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THE FIXED ALGEBRA OF A VON NEUMANN ALGEBRA UNDER AN AUTOMORPHISM GROUP

MARIE CHODA

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1. Let G be a discrete countable group of (*-)automorphisms of a von Neumann algebra $\mathscr{A}$. In [2], we called G a shift or a central shift if G satisfies the conditions in Definition 1 in the below. Denote by $\mathscr{A}^G$ the set of fixed points of $\mathscr{A}$ under every g in G and call it the fixed algebra of $\mathscr{A}$ under G. If G is a central shift of $\mathscr{A}$, then the crossed product $G \otimes \mathscr{A}$ of $\mathscr{A}$ by G is isomorphic to the tensor product $\mathscr{A}^G \otimes \mathscr{L}(l^2(G))$ of the fixed algebra $\mathscr{A}^G$ of $\mathscr{A}$ under G and the algebra $\mathscr{L}(l^2(G))$ of all bounded linear operators on $l^2(G)$, ([2: Theorem 2]). Therefore, in order to study properties of the crossed product $G \otimes \mathscr{A}$ of a von Neumann algebra $\mathscr{A}$ by a central shift G, we shall need only to do the fixed algebra $\mathscr{A}^G$ of $\mathscr{A}$ under G.

In this paper, we shall examine the fixed algebra $\mathscr{A}^G$ of a von Neumann algebra $\mathscr{A}$ under a central shift G. For a general discrete group of automorphisms of a von Neumann algebra, we shall show that the algebra is decomposed into the direct sum of the part on which the group is a central shift and the part on which the group is not a central shift at all (Theorem 2). If G is a central shift of a von Neumann algebra $\mathscr{A}$, then there exists an automorphism α of $\mathscr{A}$ which has the same fixed algebra with G (Theorem 3). For a general discrete countable group G of automorphisms of a von Neumann algebra $\mathscr{A}$, as an application of Theorem 3 we shall obtain that $\mathscr{A}^G \otimes I$, the tensor product of the fixed algebra $\mathscr{A}^G$ of $\mathscr{A}$ under G and the scalar multiples of the identity on $l^2(G)$, is the intersection of $\mathscr{A} \otimes I$ and the fixed algebra $(\mathscr{A} \otimes l^\infty(G))^\sigma$ of $\mathscr{A} \otimes l^\infty(G)$ under some automorphism σ (Theorem 5). Furthermore, it is obtained that the fixed algebra $\mathscr{A}^G$ of $\mathscr{A}$ under G is the intersection of two von Neumann algebras which are isomorphic to $\mathscr{A}$ (Theorem 6).

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2. Let G be a discrete countable group of (*-)automorphisms of a von Neumann algebra $\mathscr{A}$.

DEFINITION 1 ([2]). If there exists a projection E of $\mathscr{A}$ such that $\{g(E): g \in G\}$ is an orthogonal family and $\sum_{g \in G} g(E) = 1$, then G is called a *shift* and E is called a *shift projection* of G in $\mathscr{A}$. Especially, if a shift projection E is contained in the center of $\mathscr{A}$, then G is called a *central shift*.

A finite freely acting automorphism group is a shift in general, [5, Theorem 3]. Therefore, if G is a finite abelian group of freely acting automorphisms of a von Neumann algebra $\mathscr{A}$, then there exists a unitary representation U_γ of the dual group $\hat{G}$ of G into $\mathscr{A}$ such that

$$g(U_\gamma) = (g, \gamma) U_\gamma\,, \quad \text{for every } g \in G \text{ and } \gamma \in \hat{G}\,,$$

where (g, γ) is the value of γ at g, [1, Theorem 2]. That is, if G is a finite abelian group of freely acting automorphisms of $\mathscr{A}$, then every $g \in G$ has an eigen unitary operator U_γ with the eigen value γ for every γ in $\hat{G}$, in the sense of Størmer [8].

THEOREM. *Let G be a discrete countable group of automorphisms of a von Neumann algebra $\mathscr{A}$, then there exists a central projection E of $\mathscr{A}$ with the following properties*;

(1) *E is a fixed point under every g in G,*

(2) *G is a central shift of a reduced von Neumann algebra $\mathscr{A}_E$, and*

(3) *G is not a central shift of $\mathscr{A}_F$ for any central projection F of $\mathscr{A}_{I-E}$.*

PROOF. Put

$$\mathscr{F} = \{\text{central projection } P \text{ in } \mathscr{A};\, g(P)P = 0, \text{ for each } g(\neq 1) \text{ in } G\}\,,$$

then $\mathscr{F}$ is a partially ordered set with the ordinary order of projections. Let $\mathscr{F}_0$ be a totally ordered subset of $\mathscr{F}$. Put $Q = \sup\{P; P \in \mathscr{F}_0\}$, then we have $g(Q) = \sup\{g(P); P \in \mathscr{F}_0\}$, for each g in G. Let P and R be elements in $\mathscr{F}_0$. For each $g(\neq 1)$ in G, we have $g(P)R = 0$ because $g(P)$ (resp. $g(R)$) is orthogonal to P (resp. R) and $P \geqq R$ or $R \geqq P$. It implies that $g(Q)Q = 0$ for each $g(\neq 1)$ in G, that is, Q belongs to $\mathscr{F}$. Therefore, by Zorn's lemma, there exists a maximal element P of $\mathscr{F}$. Put $E = \sum_{g \in G} g(P)$, then E is a central projection of $\mathscr{A}$ fixed under G, and G is an automorphism group of $\mathscr{A}_E$. It is clear that G is a central shift of $\mathscr{A}_E$ with a shift projection P. By the maximality of P in $\mathscr{F}$, there does not exist a nonzero central projection R in $\mathscr{A}_{I-E}$ such that

$g(R)R = 0$ for each $g(\neq 1)$ in G. Hence E satisfies the property (3).

Let G be a countable group, then we number the elements in G such that g_0 is 1 (the unit of G). So, we can regard a countable discrete central shift of a von Neumann algebra as a group generated by a single automorphism of the von Neumann algebra in the sense of the following theorem.

THEOREM 3. *Let $\mathscr{A}$ be a von Neumann algebra and G an infinite countable discrete group of automorphisms of $\mathscr{A}$. If G is a central shift of $\mathscr{A}$, then there exists an automorphism α of $\mathscr{A}$ which has the same fixed algebra with G.*

PROOF. Let E be a shift projection of G in the center of $\mathscr{A}$. Since the family $\{g_i(E); i \in I\}$ is orthogonal and $\sum_{i \in I} g_i(E) = 1$, for every A in $\mathscr{A}$, there exists an element in $\mathscr{A}$ of the form $\sum_{i \in I} g_{i+1}(g_i^{-1}(A)E)$, where I is the set of all integers. Define a mapping α of $\mathscr{A}$ to $\mathscr{A}$ by

$$\alpha(A) = \sum_{i \in I} g_{i+1}(g_i^{-1}(A)E)\ , \qquad (A \in \mathscr{A})\ ,$$

then, for every $j \in I$, we have that

$$(*) \qquad \alpha(g_j(AE)) = g_{j+1}(AE)\ , \qquad (A \in \mathscr{A})\ .$$

In fact, by the definition of α, we have that

$$\begin{aligned}\alpha(g_j(AE)) &= \sum_{i \in I} g_{i+1}\{g_i^{-1}(g_j(AE))E\} \\ &= \sum_{i \in I} g_{i+1}\{g_i^{-1}g_j(A)g_i^{-1}g_j(E)E\} \\ &= g_{j+1}(AE)\ ,\end{aligned}$$

for every $j \in I$ and A in $\mathscr{A}$.

Since E is a central projection, it follows that α is a *-homomorphism of $\mathscr{A}$ to $\mathscr{A}$. Let A be an element in $\mathscr{A}$ such that

$$\alpha(A) = \sum_{i \in I} g_{i+1}(g_i^{-1}(A)E) = 0\ ,$$

then we have that $g_{i+1}(g_i^{-1}(A)E) = 0$ for every $i \in I$, which implies that $Ag_i(E) = 0$ for every $i \in I$. It follows that $A = 0$, so that α is an isomorphism of $\mathscr{A}$ to $\mathscr{A}$. For $A \in \mathscr{A}$, put

$$B = \sum_{i \in I} g_i(g_{i+1}^{-1}(A)E)\ ,$$

then by the equality $(*)$ we have that

$$\begin{aligned}\alpha(B)g_j(E) &= \alpha(Bg_{j-1}(E)) = \alpha(g_{j-1}(g_j^{-1}(A)E)) \\ &= g_j(g_j^{-1}(A)E) = Ag_j(E)\end{aligned}$$

for every $j \in I$, which implies that $\alpha(B) = A$.

Hence α is an automorphism of $\mathscr{A}$.

Note that A belongs to the fixed algebra $\mathscr{A}^G$ of $\mathscr{A}$ under G if and only if A is an element in $\mathscr{A}$ of the form

$$A = \sum_{i \in I} g_i(AE) .$$

Take A in $\mathscr{A}$ such that $g_i(A) = A$ for every $i \in I$, then we have that

$$\alpha(A) = \sum_{i \in I} g_{i+1}(g_i^{-1}(A)E) = \sum_{i \in I} g_{i+1}(AE) = A .$$

Conversely take A in $\mathscr{A}$ such that $\alpha^n(A) = A$ for every $n \in I$, then we have, by the equality $(*)$, that

$$Ag_{i+n}(E) = g_{i+n}(g_i^{-1}(A)E)$$

for every $i \in I$ and $n \in I$, so that

$$AE = g_i^{-1}(A)E ,$$

for every $i \in I$. It follows that

$$A = \sum_{i \in I} g_i(AE) ,$$

or A belongs to $\mathscr{A}^G$.

Therefore, the fixed algebra $\mathscr{A}^G$ of $\mathscr{A}$ under G equals the fixed algebra $\mathscr{A}^\alpha$ of $\mathscr{A}$ under α.

For a central shift G of a von Neumann algebra $\mathscr{A}$, let $G(\alpha)$ be the group generated by the automorphism α of $\mathscr{A}$ in Theorem 3. By the definition of α, $G(\alpha)$ is a central shift of $\mathscr{A}$ and the shift projection E of G is a shift projection of $G(\alpha)$, too. The crossed product $G \otimes \mathscr{A}$ of $\mathscr{A}$ by G is isomorphic to the crossed product $G(\alpha) \otimes \mathscr{A}$ of $\mathscr{A}$ by $G(\alpha)$.

3. In this section, we shall show a few results as applications of a central shift.

Let $\mathscr{A}$ be a von Neumann algebra and G a discrete countable group of automorphisms of $\mathscr{A}$ (not necessary a shift).

Denote by $l^\infty(G)$ the maximal abelian von Neumann algebra on $l^2(G)$ generated by multiplication operators by bounded complex-valued functions on G. Let $\{\varepsilon_g; g \in G\}$ be an orthonormal basis in $l^2(G)$ such that

$$\varepsilon_g(h) = \begin{cases} 1 & g = h \\ 0 & g \neq h \end{cases} \qquad (g, h \in G) .$$

For every $g \in G$, let P_g be the projection in $l^\infty(G)$ defined by ε_g. Let

W_g be the representation of G such that

$$W_g\xi(h) = \xi(g^{-1}h) \qquad (\xi \in l^2(G),\, g,\, h \in G)\,.$$

We then regard G as a group of automorphism of $l^\infty(G)$ by

$$g(A) = W_g A W_g^*\,, \qquad (g \in G,\, A \in l^\infty(G))\,.$$

It is clear that $g(P_h) = P_{gh}$ for every g and h in G, and G is a shift of $l^\infty(G)$ with a shift projection P_1, where 1 is the unit of G. For $g \in G$, let $g \otimes g$ be an automorphism of the tensor product $\mathscr{A} \otimes l^\infty(G)$ of $\mathscr{A}$ and $l^\infty(G)$ such that

$$(g \otimes g)(A \otimes B) = g(A) \otimes g(B)\,, \qquad (A \in \mathscr{A},\, B \in l^\infty(G))\,.$$

Denote by $\tilde{G}$ the automorphism group of $\mathscr{A} \otimes l^\infty(G)$ generated by $g \otimes g$, $(g \in G)$.

THEOREM 4. *Let $\mathscr{A}$ be a von Neumann algebra and G a discrete group of automorphisms of $\mathscr{A}$, then the crossed product $\tilde{G} \otimes \mathscr{A} \otimes l^\infty(G)$ of the tensor product $\mathscr{A} \otimes l^\infty(G)$ of $\mathscr{A}$ and $l^\infty(G)$ by $\tilde{G}$ is isomorphic to the tensor product $\mathscr{A} \otimes \mathscr{L}(l^2(G))$. If $\mathscr{A}$ is properly infinite and G is countable, then $\tilde{G} \otimes \mathscr{A} \otimes l^\infty(G)$ is isomorphic to $\mathscr{A}$.*

PROOF. The group $\tilde{G}$ is a central shift of $\mathscr{A} \otimes l^\infty(G)$ with a shift projection $1 \otimes P_1$. Hence by [2: Theorem 2] we have that

$$\tilde{G} \otimes \mathscr{A} \otimes l^\infty(G) = (\mathscr{A} \otimes l^\infty(G))^{\tilde{G}} \otimes \mathscr{L}(l^2(G))\,.$$

On the other hand, in the proof of [2: Theorem 2], we have that

$$(\mathscr{A} \otimes l^\infty(G))^{\tilde{G}} = (\mathscr{A} \otimes l^\infty(G))_{1 \otimes P_1}\,.$$

Therefore, we have that

$$\tilde{G} \otimes \mathscr{A} \otimes l^\infty(G) = \mathscr{A} \otimes \mathscr{L}(l^2(G))\,.$$

For a discrete group of automorphisms of a von Neumann algebra, an analogous result to Theorem 3 is obtained.

We owe the following theorem to Professor M. Takesaki.

THEOREM 5. *Let $\mathscr{A}$ be a von Neumann algebra and G a discrete countable group of automorphisms of $\mathscr{A}$, then we have that*

$$\mathscr{A}^G \otimes I = (\mathscr{A} \otimes l^\infty(G))^{\sigma^{-1}(1 \otimes \alpha)\sigma} \cap (\mathscr{A} \otimes I)\,,$$

where α (resp. σ) is an automorphism of $l^\infty(G)$ (resp. $\mathscr{A} \otimes l^\infty(G)$).

PROOF. By the definition of $\tilde{G}$, it is clear that

$$\mathscr{A}^G \otimes I = (\mathscr{A} \otimes l^\infty(G))^{\tilde{G}} \cap (\mathscr{A} \otimes I)\,.$$

Take T in $\mathscr{A}\otimes l^{\infty}(G)$, then we have that

$$T=\sum_{g\in G}T(I\otimes P_g)=\sum_{g\in G}A_g\otimes P_g\,,\qquad (A_g\in\mathscr{A})\,.$$

Define an automorphism σ of $\mathscr{A}\otimes l^{\infty}(G)$ by

$$\sigma(\sum_{g\in G}A_g\otimes P_g)=\sum_{g\in G}g^{-1}(A_g)\otimes P_g\,,$$

then we have that $g\otimes g=\sigma^{-1}(1\otimes g)\sigma$ for every $g\in G$.

In fact,

$$\begin{aligned}(g\otimes g)(\sum_{h\in G}A_h\otimes P_h)&=\sum_{h\in G}g(A_h)\otimes P_{gh}\\&=\sigma^{-1}(\sum_{h\in G}h^{-1}(A_h)\otimes P_{gh})\\&=\sigma^{-1}(1\otimes g)(\sum_{h\in G}h^{-1}(A_h)\otimes P_h)\\&=\sigma^{-1}(1\otimes g)\sigma(\sum_{h\in G}A_h\otimes P_h)\,,\quad (A_h\in\mathscr{A})\,.\end{aligned}$$

On the other hand, since G is a shift of $l^{\infty}(G)$, it follows by Theorem 3 that there exists an automorphism α of $l^{\infty}(G)$ such that $l^{\infty}(G)^{G}=l^{\infty}(G)^{\alpha}$.

Therefore, for those automorphisms α and σ, we have that

$$\mathscr{A}^{G}\otimes I=(\mathscr{A}\otimes l^{\infty}(G))^{\sigma^{-1}(1\otimes\alpha)\sigma}\cap(\mathscr{A}\otimes I)\,.$$

As another application of Theorem 3, we have the following;

THEOREM 6. *Let $\mathscr{A}$ be a von Neumann algebra and G a discrete countable group of automorphisms of $\mathscr{A}$, then the fixed algebra of $\mathscr{A}$ under G is isomorphic to the intersection of two von Neumann algebras which are isomorphic to $\mathscr{A}$.*

PROOF. Take T in $\mathscr{A}\otimes l^{\infty}(G)$, then we have that

$$T=\sum_{g\in G}A_g\otimes P_g\,,\qquad (A_g\in\mathscr{A})\,.$$

If T belongs to $(\mathscr{A}\otimes l^{\infty}(G))^{\tilde{G}}$, then we have that

$$\sum_{g\in G}h(A_g)\otimes P_{hg}=(h\otimes h)(T)=T=\sum_{g\in G}A_g\otimes P_g\,,$$

for every $h\in G$, which implies that $g(A_1)=A_g$ for every g in G.

Therefore, an operator T belongs to $(\mathscr{A}\otimes l^{\infty}(G))^{\tilde{G}}$ if and only if T has the form of

$$T=\sum_{g\in G}g(A)\otimes P_g\,,$$

for some $A\in\mathscr{A}$.

Define a mapping Φ of $\mathscr{A}$ onto $(\mathscr{A}\otimes l^{\infty}(G))^{\tilde{G}}$ by

$$\Phi(A) = \sum_{g \in G} g(A) \otimes P_g ,$$

then Φ is an isomorphism of $\mathscr{A}$ onto $(\mathscr{A} \otimes l^\infty(G))^{\tilde{G}}$.

On the other hand, we have that

$$\mathscr{A}^G \otimes I = (\mathscr{A} \otimes l^\infty(G))^{\tilde{G}} \cap (\mathscr{A} \otimes I) .$$

Therefore, the fixed algebra $\mathscr{A}^G$ is isomorphic to the intersection of two von Neumann algebras which are isomorphic to $\mathscr{A}$.

Let $\mathscr{A}$ be a von Neumann algebra and $\mathscr{B}$ a von Neumann subalgebra of $\mathscr{A}$. A positive linear mapping ψ of $\mathscr{A}$ onto $\mathscr{B}$ is called an *expectation* of $\mathscr{A}$ onto $\mathscr{B}$ if ψ satisfies that $\psi(I) = I$ and $\psi(AB) = A\psi(B)$ for all A in $\mathscr{B}$ and B in $\mathscr{A}$. An expectation ψ of $\mathscr{A}$ onto $\mathscr{B}$ is called *normal* if $A_\alpha \uparrow A$ implies $\psi(A_\alpha) \uparrow \psi(A)$ for A_α and A in $\mathscr{A}$.

REMARK 7. By an analogous definition of the above isomorphism Φ, we have an example of a normal expectation which is a homomorphism. In fact let G be a discrete countable group of automorphisms of a von Neumann algebra $\mathscr{A}$. Define a mapping ψ of $\mathscr{A} \otimes l^\infty(G)$ onto $(\mathscr{A} \otimes l^\infty(G))^{\tilde{G}}$ by

$$\psi(T) = \sum_{g \in G} g(T)(I \otimes P_g) , \qquad (T \in \mathscr{A} \otimes l^\infty(G)) ,$$

then ψ is a norm 1 projection of $\mathscr{A} \otimes l^\infty(G)$ onto $(\mathscr{A} \otimes l^\infty(G))^{\tilde{G}}$. So ψ is an expectation of $\mathscr{A} \otimes l^\infty(G)$ onto $(\mathscr{A} \otimes l^\infty(G))^{\tilde{G}}$ (cf. [10]). It is clear that ψ is a normal homomorphism. Similarly, if G is a central shift of $\mathscr{A}$, then there exists a normal expectation of $\mathscr{A}$ onto $\mathscr{A}^G$ which is a homomorphism.

As an application of Theorem 6, we have the following known result:

COROLLARY 8. *Every von Neumann algebra acting on a separable Hilbert space is isomorphic to the intersection of two type I factors.*

PROOF. Let $\mathscr{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$ and G be a countable discrete subgroup of unitaries in the commutant $\mathscr{A}'$ of $\mathscr{A}$ which generates $\mathscr{A}'$. Regard G as a group of inner automorphisms $\{g, h, \cdots\}$ of $\mathscr{L}(\mathfrak{H})$ such that

$$g(T) = U_g T U_g^* , \qquad (T \in \mathscr{L}(\mathfrak{H}),\ U_g \in G) ,$$

then $\mathscr{A}$ is the fixed algebra of $\mathscr{L}(\mathfrak{H})$ under G. Therefore, by Theorem 6, this corollary is proved.

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DEPARTMENT OF MATHEMATICS
OSAKA KYOIKU UNIVERSITY
TENNOJI, OSAKA, JAPAN