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MARTINGALE SEQUENCE OF BOUNDED VARIATION

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Introduction. D. L. Burkholder proved ([1] Theorem 5) that an L^1-bounded martingale sequence is of bounded variation a.s. on every atom of the basic probability space. This result was proved by the general convergence theorem of martingale transforms. We shall give in § 1 a direct simple proof of this theorem. In § 2 we shall give some counter examples concerning the majoration inequalities. And in § 3 we shall show that the conclusion of the Burkholder theorem is not necessarily true on the atomless part of the probability space.

1. THEOREM (Burkholder). *Let $X = \{X_n, \mathscr{F}_n, n \geqq 1\}$ be a martingale defined on a probability space $(\Omega, \mathscr{F}, \mathrm{P})$, and denote its difference sequence by $\{d_n\}$, $d_n = X_n - X_{n-1}$, $n = 1, 2, \cdots$; $X_0 = 0$. If X is L^1 bounded:*

$$\sup_n \mathrm{E}\,|X_n| = K < \infty ,$$

and if A is an atom of the probability space, then $\sum_n |d_n| < \infty$, a.s. on A.

PROOF. Every random variable is constant a.s. on every atom, so we can put $X_n = a_n$, a.s. $(n = 1, 2, \cdots)$ on the atom A where $\mathrm{P}(A) > 0$ and a_n are real constants. The sequence $\{a_n\}$ is bounded, say $|a_n| \leqq a$ for all n, as we see easily from the inequalities:

$$|a_n|\,\mathrm{P}(A) \leqq \mathrm{E}\,|X_n| \leqq \sup_n \mathrm{E}\,|X_n| = K .$$

Put $A_n = \{X_n = a_n\}$ and $A_n^* = \bigcap_{j=1}^n A_j$. Clearly $A_n, A_n^* \in \mathscr{F}_n$, $A_n^* \supset A_{n+1}^*$ and $A_n^* \supset A$ for all n, since $A_n \in \mathscr{F}_n$ and $A_n \supset A$. By the martingale equality

$$\int_{A_n^*} a_n = \int_{A_n^*} X_n = \int_{A_n^*} X_{n+1} = \int_{A_{n+1}^*} a_{n+1} + \int_{A_n^* - A_{n+1}^*} X_{n+1} ,$$

hence easily

$$(a_n - a_{n+1})\mathrm{P}(A_{n+1}^*) = \int_{A_n^* - A_{n+1}^*} (X_{n+1} - a_n) .$$

So that it follows from the submartingale inequality applied to $\{|X_n|,$

$\mathscr{F}_n, n \geqq 1\}$,

$$|a_n - a_{n+1}| \mathrm{P}(A) \leqq \int_{A_n^* - A_{n+1}^*} (|X_{n+1}| + a) \leqq \int_{A_n^* - A_{n+1}^*} (|X_N| + a)$$

for any $N > n$. Summing up for $n = 1, 2, \cdots, N-1$

$$\sum_{n=1}^{N-1} |a_n - a_{n+1}| \mathrm{P}(A) \leqq \sum_{n=1}^{N-1} \int_{A_n^* - A_{n+1}^*} (|X_N| + a)$$
$$\leqq \int_{A_1} (|X_N| + a) \leqq K + a \,,$$

hence $\sum_1^\infty |d_n| = \sum_1^\infty |a_{n+1} - a_n| < \infty$ on A a.s. q.e.d.

2. Let us suppose that the probability space is atomic, the Burkholder theorem implies $\sum |d_n| < \infty$, a.s. But, even in this case, the majoration theorem of the type

$$\mathrm{E}[\textstyle\sum |d_n|] \leqq c\, \mathrm{E}\left[\sup_n |X_n|\right] \tag{1}$$

where c is a universal constant, is not necessarily true. We shall give two examples. Note that the expectation in the right hand side of (1) can be replaced by the square function $\mathrm{E}[(\sum d_n^2)^{1/2}]$, because they are comparable (Davis [2]).

EXAMPLE 1. Let N be a positive integer. Let $\Omega = (0, 1)$ and P be the Lebesgue measure on Ω. Denote by $r_1, \cdots, r_N$ the first N Rademacher functions and put $\mathscr{F} = \sigma(r_1, \cdots, r_N)$. Clearly the probability space $(\Omega, \mathscr{F}, \mathrm{P})$ is atomic. Put, for $1 \leqq n \leqq N$,

$$d_n = r_n,\ X_n = \sum_{k=1}^n r_k,\ \mathscr{F}_n = \sigma(r_1, \cdots, r_n)\,,$$

then $\{X_n, \mathscr{F}_n, 1 \leqq n \leqq N\}$ is a martingale, and $|d_n| = 1$, a.s., so evidently

$$\mathrm{E}\left[\sum_{n=1}^N |d_n|\right] = N\,, \qquad \mathrm{E}\left[\left(\sum_{n=1}^N d_n^2\right)^{1/2}\right] = \sqrt{N}\,. \tag{2}$$

These equalities show that c in (1) could not be a universal constant.

EXAMPLE 2. We shall construct an atomic probability space and a martingale such that $\mathrm{E}[\sum |d_n|] = \infty$ and $\mathrm{E}[(\sum d_n^2)^{1/2}] < \infty$. Put $\Omega = A_1 \cup A_2 \cup \cdots$ where A_n is an atom with $\mathrm{P}(A_n) = 1/(n(n+1))$ $(n = 1, 2, \cdots)$, and for simplicity we write $\bigcup_{k=m}^n A_k = A(m, n)$.

Define for $1 \leqq n \leqq N$:

$$\mathscr{F}_n = \sigma\Big\{\Omega - A(1, 2^N),$$
$$A\Big((m-1)\frac{2^N}{2^n} + 1,\ m\frac{2^N}{2^n}\Big),\ (1 \leqq m \leqq 2^n)\Big\}\ ,$$

$$d_n = \begin{cases} 0 & \text{on } \Omega - A(1, 2^N)\ , \\ 1 & \text{on } A\Big((2m-2)\frac{2^N}{2^n} + 1,\ (2m-1)\frac{2^N}{2^n}\Big) \\ -\alpha_{n,m} & \text{on } A\Big((2m-1)\frac{2^N}{2^n} + 1,\ 2m\frac{2^N}{2^n}\Big) \qquad (1 \leqq m \leqq 2^{n-1}) \end{cases}$$

where

$$\alpha_{n,m} = \frac{\mathrm{P}\{A((2m-2)2^{N-n} + 1,\ (2m-1)2^{N-n})\}}{\mathrm{P}\{A((2m-1)2^{N-n} + 1,\ 2m \cdot 2^{N-n})\}}\ , \quad (1 \leqq m \leqq 2^{n-1})\ .$$

If we put $X_n = \sum_{k=1}^n d_k$ $(1 \leqq n \leqq N)$, we get a martingale $\{X_n, \mathscr{F}_n, 1 \leqq n \leqq N\}$ which we denote by $MG\{A_k(1 \leqq k \leqq 2^N)\}$.

Now, put $b = \max_{1 \leqq i, j \leqq 2^N}\{\mathrm{P}(A_i)/\mathrm{P}(A_j)\}$, then $1/b \leqq |\alpha_{n,m}| \leqq b$, hence $1/b \leqq |d_n| \leqq b$ $(1 \leqq n \leqq N)$ on $A(1, 2^N)$.

Therefore

$$\int_{A(1,2^N)} \sum_{n=1}^N |d_n| \geqq \frac{N}{b}\mathrm{P}(A(1, 2^N))\ ,$$

$$\int_{A(1,2^N)} \Big(\sum_{n=1}^N d_n^2\Big)^{1/2} \leqq \int_{A(1,2^N)} (Nb^2)^{1/2} \leqq b\sqrt{N}\,\mathrm{P}(A(1, 2^N))\ .$$

For an integer N, consider $MG\{A_{2^N+k}(1 \leqq k \leqq 2^N)\}$ defined as above, and let its difference sequence be $\{d_n^N, \mathscr{F}_n^N, 1 \leqq n \leqq N\}$.

Then the corresponding b, say b_N, can be estimated by

$$b_N = \mathrm{P}(A_{2^N+1})/\mathrm{P}(A_{2^{N+1}}) \leqq 4\ ,$$

and we have the following properties:

$$\bigcup_{n=1}^N \mathscr{F}_n^N = \sigma\{\Omega - A(2^N + 1, 2^{N+1}),\ A_{2^N+k}(1 \leqq k \leqq 2^N)\} = \mathscr{F}_N^N\ ,$$

$$d_n^N = 0 \text{ on } \Omega - A(2^N + 1, 2^{N+1}) \quad (1 \leqq n \leqq N)\ ,$$

$$\int_{A(2^N+1,2^{N+1})} \sum_{n=1}^N |d_n^N| \geqq \frac{N}{4}\mathrm{P}(A(2^N + 1, 2^{N+1})) \geqq \frac{1}{16}\frac{N}{2^N}\ ,$$

$$\int_{A(2^N+1,2^{N+1})} \Big\{\sum_{n=1}^N (d_n^N)^2\Big\}^{1/2} \leqq 4\sqrt{N}\,\mathrm{P}(A(2^N + 1, 2^{N+1})) \leqq \frac{2\sqrt{N}}{2^N}\ ,$$

and d_n^N $(1 \leqq n \leqq N)$ is $\mathscr{F}_k^m$-measurable for $1 \leqq k \leqq m$, $1 \leqq m \leqq N-1$.

Now we put $\mathscr{F}_1 = \mathscr{F}_1^1$, $\mathscr{F}_{N(N-1)/2+k} = \mathscr{F}_k^M \cup \mathscr{F}_{N-1}^{N-1}$ $(1 \leqq k \leqq N;\ N = 2,$

$3, \cdots)$, $d_{N(N-1)/2+k} = \alpha_N d_k^N$ $(1 \leqq k \leqq N;\ N = 1, 2, \cdots)$ where α_N is a positive constant which will be determined later, and put $X_n = \sum_{k=1}^n d_k$. Then $\{X_n, \mathscr{F}_n, n \geqq 1\}$ is a martingale defined on an atomic probability space $(\Omega, \bigvee \mathscr{F}_n, \mathrm{P})$ and we get by the above properties:

$$\mathrm{E}\left[\sum |d_n|\right] = \sum_{N=1}^{\infty} \int_{A(2^N+1, 2^{N+1})} \alpha_N \sum_{n=1}^{N} |d_n^N| \geqq \frac{1}{16} \sum_{N=1}^{\infty} \alpha_N \frac{N}{2^N},$$

$$\mathrm{E}\left[\left(\sum d_n^2\right)^{1/2}\right] = \sum_{N=1}^{\infty} \int_{A(2^N+1, 2^{N+1})} \alpha_N \left\{\sum_{n=1}^{N} (d_n^N)^2\right\}^{1/2} \leqq 4 \sum_{N=1}^{\infty} \alpha_N \frac{\sqrt{N}}{2^N}.$$

If we take $\alpha_N = 2^N N^{-2}$ we get $\mathrm{E}[\sum |d_n|] = \infty$ and $\mathrm{E}[(\sum d_n^2)^{1/2}] < \infty$.

3. By the above theorem of Burkholder, if the probability space $(\Omega, \mathscr{F}, \mathrm{P})$ is purely atomic, then every L^1 bounded martingale sequence is of bounded variation a.s. We shall show that the converse is also true, that is, if the probability space is not purely atomic, there exists an L^1 bounded martingale sequence which is not of bounded variation on the atomless part of the probability space.

To show this we shall construct a counter example. Suppose that the probability space $(\Omega, \mathscr{F}, \mathrm{P})$ is not purely atomic, then there is a decomposition of Ω into disjoint $\mathscr{F}$ measurable sets: $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \cup \cdots$ where each of the sets $\Omega_1, \Omega_2, \cdots$ are either empty or an atom, and Ω_0 is an atomless part with $\mathrm{P}(\Omega_0) > 0$. It is known that Ω_0 has the intermediate value property, that is, for every set $B \subset \Omega_0$, $B \in \mathscr{F}$ and every number b, $0 < b < \mathrm{P}(B)$ there exists a set $C \in \mathscr{F}$ such that $C \subset B$ and $\mathrm{P}(C) = b$ (Cf. e.g. [3] p. 174(2)).

For this decomposition we define a sequence of random variables $\{d_n\}$ inductively. Put $d_0 = d_0(\omega) = 1_{\Omega_0}$. By the intermediate value property there are two sets E_1 and E_2 such that $E_1 \cup E_2 = \Omega_0$, $E_1 \cap E_2 = \varnothing$, $\mathrm{P}(E_1) = \mathrm{P}(E_2) = (1/2)\mathrm{P}(\Omega_0)$. We define $d_1 = d_1(\omega) = 1/2$ for $\omega \in E_1$, $= -1/2$ for $\omega \in E_2$ and $=0$ for $\omega \notin \Omega_0$. By the same way each of the sets E_1, E_2 is decomposed into two sets such that

$$E_{1,1} \cup E_{1,2} = E_1, \qquad E_{2,1} \cup E_{2,2} = E_2,$$
$$E_{1,1} \cap E_{1,2} = \varnothing, \qquad E_{2,1} \cap E_{2,2} = \varnothing$$

and $\mathrm{P}(E_{i,j}) = (1/2)\mathrm{P}(E_i) = (1/4)\mathrm{P}(\Omega_0)$ for $i, j = 1, 2$. We define $d_2 = d_2(\omega) = 1/3$ for $\omega \in E_{1,1} \cup E_{2,1}$, $= -1/3$ for $\omega \in E_{2,1} \cup E_{2,2}$, and $=0$ for $\omega \notin \Omega_0$. In general, the sets $E_{i_1,\dots,i_n}(i_1,\dots,i_n = 1$ or 2; $n = 1, 2, \cdots)$ are defined such that

$$E_{i_1,\dots,i_{n-1},1} \cup E_{i_1,\dots,i_{n-1},2} = E_{i_1,\dots,i_{n-1}}$$
$$E_{i_1,\dots,i_{n-1},1} \cap E_{i_1,\dots,i_{n-1},2} = \varnothing$$

and $\mathrm{P}(E_{i_1,\dots,i_n}) = (1/2)\mathrm{P}(E_{i_1,\dots,i_{n-1}}) = (1/2^n)\mathrm{P}(\Omega_0)$; and we put $d_n = d_n(\omega) =$

$1/(n+1)$ for $\omega \in E_{i_1,\cdots,i_{n-1},1}$, $=-1/(n+1)$ for $\omega \in E_{i_1,\cdots,i_{n-1},2}$ and $=0$ for $\omega \notin \Omega_0(i_1, \cdots, i_{n-1} = 1$ or $2)$.

Let $\mathscr{F}_n$ be the sub-σ-field generated by $d_1, \cdots, d_n$ and write $X_n = \sum_{j=1}^n d_j$. As we see easily $X = \{X_n, \mathscr{F}_n\}$ is a martingale and L^2 bounded since $\mathrm{E}(X_n^2) = \mathrm{P}(\Omega_0) \sum_{j=1}^\infty 1/(j+1)^2 < \infty$, but $\sum_n |d_n| = \sum_n 1/(n+1) = \infty$ on Ω_0, that is, X is not of bounded variation on Ω_0.

We remark finally that, combining the above example and the Burkholder theorem we get the following theorem which is of similar form to a result of E. Marczewski [4] and Thomasian [5].

THEOREM. *For a probability space* $(\Omega, \mathscr{F}, \mathrm{P})$, *the following statements are equivalent*:

(1) *Ω is a sum of disjoint atoms,*

(2) *Any L^1-bounded martingale sequence is of bounded variation a.s.*

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