0, 1] \bigcup_f G \cup A \cup$ cone of $\tilde{\Sigma}^p; \Sigma^m \times [0, 3] \cup$ cone of $(\Sigma^m \times \{3\}), \Sigma^n \times [0, 1] \bigcup_g G \,|\, (\Sigma^m \cap \Sigma^n) \cup C \cup$ cone of $\tilde{\Sigma}^{n\prime})$, where

$$\tilde{\Sigma}^p = \partial(\Sigma^p \times [0, 1] \cup G \cup A) - \Sigma^p \times \{0\}$$

and

$$\tilde{\Sigma}^n = \partial(\Sigma^n \times [0, 1] \cup G \,|\, (\Sigma^m \cap \Sigma^n) \cup C) - \bar{\Sigma}^n \times \{0\}\,.$$

This is a disk triple required. Thus we have showen Proposition $2_{(p)}$.

3. Representation of homology classes.

DEFINITION. We say that s cohomology class $u \in H^k(M; Z_2)$ of a homology manifold M is HML-realizable if there exists an h-cobordism $(W; M, M')$ and there exists a map $f\colon M' \to T(EHML(k))$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class U_k of the Thom complex $T(EHML(k))$. We say that a cohomology class $u \in H^k(M; Z)$ of an oriented homology manifolds M is $SHML(k)$-realizable if there exists an h-cobordism $(W; M, M')$ and there exists a maps $f\colon m \to T(ESHML(k))$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class U_k of the Thom complex $T(ESHML(k))$.

THEOREM. *Let M be a closed homology manifold of dimension m.*

a) *In order that there may exist an H-cobordism $(W; M, M')$ such that a homology class $z \in H_{m-k}(M, Z_2)$ $k > 0$, can be realized by a homology submanifold N^{m-k} which has a normal homology bundle in M', it is necessary and sufficient that the cohomology class $u \in H^k(M; Z_2)$, corresponding to z by Poincaré duality, is $HML(k)$-realizable.*

b) *Let M be oriented. In order that there may exist an H-cobordism $(W; M, M')$ such that a homology class $z \in H_{m-k}(M; Z)$ $k > 0$, can be realized by an oriented homology submanifold N^{m-k} which has an orientable normal homology bundle in M', it is necessary and sufficient that the cohomology class $u \in H^k(M; Z)$, corresponding to z by Poincaré duality, is $SHML(k)$-realizable.*

We will prove the case a) of the theorem. The case b) can be proved by the same way.

PROOF. i) Necessity. Suppose that there exists a homology submanifold N^{m-k} in M which represent z and has a normal homology bundle of dimension k. There exists a map $g: N \to BHML(k)$ such that $G: \cong g^{\#}\gamma(k)$. We define W by $M \times I \bigcup_{E(\xi)} G$; such that $(W; M, M')$ where $M' = \partial M - M$ and $(V; N, N')$, where $N = \partial V - N'$ are H-cobordisms. Then we have a map $f: M' \to T(EHML(k))$. And we have a following diagram.

$$
\begin{array}{ccccc}
H_{m-k}(M'; Z_2) & \xrightarrow{p_2} & H^k(M'; Z_2) & \xleftarrow{\quad f^* \quad} & \\
\uparrow & & \uparrow & & \\
H_{m-k}(N'; Z_2) & \xrightarrow{p^{-1}\circ p_3 \circ i_*} & H^k(M', M' - E(g^{\#}(\gamma)); Z_2) & \longleftarrow & H^k(T(EHML(k)); Z_2) \\
& \searrow^{p_1} & \downarrow t_2 & & \downarrow t_1 \\
& & H^0(N'; Z_2) & \xleftarrow{\quad g^* \quad} & H^0(BHML(k); Z_2)
\end{array}
$$

$$H_{m-k}(N'; Z_2) \xrightarrow{p_1} H^0(N'; Z_2),\ H_{m-k}(M'; Z_2) \xrightarrow{p_2} H^k(M'; Z_2)$$

and

$$H_{m-k}(E(g^{\#}(\gamma)); Z_2) \xrightarrow{p_3} H^k(E(g^{\#}(\gamma)), \partial E(g^{\#}(\gamma)); Z_2)$$

are isomorphisms by Poincaré duality.

$$H^k(M', M' - E(g^{\#}(\gamma)); Z_2) \xrightarrow{p} H^k(E(g^{\#}(\gamma)), \partial E(g^{\#}(\gamma)); Z_2)$$

is an isomorphism by the excision theorem.

$$H_{m-k}(N'; Z_2) \xrightarrow{i_*} H_{m-k}(E(g^\sharp(\gamma)); Z_2)$$

is an isomorphism induced by the inclusion maps.

$$H^k(T(BHML(k); Z_2) \xrightarrow{t_1} H^0(BHML(k); Z_2)$$

and

$$H^k(M', M' - E(g^\sharp(\gamma)); Z_2) \xrightarrow{t_2} H^0(N'; Z_2)$$

are Thom isomorphisms.

α is a generator of $H^0(BHML(k); Z_2)$ and β is a generator of $H^0(N'; Z_2)$ such that $\beta = g^*(\alpha)$, $\beta = p_1([N'])$ and $U_k = t_1^{-1}(\alpha)$.

We have,

$$\begin{aligned} u &= p_2 \circ i_*([N]) = p_2 \circ i_* \circ p_1^{-1}(\beta) = p_2 \circ i_* \circ p_1^{-1} \circ g^*(\alpha) \\ &= f^* \circ t_1^{-1}(\alpha) = f^*(U_k)\,. \end{aligned}$$

Then we have $f^*(U_k) = u$.

ii) Sufficiency. By the transversality theorem of homology manifolds, we have a homology submanifold N^{m-k} realizing the homology class z.

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