Tôhoku Math. Journ.
28 (1976), 257-266.

SURJECTIVITY OF THE BERS MAP

MASAMI NAKADA

(Received February 18, 1975)

Preliminaries. Let G be a non-elementary finitely generated Kleinian group with the region of discontinuity Ω and let $B_q(\Omega, G)$ be the space of bounded holomorphic automorphic forms of weight $-2q$ for G operating on Ω, where $q(\geqq 2)$ is an integer. We denote by Π_{2q-2} the vector space of complex polynomials in one variable of degree at most $2q-2$. Clearly Π_{2q-2} is a G-module with $(v \cdot \gamma)(z) = v(\gamma(z))\gamma'(z)^{1-q}$ for $v \in \Pi_{2q-2}$ and $\gamma \in G$.

A mapping $p: G \rightarrow \Pi_{2q-2}$ is a cocycle if

$$p(\gamma_1 \circ \gamma_2) = p(\gamma_1) \cdot \gamma_2 + p(\gamma_2)$$

for any $\gamma_1, \gamma_2 \in G$. This mapping is a coboundary if

$$p(\gamma) = v \cdot \gamma - v$$

for some polynomial $v \in \Pi_{2q-2}$.

We denote by $H^1(G, \Pi_{2q-2})$ the first cohomology space of G with coefficients in Π_{2q-2}, that is, $H^1(G, \Pi_{2q-2})$ is the cocycles factored by the coboundaries. An element of $H^1(G, \Pi_{2q-2})$ with a representative p will be denoted by $\{p\}$.

Let Δ be any G-invariant union of components of Ω. We denote by $B_q(\Delta, G)$ the space of all elements in $B_q(\Omega, G)$ which vanish in $\Omega - \Delta$. Clearly we see that $B_q(\Delta, G)$ is the space of bounded holomorphic automorphic forms of weight $-2q$ for G operating on Δ.

The Δ-parabolic cohomology space $PH^1_\Delta(G, \Pi_{2q-2})$ is the subspace of $H^1(G, \Pi_{2q-2})$ such that each element $\{p\} \in PH^1_\Delta(G, \Pi_{2q-2})$ satisfies the condition that at every parabolic cusp in Δ, there exists a polynomial $v \in \Pi_{2q-2}$ with the property

$$p(\gamma) = v \cdot \gamma - v$$

for some (and hence for every) cocycle $p \in \{p\}$ and for any γ in the parabolic cyclic subgroup G_0 of G corresponding to the cusp. In particular, if the above condition is satisfied for every parabolic cyclic subgroup G_0 of G, then we say that $\{p\}$ belongs to $PH^1(G, \Pi_{2q-2})$, the space of parabolic cohomology.

As is well known, Bers [1] introduced the anti-linear map $\beta^*: B_q(\Omega, G)$

$\rightarrow H^1(G, \Pi_{2q-2})$ and proved that this mapping is injective. The map β^* is called the Bers map and the image $\beta^*(\psi)$ of $\psi \in B_q(\Omega, G)$ under β^* is called a Bers cohomology class of ψ. In [2], Kra proved the inclusion relation $\beta^*(B_q(\Omega, G)) \subset PH^1(G, \Pi_{2q-2})$.

A holomorphic function F on Δ is called an Eichler integral (of order $1-q$) on Δ, if

$$F \cdot \gamma - F \in \Pi_{2q-2}$$

for any $\gamma \in G$, where $F \cdot \gamma = F(\gamma(z))\gamma'(z)^{1-q}$. We denote by $E_{1-q}(\Delta, G)$ the space of Eichler integrals (of order $1-q$) on Δ modulo Π_{2q-2}.

For a holomorphic function F on Δ, we define an operator D^{2q-1} by the equality

$$(D^{2q-1}F)(z) = \frac{d^{2q-1}}{dz^{2q-1}} F(z) .$$

Bol's identity shows that, for an Eichler integral F on Δ, $D^{2q-1}F$ is a holomorphic automorphic form of weight $-2q$ for G operating on Δ.

We denote by $E^b_{1-q}(\Delta, G)$ the subspace of $E_{1-q}(\Delta, G)$ whose element f satisfies $D^{2q-1}f \in B_q(\Delta, G)$. We note that if G has no parabolic elements, then $PH^1(G, \Pi_{2q-2}) = PH^1_\Delta(G, \Pi_{2q-2}) = H^1(G, \Pi_{2q-2})$ and $E^b_{1-q}(\Delta, G) = E_{1-q}(\Delta, G)$. An Eichler integral F on Δ is called trivial if $D^{2q-1}F = 0$. The space of all images of trivial Eichler integrals on Δ in $E_{1-q}(\Delta, G)$ is denoted by $E^0_{1-q}(\Delta, G)$.

In his paper [2], Kra proved that the mapping

$$\alpha\colon E^b_{1-q}(\Delta, G) \rightarrow PH^1_\Delta(G, \Pi_{2q-2})$$

defined by $\alpha(f) = \{p\}$, where $p(\gamma) = F \cdot \gamma - F$ for a representative p of $\{p\}$ and a representative F of f, is injective. The image $\alpha(f)$ of $f \in E^b_{1-q}(\Delta, G)$ is called an Eichler cohomology class of f. Kra also proved the following

THEOREM (Kra [2]). *Let G be a non-elementary finitely generated Kleinian group with an invariant component Δ_0 such that $\Omega - \Delta_0 \neq \emptyset$. Then every cohomology class $\{p\} \in PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$ can be written as a sum of a Bers cohomology class of some $\psi \in B_q(\Omega, G)$ and an Eichler cohomology class of some $f \in E^0_{1-q}(\Omega - \Delta_0, G)$.*

From this Theorem, we can easily obtain the following Proposition.

PROPOSITION. *If $\Omega - \Delta_0 \neq \emptyset$ and if $E^0_{1-q}(\Omega - \Delta_0, G) = 0$, then $\beta^*(B_q(\Omega, G)) = PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$, so $\beta^*\colon B_q(\Omega, G) \rightarrow PH^1(G, \Pi_{2q-2})$ is surjective.*

This can be considered as a proposition which gives a sufficient condition for surjectivity of the Bers map. In this article we shall give another sufficient condition. We shall also treat a converse problem "Does surjectivity of the Bers map imply $E^0_{1-q}(\Omega - \Delta_0, G) = 0$?" and give a negative answer.

1. First we shall prove two lemmas.

LEMMA 1. *Let G be a non-elementary Kleinian group with N generators $\gamma_1, \cdots, \gamma_N$. If one of generators, γ_i, is elliptic of order ν_i, then*

$$\dim H^1(G, \Pi_{2q-2}) \leqq (2q-1)(N-1) - \left\{2\left[\frac{q-1}{\nu_i}\right] + 1\right\},$$

where $[x]$ is the integral part of x.

PROOF. We denote by $Z^1(G, \Pi_{2q-2})$ the space of cocycles. Since a cocycle is uniquely determined by its values on generators of G, we have $\dim Z^1(G, \Pi_{2q-2}) \leqq (2q-1)N$ (see Bers [1]).

From the fact that the dimension of $Z^1(G, \Pi_{2q-2})$ is invariant under conjugation by a linear transformation, we may assume that $\gamma_i(z) = \lambda z$, $\lambda^{\nu_i} = 1$, $\lambda \neq 1$. For simplicity we set $\gamma_i(z) = \gamma(z)$ and $\nu_i = \nu$.

Let p be a cocycle and set $p(\gamma) = \sum_{k=0}^{2q-2} a_k z^k$. Since $\gamma^\nu = id$, we have

$$0 = p(\gamma^\nu) = \sum_{k=0}^{2q-2} a_k(1 + \lambda^{k+1-q} + \cdots + \lambda^{(\nu-1)(k+1-q)})z^k .$$

If $\lambda^{k+1-q} \neq 1$, then $1 + \lambda^{k+1-q} + \cdots + \lambda^{(\nu-1)(k+1-q)} = 0$. Hence, for k satisfying $\lambda^{k+1-q} = 1$, we have $a_k = 0$. Therefore, it is clear that

$$\dim Z^1(G, \Pi_{2q-2}) \leqq (2q-1)N - t ,$$

where t is the number of integers k for which $\lambda^{k+1-q} = 1$.

Since $\lambda^\nu = 1$, we see that, t equals the number of integers j such that $k + 1 - q = \nu j$ for k, $0 \leqq k \leqq 2q-2$. Such an integer j satisfies $0 \leqq \nu j + q - 1 \leqq 2q - 2$, so

$$-\frac{q-1}{\nu} \leqq j \leqq \frac{q-1}{\nu} ,$$

from which we have $t = 2[(q-1)/\nu] + 1$.

On the other hand, it is well known that the dimension of the space of coboundaries is $2q-1$. Therefore we have our lemma.

REMARK. Under the assumption of Lemma 1, if generators γ_i $(i = 1, \cdots, m \leqq N)$ are elliptic of order ν_i, then

$$\dim H^1(G, \Pi_{2q-2}) \leqq (2q-1)(N-1) - \left\{2\sum_{i=1}^{m}\left[\frac{q-1}{\nu_i}\right] + m\right\} .$$

LEMMA 2. *Under the same assumption as in Lemma* 1, *if one of generators*, γ_i, *is parabolic, then*

$$\dim PH^1(G, \Pi_{2q-2}) \leqq (2q-1)(N-1) - 1 .$$

PROOF. Let p be an element of $PZ^1(G, \Pi_{2q-2})$, the space of parabolic cocycles for G. By conjugation by a linear transformation, we may assume that $\gamma_i(z) = z + 1$. For simplicity we set $\gamma_i(z) = \gamma(z)$. By definition, $p(\gamma) = v \cdot \gamma - v$ for some polynomial $v \in \Pi_{2q-2}$. If $p(\gamma) = \sum_{k=0}^{2q-2} a_k z^k$, then $\sum_{k=0}^{2q-2} a_k z^k = v \cdot \gamma - v = v(z+1) - v(z)$. Since $v(z+1) - v(z) \in \Pi_{2q-3}$, we have $a_{2q-2} = 0$, that is

$$\dim PZ^1(G, \Pi_{2q-2}) \leqq (2q-1)N - 1 .$$

By the same reasoning as in the proof of Lemma 1, we have our lemma.

REMARK. Under the assumption of Lemma 2, if generators γ_i $(i = 1, \cdots, n \leqq N)$ are parabolic, then

$$\dim PH^1(G, \Pi_{2q-2}) \leqq (2q-1)(N-1) - n .$$

Now we can prove following for a quasi-Fuchsian group which is a quasi-conformal deformation of a Fuchsian group.

THEOREM 1. *For any non-elementary finitely generated quasi-Fuchsian group G, the Bers map $\beta^*: B_q(\Omega, G) \to PH^1(G, \Pi_{2q-2})$ is surjective.*

PROOF. For a quasi-Fuchsian group G with two invariant components Kra [2] proved surjectivity of the Bers map β^*. (See also Remark of Theorem 2 stated later.) So it is sufficient to prove Theorem for a quasi-Fuchsian group G with the connected region of discontinuity, that is, for a quasi-Fuchsian group G of the second kind.

There exist a Fuchsian group Γ of the second kind and a quasi-conformal mapping $w: \hat{C} \to \hat{C}$ such that $G = w \circ \Gamma \circ w^{-1}$. Let $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_m, \delta_1, \cdots, \delta_n, \eta_1, \cdots, \eta_k$ be standard generators of Γ with the defining relations

$$[\alpha_1, \beta_1] \circ \cdots \circ [\alpha_g, \beta_g] \circ \gamma_1 \circ \cdots \circ \gamma_m \circ \delta_1 \circ \cdots \circ \delta_n \circ \eta_1 \circ \cdots \circ \eta_k = id$$

and

$$\gamma_i^{\nu_i} = id \qquad (i = 1, \cdots, m) ,$$

where $\alpha_i, \beta_i, \eta_i$ are hyperbolic, γ_i are elliptic of order ν_i, δ_i are parabolic and $[\alpha_i, \beta_i]$ means a commutator $\alpha_i \circ \beta_i \circ \alpha_i^{-1} \circ \beta_i^{-1}$.

Since Γ is generated by $(2g+m+n+k-1)$ elements $\alpha_1, \cdots, \eta_{k-1}$ with relations $\gamma_i^{\nu_i} = id$, the quasi-Fuchsian group G is generated by $(2g+m+n+k-1)$ elements $w \circ \alpha_i \circ w^{-1}$, $w \circ \beta_i \circ w^{-1}$ $(1 \leqq i \leqq g)$, $w \circ \gamma_i \circ w^{-1}$ $(1 \leqq i \leqq m)$, $w \circ \delta_i \circ w^{-1}$ $(1 \leqq i \leqq n)$, $w \circ \eta_i \circ w^{-1}$ $(1 \leqq i \leqq k-1)$ with relations $(w \circ \gamma_i \circ w^{-1})^{\nu_i} = id$, where $w \circ \delta_i \circ w^{-1}$ are parabolic. Hence, from Remarks of Lemma 1 and Lemma 2, we have

$$\begin{aligned}\dim PH^1(G, \Pi_{2q-2}) \leqq &(2q-1)(2g+m+n+k-1-1)\\ &-\left\{2\sum_{i=1}^{m}\left[\frac{q-1}{\nu_i}\right]+m\right\}-n\,.\end{aligned}$$

We denote by $\Omega(G)$ and $\Omega(\Gamma)$ the regions of discontinuity of G and Γ, respectively. Since $G = w \circ \Gamma \circ w^{-1}$, we have

$$\dim B_q(\Omega(G), G) = \dim B_q(\Omega(\Gamma), \Gamma)\,,$$

whence

$$\begin{aligned}\dim B_q(\Omega(G), G) = &(2q-1)(2g+k-1-1)\\ &+2\sum_{i=1}^{m}\left[q-\frac{q}{\nu_i}\right]+2n(q-1)\,.\end{aligned}$$

Consequently

$$\begin{aligned}&\dim B_q(\Omega(G), G) - \dim PH^1(G, \Pi_{2q-2})\\ &\geqq 2\sum_{i=1}^{m}\left\{\left[q-\frac{q}{\nu_i}\right]+\left[\frac{q-1}{\nu_i}\right]-(q-1)\right\}.\end{aligned}$$

It is easily seen that $[q - q/\nu_i] + [(q-1)/\nu_i] = q-1$. Therefore we obtain

$$\dim B_q(\Omega(G), G) \geqq \dim PH^1(G, \Pi_{2q-2})\,. \tag{1}$$

Since $\beta^*(B_q(\Omega(G), G)) \subset PH^1(G, \Pi_{2q-2})$ and since β^* is injective, we have the converse inequality for (1). Hence

$$\dim B_q(\Omega(G), G) = \dim PH^1(G, \Pi_{2q-2})\,,$$

which shows

$$\beta^*(B_q(\Omega(G), G)) = PH^1(G, \Pi_{2q-2})\,.$$

This completes the proof of Theorem 1.

COROLLARY. *For any non-elementary finitely generated quasi-Fuchsian group G, it holds that $E^b_{1-q}(\Omega, G) = 0$.*

PROOF. This is an immediate consequence of Kra's unique decomposition (Corollary 1 to Theorem 4 in [2]) and Theorem 1.

2. Now we consider the problem whether surjectivity of the Bers map imply $E_{1-q}^{0}(\Omega-\Delta_0, G)=0$ or not.

For the purpose we shall first treat a non-elementary finitely generated Kleinian group G with an invariant component Δ_0 such that $\Omega-\Delta_0\neq\emptyset$ and such that $E_{1-q}^{0}(\Omega-\Delta_0, G)=0$.

We can prove the following lemma.

LEMMA 3. *Let G be a non-elementary finitely generated Kleinian group with an invariant component Δ_0 such that $\Omega-\Delta_0\neq\emptyset$ and let $q(\geqq 2)$ be an integer. If $E_{1-q}^{0}(\Omega-\Delta_0, G)=0$, then*

$$\dim B_q(\Delta_0, G)=\dim B_q(\Omega-\Delta_0, G)\,.$$

PROOF. Since $E_{1-q}^{0}(\Omega-\Delta_0, G)=0$, Kra's theorem stated in the above preliminaries and the injectivity of the Bers map $\beta^*\colon B_q(\Omega, G)\rightarrow H^1(G, \Pi_{2q-2})$ imply

$$(2)\qquad \dim PH_{\Omega-\Delta_0}^{1}(G, \Pi_{2q-2})=\dim B_q(\Omega, G)\,.$$

On the other hand, Kra's unique decomposition (Corollary 1 to Theorem 4 in [2]) implies

$$PH_{\Omega-\Delta_0}^{1}(G, \Pi_{2q-2})=\beta^*(B_q(\Omega-\Delta_0, G))\oplus\alpha(E_{1-q}^{b}(\Omega-\Delta_0, G))\,,$$

where α and β^* are injective and the notation $\oplus$ means the direct sum. Therefore

$$(3)\qquad \dim PH_{\Omega-\Delta_0}^{1}(G, \Pi_{2q-2})=\dim B_q(\Omega-\Delta_0, G)+\dim E_{1-q}^{b}(\Omega-\Delta_0, G)\,.$$

Since $E_{1-q}^{0}(\Omega-\Delta_0, G)$ is the kernel of the operator D^{2q-1}, we have

$$D^{2q-1}\colon E_{1-q}^{b}(\Omega-\Delta_0, G)\rightarrow B_q(\Omega-\Delta_0, G)$$

is injective and hence we see

$$(4)\qquad \dim E_{1-q}^{b}(\Omega-\Delta_0, G)\leqq\dim B_q(\Omega-\Delta_0, G)\,.$$

From (2), (3) and (4) we have

$$\dim B_q(\Omega, G)\leqq 2\dim B_q(\Omega-\Delta_0, G)\,,$$

which yields

$$\dim B_q(\Delta_0, G)\leqq\dim B_q(\Omega-\Delta_0, G)\,.$$

Further, from the assumption we have $\dim B_q(\Delta_0, G)\geqq\dim B_q(\Omega-\Delta_0, G)$ (see [1]). Therefore

$$\dim B_q(\Delta_0, G)=\dim B_q(\Omega-\Delta_0, G)\,,$$

which proves our lemma.

As an application of Lemma 3, we have the following characterization of a quasi-Fuchsian group.

THEOREM 2. *Let G be a non-elementary finitely generated Kleinian group with an invariant component Δ_0 such that $\Omega - \Delta_0 \neq \emptyset$. Then $E^0_{1-q}(\Omega - \Delta_0, G) = 0$ for an even integer q if and only if G is a quasi-Fuchsian group.*

PROOF. The proof of the if part of our Theorem is easily obtained by noting that $\Omega - \Delta_0$ is connected. On the other hand, under the same assumption on G in our Theorem, Maskit [3] proved that $\dim B_q(\Delta_0, G) = \dim B_q(\Omega - \Delta_0, G)$ for some even integer q if and only if G is a quasi-Fuchsian. Thus Lemma 3 gives a proof of the only if part of our Theorem.

REMARK. Proposition stated in preliminaries and Theorem 2 imply surjectivity of the Bers map for a quasi-Fuchsian group G under the assumption $\Omega - \Delta_0 \neq \emptyset$.

Next we prove the following lemma.

LEMMA 4. *Let G_1 and G_2 be non-elementary finitely generated Kleinian groups and let $G = \langle G_1, G_2 \rangle$ be the Kleinian group generated by G_1 and G_2. Then*

$$\begin{aligned}\dim PH^1(G, \Pi_{2q-2}) \leqq \dim PH^1(G_1, \Pi_{2q-2}) \\ + \dim PH^1(G_2, \Pi_{2q-2}) + (2q-1)\,,\end{aligned}$$

where equality holds whenever $\langle G_1, G_2 \rangle$ is the free product of G_1 and G_2.

PROOF. We denote by $PZ^1(\Gamma, \Pi_{2q-2})$ the space of parabolic cocycles for a Kleinian group Γ and by $B^1(\Gamma, \Pi_{2q-2})$ the space of coboundaries for Γ.

Consider the linear map Φ

$$\Phi\colon PZ^1(G, \Pi_{2q-2}) \to PZ^1(G_1, \Pi_{2q-2}) \times PZ^1(G_2, \Pi_{2q-2})$$

defined by $\Phi(p) = (p_1, p_2)$, where p_i is the restriction of p to G_i for $i = 1, 2$.

Since G is generated by G_1 and G_2, we see that the map Φ is injective. Therefore we have

$$(5)\qquad \dim PZ^1(G, \Pi_{2q-2}) \leqq \dim PZ^1(G_1, \Pi_{2q-2}) + \dim PZ^1(G_2, \Pi_{2q-2})\,.$$

From the assumption that G_1 and G_2 are non-elementary, we see that G is non-elementary. Hence, by Bers [1],

$$(6)\quad \dim B^1(G, \Pi_{2q-2}) = \dim B^1(G_1, \Pi_{2q-2}) = \dim B^1(G_2, \Pi_{2q-2}) = 2q-1\,.$$

Combining (5) and (6) we have

$$\dim PH^1(G, \Pi_{2q-2}) \leqq \dim PH^1(G_1, \Pi_{2q-2}) + \dim PH^1(G_2, \Pi_{2q-2}) + 2q - 1 .$$

In particular, if $\langle G_1, G_2\rangle$ is the free product of G_1 and G_2, then Φ is surjective. Therefore, in this case, we have the equality in (5) and hence the equality holds in our lemma. This completes the proof of Lemma 4.

We shall prove the following

THEOREM 3. *There exists a non-elementary finitely generated Kleinian group G with an invariant component Δ_0 such that $\Omega - \Delta_0 \neq \emptyset$ and $E^0_{1-q}(\Omega - \Delta_0, G) \neq 0$ and such that $\beta^*(B_q(\Omega, G)) = PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$ for any integer q.*

PROOF. Let S_0 be a Riemann surface of type $(g_0, m+n)$, where g_0 is the genus of S_0 and $m+n$ is the number of punctures of S_0. We assume that $3g_0 - 3 + (m+n) > 0$. Now, associate with an integer $\nu_i(\nu_i \geqq 2,\ i = 1, \cdots, m)$ m punctures and associate with ∞ the remainder n punctures. Let C be a simple loop on S_0 which bounds neither a disk nor a punctured disk on S_0 and which separates S_0 into two pieces, which we denote by S_1' and S_2'. We attach a disk along C to S_i' for $i = 1, 2$ and we denote the resulting surfaces by S_1 and S_2. We give a conformal structure to S_i so as to be a finite Riemann surface and we denote by S_i^+ this Riemann surface for $i = 1, 2$. Let (g_i, t_i) be type of S_i^+ $(i = 1, 2)$. If g_0 is sufficiently large, we may choose C such that $g_1 \geqq 2$ and $g_2 \geqq 2$. Here we note that $g_0 = g_1 + g_2$ and $m + n = t_1 + t_2$.

As Maskit [3] has stated, we can construct a Kleinian group G with an invariant component Δ_0 such that $\Delta_0'/G = S_0$ and $(\Omega - \Delta_0)'/G = S_1^+ + S_2^+$, where Ω is the region of discontinuity of G and $\Delta_0' = \Delta_0 - \{$all elliptic fixed points of $G\}$, $(\Omega - \Delta_0)' = (\Omega - \Delta_0) - \{$all elliptic fixed points of $G\}$. For a component Δ_i of $\Omega - \Delta_0$, we have $S_i^+ = \Delta_i'/G_i$, where $G_i = \{\gamma \in G; \gamma(\Delta_i) = \Delta_i\}$, $\Delta_i' = \Delta_i - \{$all elliptic fixed points of $G\}$ for $i = 1, 2$ and G is generated by G_1 and G_2.

From Lemma 4 we have

$$\dim PH^1(G, \Pi_{2q-2}) \leqq \dim PH^1(G_1, \Pi_{2q-2}) + \dim PH^1(G_2, \Pi_{2q-2}) + (2q-1) .$$

Since G is a finitely generated Kleinian group with an invariant component Δ_0, we see that G_1 and G_2 are finitely generated quasi-Fuchsian groups with an invariant component Δ_1 and Δ_2, respectively. Therefore

$$\dim PH^1(G_i, \Pi_{2q-2}) = 2 \dim B_q(\Delta_i, G_i)$$

for $i=1, 2$. Since $\Delta_i'/G_i = S_i^+$, we have

$$\dim B_q(\Delta_i, G_i) = (2q-1)(g_i-1) + \sum_{x \in \bar{S}_i^+ - S_i^+} \left[q - \frac{q}{\nu(x)}\right],$$

where $\nu(x)$ equals ν_i or ∞ and $[q - q/\nu(x)] = q-1$ when $\nu(x) = \infty$. Hence we see that

$$\begin{aligned} \dim PH^1(G, \Pi_{2q-2}) &\leqq 2\{\dim B_q(\Delta_1, G_1) + \dim B_q(\Delta_2, G_2)\} + (2q-1) \\ &= 2\left\{(2q-1)(g_1+g_2-2) + \sum_{x \in \bar{S}_0 - S_0} \left[q - \frac{q}{\nu(x)}\right]\right\} \\ &\quad + (2q-1) \\ &= (2q-1)(2g_0-3) + 2\sum_{i=1}^{m}\left[q - \frac{q}{\nu_i}\right] + 2n(q-1)\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} \dim B_q(\Omega, G) &= \dim B_q(\Delta_0, G) + \dim B_q(\Delta_1, G_1) + \dim B_q(\Delta_2, G_2) \\ &= (2q-1)(2g_0-3) + 2\sum_{i=1}^{m}\left[q - \frac{q}{\nu_i}\right] + 2n(q-1)\,. \end{aligned}$$

Consequently, we have

$$\dim PH^1(G, \Pi_{2q-2}) \leqq \dim B_q(\Omega, G)\,.$$

Since $\beta^*(B_q(\Omega, G)) \subset PH^1(G, \Pi_{2q-2})$ and since β^* is injective, we have the converse inequality. Hence

$$\dim PH^1(G, \Pi_{2q-2}) = \dim B_q(\Omega, G)\,, \text{ or, } \beta^*(B_q(\Omega, G)) = PH^1(G, \Pi_{2q-2})\,.$$

Since $\Omega'/G = S_0 + S_1^+ + S_2^+$, we see that G is not a quasi-Fuchsian group, where $\Omega' = \Omega - \{\text{all elliptic fixed points of } G\}$.

Let $\{p\}$ be an element of $PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$. Then, for any parabolic element γ_0 belonging to G_1 or G_2, $p(\gamma_0) = v\cdot\gamma_0 - v$ for some $v \in \Pi_{2q-2}$. Take an arbitrary parabolic $\gamma \in G$. Then $\gamma = \alpha\circ\gamma_0\circ\alpha^{-1}$ for some γ_0 and some $\alpha \in G$. (See Maskit [3].) Hence we have $p(\gamma) = p(\alpha\circ\gamma_0\circ\alpha^{-1}) = V\cdot\gamma - V$ for $V = v\cdot\alpha^{-1} - p(\alpha^{-1}) \in \Pi_{2q-2}$. Hence $\{p\} \in PH^1(G, \Pi_{2q-2})$, that is, $PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2}) \subset PH^1(G, \Pi_{2q-2})$. On the other hand, obviously $PH^1(G, \Pi_{2q-2}) \subset PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$. Therefore we have $PH^1(G, \Pi_{2q-2}) = PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$, which shows $\beta^*(B_q(\Omega, G)) = PH^1_{\Omega-\Delta_0}(G, \Pi_{2q-2})$.

Now we have only to show $E^0_{1-q}(\Omega - \Delta_0, G) \neq 0$. For our group G, we see that

$$\dim B_q(\Delta_0, G) > \dim B_q(\Delta_1, G_1) + \dim B_q(\Delta_2, G_2) = \dim B_q(\Omega - \Delta_0, G)\,.$$

Hence, by Lemma 3 we have $E^0_{1-q}(\Omega - \Delta_0, G) \neq 0$. Thus the proof of our Theorem is complete.

REFERENCES

[1] L. BERS, Inequalities for finitely generated Kleinian groups, J. d'Anal. Math., 18 (1967), 23-41.
[2] I. KRA, On cohomology of Kleinian groups: II, Ann. of Math., 90 (1969), 575-589.
[3] B. MASKIT, Decomposition of certain Kleinian groups, Acta Math., 130 (1973), 243-263.

DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA, JAPAN