Tôhoku Math. Journ.
28 (1976), 443-457.

ON THE DEFICIENCIES OF MEROMORPHIC FUNCTIONS

SEIKI MORI

(Received October 13, 1975)

1. Introduction. The Nevanlinna theory for meromorphic functions in $|z| < +\infty$ was extended by Hällström [2] and Tsuji [3] to meromorphic functions defined in $\hat{C} - E$, where E is a bounded closed set of capacity zero in the complex plane C and $\hat{C}$ denotes the extended complex plane. This was done by using the level curves of the so called Evans' function.

In this paper, we treat special cases that E is a finite set or a countable set, and prove relations between the order and the deficiencies of a meromorphic function in $\hat{C} - E$. These relations are closely related with a theorem due to Edrei-Fuchs [1]. However, our result (Theorem 1) can not be obtained from properties of the function in a neighbourhood of an isolated singularity.

Let E be a bounded closed set of capacity zero on C, $u(z)$ be an Evans' function with respect to E, and let $v(z)$ be its conjugate harmonic function. The level curve C_r: $u(z) = \log r$ consists of a finite number of analytic Jordan curves clustering to E as $r \to +\infty$. Let Δ_r be the unbounded domain surrounded by C_r. It is well known that $\int_{C_r} dv = 2\pi$.

For a single-valued meromorphic function $w = f(z)$ in $\hat{C} - E$ with an essential singularity at every point of E, we put

$$m(r, w) = \frac{1}{2\pi}\int_{C_r} \log \frac{1}{[f(z), w]} dv(z) \geqq 0 ,$$

where $[w_1, w_2]$ denotes the spherical distance between w_1 and w_2. For a fixed $r_0 > 0$, we write

$$N(r, w) = \int_{r_0}^{r} n(t, w)\frac{dt}{t} - m(r_0, w) + k_0(w) \log\left(\frac{r}{r_0}\right) ,$$

where $n(t, w)$ denotes the number of zero points of $f(z) - w$ in $\Delta_t \backslash \bar{\Delta}_{r_0}$ and $k_0(w) = (1/2\pi)\int_{C_{r_0}} d \arg (f(z) - w)$.

We now write

$$T(r, f) = m(r, w) + N(r, w)$$

which is independent of w, and we call $T(r, f)$ the characteristic function of $f(z)$. We note that $T(r, f)$ depends on the choice of an Evans' function $u(z)$.

The order λ_u and the lower order μ_u of $f(z)$ with respect to $u(z)$ are defined as

$$\lambda_u = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \mu_u = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} .$$

Again the order λ and the lower order μ of $f(z)$ are defined as

$$\lambda = \inf_u \lambda_u \text{ and } \mu = \inf_u \mu_u .$$

We put

$$m^*(r, w) = \frac{1}{2\pi}\int_{C_r} \log^+ \left|\frac{1}{f(z) - w}\right| dv(z)$$

and

$$N^*(r, w) = \int_{r_0}^{r} n(t, w)\frac{dt}{t} .$$

Then we see $m(r, w) - m^*(r, w) = O(1)$ and $N(r, w) - N^*(r, w) = O(\log r)$.

Denoting by $n^0(r)$ the number of components of C_r, we write

$$F(r) = \int_{r_0}^{r} n^0(t)\frac{dt}{t} , \quad \xi = \limsup_{r\to\infty} \frac{F(r)}{T(r, f)}$$

and

$$\delta(w, f) = \liminf_{r\to\infty} \frac{m(r, w)}{T(r, f)}\left(= 1 - \limsup_{r\to\infty} \frac{N(r, w)}{T(r, f)}\right) .$$

Hällström [2] proved a defect relation:

DEFECT RELATION. Suppose that $f(z)$ is a meromorphic function in $\hat{C} - E$ with an essential singularity at every point of E. Then, for any q distinct complex numbers $w_1, \cdots, w_q$,

$$\sum_{j=1}^{q} \delta(w_j, f) \leqq 2 + \xi .$$

REMARK. If E is a finite set, then ξ is always zero. However, if E consists of an infinite number of points, ξ need not be finite.

2. Edrei and Fuchs [1] showed that every meromorphic function defined in C having more than one deficient value is of positive lower order. Here we shall prove that this property still holds for holomor-

phic functions defined in $\hat{C}-E$, when E is a finite set, and that this property need not be true for meromorphic functions in $\hat{C}-E$, when E consists of more than one point. Further, we can prove that if E consists of an infinite number of points, then the above property need not be true for holomorphic function in $\hat{C}-E$.

3. First we prove the following lemma.

LEMMA. *Let $f(z)$ be a single valued holomorphic function in $\hat{C}-\{z_1, \cdots, z_n\}$. If $f(z)$ has no zero point, then $f(z)$ can be expressed as follows:*

$$f(z) = c\prod_{j=1}^{n}(z-z_j)^{\nu_j}\exp\Big(\sum_{j=1}^{n}\psi_j(z)\Big),$$

where $\psi_j(z)$ is holomorphic in $\hat{C}-\{z_j\}$ $(j=1, \cdots, n)$, c is a constant and ν_j is some integer.

PROOF. Clearly we see $f'(z)/f(z) = \sum_{j=1}^{n}\phi_j(z)$, where $\phi_j(z)$ is holomorphic in $\hat{C}-\{z_j\}$ $(j=1, \cdots, n)$, since $f'(z)/f(z)$ is holomorphic in $\hat{C}-\{z_1, \cdots, z_n\}$. We expand $\phi_j(z)$ in the Laurent series of power of $z-z_j$, that is

$$\phi_j(z) = \sum_{\nu=0}^{\infty}\frac{a^j_{-\nu}}{(z-z_j)^\nu} = \sum_{\substack{\nu\neq 1\\ \nu\geqq 0}}\frac{a^j_{-\nu}}{(z-z_j)^\nu} + \frac{a^j_{-1}}{(z-z_j)},$$

where

$$a^j_\nu = \frac{1}{2\pi i}\int_{|\zeta-z_j|=r}\frac{\phi_j(\zeta)}{(\zeta-z_j)^{\nu+1}}d\zeta.$$

Put

$$h(z) = \int_{z_0}^{z}\frac{f'(z)}{f(z)}dz$$

for a fixed $z_0 \neq z_j$ $(j=1, \cdots, n)$. Then

$$\begin{aligned} h(z) &= \sum_{j=1}^{n}\int_{z_0}^{z}\Big(\sum_{\substack{\nu\neq 1\\ \nu\geqq 0}}\frac{a^j_{-\nu}}{(z-z_j)^\nu} + \frac{a^j_{-1}}{(z-z_j)}\Big)dz \\ &= \sum_{j=1}^{n}\{\psi_j(z) + a^j_{-1}(\log(z-z_j) + 2m_j\pi i)\}, \end{aligned}$$

where $\psi_j(z)$ is holomorphic in $\hat{C}-\{z_j\}$. We note that a^j_{-1} is an integer. In fact, we have for a sufficiently small r,

$$a_{-1}^{j}=\frac{1}{2\pi i}\int_{|\zeta-z_j|=r}\phi_j(\zeta)d\zeta=\frac{1}{2\pi i}\int_{|\zeta-z_j|=r}\left(\frac{f'(\zeta)}{f(\zeta)}-\sum_{k\neq j}\phi_k(\zeta)\right)d\zeta$$
$$=\frac{1}{2\pi i}\int_{|\zeta-z_j|=r}\frac{f'(\zeta)}{f(\zeta)}d\zeta$$

and therefore a_{-1}^{j} is equal to an integer ν_j. Thus we have

$$h(z)=\sum_{j=1}^{n}\psi_j(z)+\nu_j\log(z-z_j)+2m_j\pi i,\ (\nu_j,\ m_j\in\boldsymbol{Z}).$$

It is easy to see that $f(z)=c\cdot\exp h(z)$ for some constant c. Therefore we obtain $f(z)=c\cdot\prod_{j=1}^{n}(z-z_j)^{\nu_j}\exp\{\sum_{j=1}^{n}\psi_j(z)\}$, where $\psi_j(z)$ is holomorphic in $\hat{C}-\{z_j\}$ $(j=1,\cdots,n)$, and ν_j is an integer.

Now we can prove the following.

THEOREM 1. *Let E be a finite set, $E=\{z_1,\cdots,z_n\}$, $u(z)$ be an Evans' function with respect to E and let $w=f(z)$ be a meromorphic function in $\hat{C}-E$ with an essential singularity at every point of E. Suppose that there exist two values α and β $(\alpha\neq\beta)$ such that $\delta(\alpha,f)+\delta(\beta,f)>1$ for some $u(z)$. Then the lower order μ of $f(z)$ is positive.*

PROOF. We may assume that $\alpha=\infty$ and $\beta=0$. Let $\{a_\nu^j\}_{\nu=1}^{\infty}$ be the zeros and $\{b_\nu^j\}_{\nu=1}^{\infty}$ the poles of $f(z)$ in the component $D_{r_0}^{j}$ of D_{r_0} containing $z=z_j$ $(j=1,\cdots,n)$, where $D_{r_0}=\{z;\log r_0<u(z)<+\infty$ for a fixed $r_0>0$ such that D_{r_0} consists of n components$\}$, and let $\{c_\nu\}_{\nu=1}^{N}$ be the zeros and $\{d_\nu\}_{\nu=1}^{M}$ the poles of $f(z)$ in $\{z;u(z)<\log r_0\}$.

Let $\pi_j^0(z)$ and $\pi_j^\infty(z)$ be the canonical products formed by the zeros $\{a_\nu^j\}_{\nu=1}^{\infty}$ and the poles $\{b_\nu^j\}_{\nu=1}^{\infty}$, respectively.

Let $R(z)=\prod_{\nu=1}^{N}(z-c_\nu)/\prod_{\nu=1}^{M}(z-d_\nu)$ be a rational function formed by zeros $\{c_\nu\}_{\nu=1}^{N}$ and poles $\{d_\nu\}_{\nu=1}^{M}$. Then $g(z)=f(z)\cdot\prod_{j=1}^{n}\pi_j^\infty(z)/R(z)\cdot\prod_{j=1}^{n}\pi_j^0(z)$ is holomorphic and not zero in $\hat{C}-\{z_1,\cdots,z_n\}$. By Lemma, we can express $g(z)$ as follows:

$$g(z)=c\prod_{j=1}^{n}(z-z_j)^{\nu_j}\exp\left\{\sum_{j=1}^{n}\psi_j(z)\right\},$$

where $\psi_j(z)$ is holomorphic in $\hat{C}-\{z_j\}$ and c is a constant, so we can write

$$f(z)=\prod_{j=1}^{n}f_j(z),\tag{1}$$

where $f_j(z)$ is meromorphic in $\hat{C}-\{z_j\}$ $(j=1,\cdots,n)$.

We next show that for any $\sigma>1$, the inequality

$$T(r, f) \leqq \frac{4}{\sigma - 1} T(\sigma' r, f) + N(\sigma' r, 0, f) + N(\sigma' r, \infty, f) + O(\log r)$$

holds for all sufficiently large values of r, where $\sigma' = L\sigma$ for some constant L.

From (1), we have

$$\begin{aligned} T(r, f) &= \frac{1}{2\pi}\int_{C_r} \log^+ |f(z)|\, dv(z) + \int_{r_0}^{r} \frac{n(t, \infty, f)}{t} dt + O(\log r) \\ &= \sum_{j=1}^{n} \frac{1}{2\pi}\int_{C_r^j} \log^+ |f(z)|\, dv(z) + \sum_{j=1}^{n} \int_{r_0}^{r} \frac{n^j(t, \infty, f)}{t} dt + O(\log r) \\ &= \sum_{j=1}^{n} \frac{1}{2\pi}\int_{C_r^j} \log^+ |f_j(z)|\, dv(z) + \sum_{j=1}^{n} \int_{r_0}^{r} \frac{n^j(t, \infty, f)}{t} dt + O(\log r) . \end{aligned}$$

Here C_r^j is a component of C_r surrounding $z = z_j$ and $n^j(t, \infty, f)$ denotes the number of poles of $f(z)$ in $D_{r_0}^j \cap \varDelta_r \equiv \varDelta_t^j$.

We write

$$T^j(r, f) = \frac{1}{2\pi}\int_{C_r^j} \log^+ |f(z)|\, dv(z) + \int_{r_0}^{r} \frac{n^j(t, \infty, f)}{t} dt$$

and

$$T_0^j(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(z_j + \frac{1}{r} e^{i\theta} \right) \right| d\theta + \int_{r_0}^{r} \frac{N_0^j(t, \infty, f)}{t} dt .$$

Note that for a finite set $E = \{z_1, \cdots, z_n\}$, its Evans' function can be written in the form

$$u(z) = \sum_{j=1}^{n} p_j \log \frac{1}{|z - z_j|} ,$$

where $0 < p_j \leqq 1$ and $\sum_{j=1}^{n} p_j = 1$. We see $C_r^j \subset \{z; 1/Ar \leqq |z - z_j|^{p_j} \leqq B/r\}$, $(j = 1, \cdots, n)$ for all sufficiently large values of r, where A and B are constants depending only on $|z_j - z_k|$, $(j \neq k)$ and p_j $(j = 1, \cdots, n)$. Hence we have

$$\begin{aligned} T_0^j\left(\left(\frac{r}{B} \right)^{1/p_j}, f_j \right) - O(\log r) \leqq \frac{1}{p_j} T^j(r, f_j) \leqq T_0^j((Ar)^{1/p_j}, f_j) \\ + O(\log r) . \end{aligned}$$

Since, by Edrei and Fuchs [1, p. 310-311],

$$\begin{aligned} T_0^j((Ar)^{1/p_j}, f_j) \leqq \frac{4}{\sigma - 1} T_0^j(\sigma (Ar)^{1/p_j}, f_j) + N_0^j(\sigma (Ar)^{1/p_j}, 0, f_j) \\ + N_0^j(\sigma (Ar)^{1/p_j}, \infty, f_j) , \end{aligned}$$

for any fixed $\sigma > 1$, we have

$$T^j(r, f_j) \leqq \frac{4}{\sigma - 1} T^j(\sigma'_j r, f_j) + N^j(\sigma'_j r, 0, f_j) + N^j(\sigma'_j, \infty, t_j) + O(\log r) ,$$

where $N^j_o(r, 0, f_j) = \int_{r_0}^{r} (n^j_o(t, 0, f_j)/t)dt$, $n^j_o(t, 0, f_j)$ denotes the number of zeros of f_j in $1/r < |z - z_j| < 1/r_0$, and $\sigma'_j = \sigma^{p_j} AB$. Therefore we have

$$\begin{aligned} T(r, f) &= \sum_{j=1}^{n} T^j(r, f_j) + O(\log r) \\ &\leqq \frac{4}{\sigma - 1} T(\sigma' r, f) + N(\sigma' r, 0, f) + N(\sigma' r, \infty, f) + O(\log r) , \end{aligned}$$

where $\sigma' = \max_{1 \leqq j \leqq n} \sigma'_j$. Since $\delta(0, f) + \delta(\infty, f) > 1$, by a similar argument to that of Edrei and Fuchs [1, p. 316-317] we have $\mu_u(f) > 0$.

We note that if $\tilde{u}(z)$ is another Evans' function with respect to E and if $\tilde{C}_r$ is the level curve with respect to $\tilde{u}(z)$, then $T_u(r, f) \leqq \max_j (p_j/\tilde{p}_j) T_{\tilde{u}}(Kr^s, f)$, for all sufficiently large values of r, where K is a constant independent of r and $s = \max_j (\tilde{p}_j/p_j)$. In fact, $\tilde{u}(z)$ has the form

$$\tilde{u}(z) = \sum_{j=1}^{n} \tilde{p}_j \log \frac{1}{|z - z_j|} , \qquad 0 < \tilde{p}_j \leqq 1 \ \text{ and } \ \sum_{j=1}^{n} \tilde{p}_j = 1 .$$

Further we see that there exist four constants A, $\tilde{A}$, B and $\tilde{B}$ such that

$$C_r \subset \left\{z; \frac{1}{Ar} \leqq |z - z_j|^{p_j} \leqq \frac{B}{r}\right\} \quad \text{and} \quad \tilde{C}_r \subset \left\{z; \frac{1}{\tilde{A}r} \leqq |z - z_j|^{\tilde{p}_j} \leqq \frac{\tilde{B}}{r}\right\} .$$

Thus, for any j, we have $T^j_u(r, f) \leqq (p_j/\tilde{p}_j) T^j_{\tilde{u}}(\tilde{B}(Ar)^{\tilde{p}_j/p_j}, f)$ and so $T_u(r, f) \leqq \max_j (p_j/\tilde{p}_j) T_{\tilde{u}}(\tilde{B}(Ar)^s, f)$ for all sufficiently large values of r. Therefore we obtain

$$\mu_u \leqq \liminf_{r \to \infty} \frac{\log T_{\tilde{u}}(Kr^s, f)}{\log r} = s\mu_{\tilde{u}} ,$$

so that

$$\mu_{\tilde{u}} \geqq \frac{1}{s} \mu_u = \min_j (p_j/\tilde{p}_j) \mu_u \geqq \min_j (p_j) \mu_u > 0 .$$

4. Remark. Let $F(\zeta)$ be a meromorphic function defined on $\hat{C} - \{0, 1\}$ and having an essential singularity at each point of $\{0, 1\}$. We can establish the (local) Nevanlinna theory in D_0: $0 < |\zeta| < 1/2$ or D_1: $0 < |\zeta - 1| < 1/2$. Denote by $\delta^j(a, F)$ the deficiency appearing in the (local) Nevanlinna theory in $D_j (j = 0, 1)$. By giving an example, we can see

that there exists an $F(\zeta)$ with the following property: $F(\zeta)$ has only one (local) deficient value 0 in D_0 and only one (local) deficient value ∞ in D_1 such that $\delta^0(0, F) = 1$ and $\delta^1(\infty, F) = 1$ and $F(\zeta)$ has two deficient values such that $\delta(0, F) = \delta(\infty, F) = 1$ with respect to an Evans' function $u(\zeta)$. Hence the fact that $\delta(\alpha, F) + \delta(\beta, F) > 1$ implies $\mu_F > 0$, does not follow from a local argument.

EXAMPLE. We determine the sequences of positive numbers $\{s_n\}$, $\{t_n\}$, $\{x_n\}$, $\{y_n\}$ and positive integers $\{\eta_n\}$, $\{\nu_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$ such that

$$y_n < e^{(\log y_n)^{3/2}} < t_n < 4t_n < x_n < e^{(\log x_n)^{3/2}} < s_{n+1} < 4s_{n+1} < y_{n+1}$$

and such that

$$\begin{aligned}
e^{(\log r)^2} \leqq T_o(r, f_0) &\leqq e^{(\log r)^3} \quad (x_n + 1 \leqq r \leqq e^{(\log x_n)^{3/2}}) ,\\
T_o(r, f_0) &\leqq r \quad (s_{n+1} \leqq r \leqq x_{n+1}) ,\\
e^{(\log r)^2} \leqq T_o(r, f_1) &\leqq e^{(\log r)^3} \quad (y_n + 1 \leqq r \leqq e^{(\log y_n)^{3/2}}) ,\\
T_o(r, f_1) &\leqq r \quad (t_n \leqq r \leqq y_{n+1}) ,\\
T_o(r, g_0) &\geqq e^{(\log r)^5} \quad (4s_n \leqq r \leqq 4t_n) ,\\
T_o(r, g_0) &\leqq e^{(\log r)^{3/2}} \quad (x_n \leqq r \leqq s_{n+1}) ,\\
T_o(r, g_1) &\geqq e^{(\log r)^5} \quad (4t_n \leqq r \leqq 4s_{n+1})
\end{aligned}$$

and

$$T_o(r, g_1) \leqq e^{(\log r)^{3/2}} \quad (y_{n+1} \leqq r \leqq t_{n+1})$$

for functions $f_0(z)$, $f_1(z)$, $g_0(z)$ and $g_1(z)$ defined as follows:

$$f_0(z) = \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{x_n}\right)^{\lambda_n}\right), \quad f_1(z) = \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{y_n}\right)^{\mu_n}\right),$$

$$g_0(z) = \exp \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{s_n}\right)^{\eta_n}\right) \quad \text{and} \quad g_1(z) = \exp \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{t_n}\right)^{\nu_n}\right).$$

These functions have an essential singularity at infinity and $T_o(r, *)$ denotes the usual Nevanlinna characteristic function.

We write $G^0(z) = (f_0(z)\cdot g_0(z))^{-1}$ and $G^1(z) = f_1(z)\cdot g_1(z)$. For any $K > 0$, there exists an n_o such that the inequality

$$\log |f_0(z)| \geqq K \log \max_{|z|=r} \left|\frac{1}{g_0(z)}\right|$$

holds for $z \in \{z; 2x_n \leqq |z| \leqq (1/2)e^{(\log x_n)^{3/2}}, \ n \geqq n_o\}$. Hence we have $|f_0(z)\cdot g_0(z)| > e^{K-1}$, for $z \in \{z; 2x_n \leqq |z| \leqq (1/2)e^{(\log x_n)^{3/2}}, n \geqq n_o\}$. Similarly, there exists an n_1 such that $|f_1(z)\cdot g_1(z)| > e^{K-1}$, for $z \in \{z; 2y_n \leqq |z| \leqq (1/2)e^{(\log x_n)^{3/2}}, \ n \geqq n_1\}$.

We now consider

$$F(\zeta)=G^{0}\Big(-\frac{1}{\zeta}\Big)\cdot G^{1}\Big(\frac{1}{\zeta-1}\Big).$$

Obviously $F(\zeta)$ is meromorphic in $\hat{C}-\{0,1\}$ and has essential singularities at two points $\zeta=0$ and $\zeta=1$.

From the above, we can easily see that $\delta^{0}(0,F)=1$, $\delta^{0}(\tau,F)=0$ $(\tau\neq 0)$, $\delta^{1}(\infty,F)=1$ and $\delta^{1}(\tau,F)=0$ $(\tau\neq\infty)$. On the other hand, if we use the level curve of an Evans' function $u(\zeta)=(1/2)\log(1/|\zeta(\zeta-1)|)$, then we can easily see that

$$\max(N(r,0,F)\,,\quad N(r,\infty,F))=o(T(r,F))\,,$$

as $r\to\infty$. Hence we have $\delta(0,F)=\delta(\infty,F)=1$, where $N(r)$, $T(r)$ and δ denote the counting function, the characteristic function and the deficiency with respect to the Evans' function $u(\zeta)$, respectively. Therefore we obtain a desired example.

5. By giving an example, we next show that the assertion of Theorem 1 is not true for meromorphic function having no deficient values α and β with $\delta(\alpha,f)+\delta(\beta,f)>1$ and being defined on $\hat{C}-E$, where E consists of two points.

Let E be the set $\{z_1,z_2\}$, $z_1=0$, $z_2=1/2$. Put

$$u(z)=\frac{1}{2}\log\frac{1}{|z|}+\frac{1}{2}\log\frac{1}{|z-1/2|}\,,$$

and denote by C_r the level curve $u(z)=\log r$. Then we see easily $C_r\subset\{(1/r^2\leqq|z|\leqq 3/r^2)\cup(1/r^2\leqq|z-1/2|\leqq 3/r^2)\}$ for all sufficiently large values of r.

Consider the function

$$f(z)=\prod_{\nu=1}^{\infty}\Big(1-\frac{1}{z\cdot 2^{2^{\nu}}}\Big)\Big/\prod_{\nu=1}^{\infty}\Big(1-\frac{1}{(z-1/2)2^{2^{\nu}}}\Big)=\frac{f_1(z)}{f_2(z)}\,,$$

say. Then we have

$$T(r,f(z))=\frac{1}{2\pi}\int_{C_r^0}\log^{+}|f(z)|dv(z)+\frac{1}{2\pi}\int_{C_r^{1/2}}\log^{+}|f(z)|\,dv(z)$$
$$+\int_{r_0}^{r}\frac{n(t,\infty,f)}{t}dt+O(\log r)\,,$$

where C_r^0 and $C_r^{1/2}$ denote components of C_r surrounding $z=0$ and $z=1/2$, respectively. We write $n(t,0)$ for the number of zeros of $f(z)$ in Δ_t. Then $(1/\log 2)\log\log(t/\sqrt{3})+1<n(t,0)<(1/\log 2)\log\log t+2$. Hence we have

$$\int_{r_0}^{r}\frac{n(t,0)}{t}dt = (1/\log 2)\log r\cdot\log\log r + O(\log r)\ .$$

Next, we estimate $|f(z)|$ on C_r^0. Now, we see

$$\log\prod_{\nu=1}^{\infty}\left|1-\frac{1}{z\cdot 2^{2^\nu}}\right| \leqq \sum_{\nu=1}^{\infty}\log\left(1+\frac{r^2}{2^{2^\nu}}\right) \leqq \int_{2^2}^{r^2}\frac{n(t,0)}{t}dt + O(1)$$
$$\leqq (2/\log 2)\log r\cdot\log\log r + O(\log r)\ .$$

On the other hand, we have

$$\log\prod_{\nu=1}^{\infty}\left|1-\frac{1}{z\cdot 2^{2^\nu}}\right| \geqq \sum_{\nu=1}^{\infty}\log\frac{\sqrt{|z|^2+(1/2^{2^\nu})^2}}{|z|} \geqq \sum_{\nu=1}^{\infty}\frac{1}{2}\log\left(1+\left(\frac{r}{3\cdot 2^{2^\nu}}\right)^2\right)$$
$$\geqq (2/\log 2)\log r\cdot\log\log r + O(\log r)$$

for $z\in C_r^0\cap(\mathrm{Re}\, z<0)$.

We note again that $|\log\prod_{\nu=1}^{\infty}|1-1/(z-1/2)\cdot 2^{2^\nu}||=O(1)$ for $z\in C_r^0$, $(r>5)$. Thus we have

$$\log^+|f(z)| \leqq (2/\log 2)\log r\cdot\log\log r + O(\log r)\ ,$$

on C_r^0 and

$$\log^+|f(z)| \geqq (2/\log 2)\log r\cdot\log\log r - O(\log r)\ ,$$

on $C_r^0\cap(\mathrm{Re}\, z<0)$. Hence we have

$$(1/2\log 2)\log r\cdot\log\log r - O(\log r) \leqq \frac{1}{2\pi}\int_{C_r^0}\log^+|f(z)|\,dv$$
$$\leqq (1/\log 2)\log r\cdot\log\log r + O(\log r)$$

for all sufficiently large values of r. We note that $\int_{C_r^{1/2}}\log^+|f(z)|\,dv$ is not so large. Therefore we obtain

$\delta(0,f)\geqq$

$$1-\limsup_{r\to\infty}\frac{\frac{1}{\log 2}\log r\cdot\log\log r + O(\log r)}{\left\{\frac{1}{\log 2}\log r\cdot\log\log r+O(\log r)\right\}+\left\{\frac{1}{2\log 2}\log r\cdot\log\log r+O(\log r)\right\}}$$
$$=1-\frac{2}{3}=\frac{1}{3}\ .$$

Similarly we obtain $\delta(\infty,f)\geqq 1/3$. Moreover, we can easily see that the order of $f(z)$ with respect to $u(z)$ is zero.

6. Here we show an example for which the assertion of Theorem 1 does not hold for an infinite set E, even if $f(z)$ is holomorphic in $\hat{C} - E$.

Let E be the set $\{z_k\}_{k=0}^{\infty}$, where $z_0 = 0$, $z_k = 1/2^{2^k}$ $(k = 1, 2, \cdots)$ and put

$$u(z) = \sum_{k=0}^{\infty} p_k \log \frac{1}{|z - z_k|}, \qquad \text{where} \quad p_0 = \frac{1}{2} \quad \text{and} \quad p_k = \frac{1}{2^{k+1}} \quad (k = 1, 2, \cdots).$$

Clearly $u(z)$ is an Evans' function with respect to E. Let C_r be the level curve $u(z) = \log r$. Then $C_r = \bigcup_{j=0}^{N(r)} C_r^j$, where C_r^0 is a component of C_r surrounding $z = 0$ and C_r^j is a component of C_r not surrounding $z = 0$, $(j = 1, \cdots, N(r))$. Consider the function

$$f(z) = \prod_{k=0}^{\infty} f_k(z),$$

where

$$f_0(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{1}{2^{2^\nu} \cdot z}\right)^{\nu} \quad \text{and} \quad f_k(z) = \prod_{\nu=1}^{\infty} \left(1 - \frac{1}{2^{2^\nu / p_k} \cdot (z - z_k)}\right)$$

for $k = 1, 2, \cdots$.

We show that $f(z)$ is of order zero with respect to $u(z)$ and $\delta(0, f) = \delta(\infty, f) = 1$.

(i) We note that

$$\bigcap_{k=0}^{\infty} \left(|z - z_k| > \frac{1}{r}\right) \cap C_r = \emptyset \quad \text{and} \quad \bigcup_{k=0}^{\infty} \left(|z - z_k| < \left(\frac{1}{r}\right)^{1/p_k}\right) \cap C_r = \emptyset,$$

so that $C_r \subset (|z| < 1)$ for all sufficiently large values of r. We note again that $(\operatorname{Re} z < 0) \cap C_r \subset \{z; 1/r^2 < |z| < 2/r^2\}$.

(ii) We next show that the order of $f(z)$ with respect to $u(z)$ is zero. If $z \in C_r$, then $|z - z_0| = |z| > 1/r^2$, so

$$\begin{aligned} \log |f_0(z)| &\leqq \sum_{\nu=1}^{\infty} \log (1 + (r^2/2^{2^\nu})^{\nu}) \\ &\leqq \sum_{2^{2^\nu} < r^2} \nu \log (r^2/2^{2^\nu}) + \sum_{2^{2^\nu} < r^2} \nu (2^{2^\nu}/r^2) \\ &\quad + \sum_{r^2 < 2^{2^\nu}} \nu (r^2/2^{2^\nu}) \\ &= I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

Now, we denote by $n_0(t)$ the number of zeros of $f_0(z)$ in $|z| > 1/t$. Then the integer ν such that $2^{2^\nu} \leqq t < 2^{2^{\nu+1}}$ belongs to the interval

$$\frac{\log \log t - \log \log 2}{\log 2} - 1 < \nu \leqq \frac{\log \log t - \log \log 2}{\log 2},$$

so that $n_0(t) < 1/2\,(\log\log t/\log 2 + 2)^2$.

Hence we have

$$I_1 = \int_2^{r^2} \frac{n_0(t)}{t} dt = O(\log r(\log\log r)^2) .$$

Next, the integer ν satisfying $2^{2^\nu} < r^2$ fulfils $\nu < \log\log r/\log 2 + 2$, so that

$$I_2 = \sum_{2^{2^\nu} < r^2} \nu(2^{2^\nu}/r^2) \leqq O(\log\log r) .$$

Further, if $\nu_0 = [\log\log r/\log 2] + 2$, then

$$I_3 = \sum_{2^{2^\nu} > r^2} \nu(r^2/2^{2^\nu}) = \sum_{\nu=\nu_0}^{\infty} \nu(r^2/2^{2^\nu}) = O(\log\log r) .$$

Therefore we have

$$\log|f_0(z)| = O(\log r(\log\log r)^2) , \quad (z \in C_r) .$$

Next, we estimate $\prod_{k=1}^{\infty} |f_k(z)|$ on C_r. Let $k = k_0$ be the largest integer satisfying $z_k \in (|z| > 1/r^2)$. Then $k_0 < [\log\log r/\log 2] + 2$. If $k > k_0 + 1$, then

$$(|z - z_k|\cdot 2^{2^{\nu/p_k}})^{-1} < 1 ,$$

and hence $\log|f_k(z)| < (2^{2^{2k-2}})^{-1}$, so $\log \prod_{k>k_0+1} |f_k(z)| < 1$. If $k \leqq k_0 + 1$, then $|z - z_k| > (1/r)^{1/p_k}$ for $z \in C_r$. Thus we have

$$\begin{aligned}\log|f_k(z)| &\leqq \sum_{\nu=1}^{\infty} \log\{1 + (|z - z_k|\cdot 2^{2^\nu p_k})^{-1}\} \\ &\leqq \sum_{\nu=1}^{\infty} \log\{1 + (r^{1/p_k})\cdot(2^{2^\nu p_k})^{-1}\} \\ &\leqq \frac{2p_k}{\log 2} \log r^{1/p_k} \log\log r^{1/p_k} + O(1) \\ &= O(\log r \log\log r) ,\end{aligned}$$

where $n_k(t)$ denotes the number of zeros of $f_k(z)$ in $|z - z_k| > t$, since $k < k_0 + 1 < \log\log r/\log 2 + 3$. Hence we have

$$\log \prod_{k=1}^{k_0+1} |f_k(z)| \leqq O(k_0 \log r\cdot\log\log r) = O(\log r(\log\log r)^2) .$$

Therefore we have

$$\log|f(z)| = \log \prod_{k=0}^{\infty} |f_k(z)| = O(\log r(\log\log r)^2)$$

for $z \in C_r$. Thus we obtain

$$T(r, f) = \frac{1}{2\pi}\int_{C_r} \log^+ |f(z)|\, dv(z) + O(\log r) = O(\log r(\log\log r)^2)\ ,$$

where $v(z)$ is a conjugate harmonic function of $u(z)$, and we also see

$$\mu \leqq \lambda \leqq \lambda_u(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = 0\ .$$

Therefore $f(z)$ is of order zero with respect to $u(z)$.

(iii) If $z \in C_r \cap (\mathrm{Re}\, z < 0)$, then $|z| < 2/r^2$ and

$$\left|z - \frac{1}{2^{2^\nu}}\right| > \sqrt{\left(\frac{2}{r^2}\right)^2 + \left(\frac{1}{2^{2^\nu}}\right)^2}\ ,$$

and hence we have

$$\begin{aligned} \log|f_0(z)| &= \sum_{\nu=1}^{\infty} \log|1 - (2^{2^\nu}\cdot z)^{-\nu}| \\ &\geqq \sum_{\nu=1}^{\infty} \frac{\nu}{2} \log\left\{1 + \left(\frac{r^2}{2}\right)^2 \cdot (2^{2^\nu})^{-2}\right\} \\ &\geqq \frac{2}{3} \log r(\log\log r)^2\ . \end{aligned}$$

We also see $|z - z_k| \geqq \sqrt{|z_k|^2 + (1/r^2)^2} \geqq \max(|z_k|,\ 1/r^2)$ for $z \in C_r \cap (\mathrm{Re}\, z < 0)$, so that $|z - z_k|\cdot 2^{2^{\nu} p_k} > 4$. Thus we deduce that for $f_k(z)$ $(k = 1, 2, \cdots)$,

$$\log\left|\frac{1}{f_k(z)}\right| < (2^{2^{2k-1}})^{-1}\ ,$$

whence

$$\log \frac{1}{\prod_{k=1}^{\infty} |f_k(z)|} = \sum_{k=1}^{\infty} \log \frac{1}{|f_k(z)|} \leqq \sum_{k=1}^{\infty} (2^{2^{2k-1}})^{-1} < 1\ .$$

Therefore we obtain

$$\begin{aligned} T(r, f) = m(r, f) &= \frac{1}{2\pi}\int_{C_r} \log^+ |f(z)|\, dv(z) + O(1) \\ &\geqq \frac{1}{2\pi}\int_{C_r\cap(\mathrm{Re}\, z<0)} \log^+ |f(z)|\, dv(z) + O(1) \\ &= \frac{1}{2\pi}\int_{C_r\cap(\mathrm{Re}\, z<0)} \log^+ |f_0(z)|\, dv(z) + O(1) \\ &= K \log r(\log\log r)^2 \end{aligned}$$

for some constant $K(> 0)$ independent from r.

(iv) Finally we estimate the number of zeros of $f(z)$ in Δ_r. We note that $C_r \subset (|z| > 1/r^2) \cap (\bigcup_{k=1}^{\infty} (|z - z_k| > (1/r)^{1/p_k}))$ and that the number N_r of $z_k \in (|z| > 1/r^2)$ satisfies $N_r < \log\log r/\log 2 + 2$. For each k, the number of zeros of $f_k(z)$ in Δ_r is less than $p_k \cdot \log\log r + 1$. Thus we see that the number $n(r, 0)$ of zeros of $f(z) = \prod_{k=0}^{\infty} f_k(z)$ satisfies

$$n(r, 0) < \sum_{k=1}^{N_r} (p_k \log\log r + 1) < 2 \log\log r \ .$$

Hence we obtain

$$N(r, 0, f) \leqq 4 \log r \cdot \log\log r$$

for all sufficiently large values of r. Therefore we have

$$1 \geqq \delta(0, f) = 1 - \limsup_{r\to\infty} \frac{N(r, 0, f)}{T(r, f)} = 1 \ ,$$

so $\delta(0, f) = 1$. Clearly, we see $\delta(\infty, f) = 1$.

7. **Remark.** For the set E and the Evans' function $u(z)$ in the above example, we see that the number $n^*(r)$ of components of C_r satisfies $n^*(r) < \log\log r/\log 2 + 2$, since $C_r \cap (|z| < 1/r_2) = \varnothing$. Hence we have

$$F(r) = \int_{r_0}^{r} \frac{n^*(t)}{t} dt = O(\log r \cdot \log\log r) \ .$$

Thus for the holomorphic function $f(z)$ in the above example,

$$\xi = \limsup_{r\to\infty} \frac{F(r)}{T(r, f)} = 0 \ ,$$

since $T(r, f) > K \log r (\log\log r)^2$ for a constant K (>0). Hence $\sum_{j=1}^{\infty} \delta(a_j, f) \leqq 2$ for any distinct complex numbers a_j. On the other hand, since $\delta(0, f) = \delta(\infty, f) = 1$ in this example, we see that $\delta(a, f) = 0$ for all $a \neq 0, \infty$.

8. What can we say about the order (or the lower order) of $f(z)$ under more stronger condition than Theorem 1? We obtain the following result about this problem.

THEOREM 2. *Suppose that $f(z)$ is a meromorphic function in $\hat{C} - E$ with an essential singularity at each point of the set $E = \{z_1, \cdots, z_n\}$ and that $f(z)$ satisfies*

$$(2) \qquad \max(\lambda_u(N(r, 0, f)) \ , \ \lambda_u(N(r, \infty, f))) < \lambda_u(f) < \infty$$

for every Evans' function $u(z)$ with respect to E. Then the order of

$f(z)$ is a positive integer and $\lambda = \mu$, where λ and μ are the order and lower order of $f(z)$, respectively.

PROOF. Let two canonical products $\pi_j^0(z)$, $\pi_j^\infty(z)$ ($j = 1, \cdots, n$) and a rational function $R(z)$ be as in the proof of Theorem 1. Then

$$g(z) = f(z)\cdot \prod_{j=1}^{n} \pi_j^\infty(z)/R(z)\cdot \prod_{j=1}^{n} \pi_j^0(z)$$

is holomorphic and not zero in $\hat{C} - E$. Thus, by Lemma,

$$g(z) = c\cdot \prod_{j=1}^{n} (z - z_j)^{\nu_j} \exp\left\{\sum_{j=1}^{n} \psi_j(z)\right\},$$

where $c \in C$, ν_j is an integer and $\psi_j(z)$ is a holomorphic function in $\hat{C} - \{z_j\}$. Since $f(z)$ is finite order, we see $\psi_j(z)$ is a polynomial of $1/(z - z_j)$ of degree k_j ($0 \leqq k_j < \infty$). Hence there exist non-negative constants α_j and β_j ($j = 1, \cdots, n$) such that

$$\begin{aligned}\left(\beta_j\left(\frac{r}{B}\right)^{k_j/p_j}\right) &\leqq T_0^j\left(\left(\frac{r}{B}\right)^{1/p_j}, g\right) \leqq \frac{1}{p_j} T^j(r, g) + O(\log r)\\ &\leqq T_0^j((Ar)^{1/p_j}, g) + O(\log r) \leqq (\alpha_j (Ar)^{k_j/p_j}) + O(\log r)\end{aligned}$$

for all sufficiently large values of r. Hence we have

$$\sum_{j=1}^{n} p_j\left\{\beta_j\left(\frac{r}{B}\right)^{k_j/p_j}\right\} \leqq T(r, f) \leqq \sum_{j=1}^{n} p_j\{\alpha_j (Ar)^{k_j/p_j}\} + O(\log r),$$

so that

$$\limsup_{r\to\infty} \frac{\log T(r, g)}{\log r} = \liminf_{r\to\infty} \frac{\log T(r, g)}{\log r} = \max_{1\leqq j\leqq n}\left(\frac{k_j}{p_j}\right).$$

Again, by $f(z) = g(z)R(z)\pi_0(z)/\pi_\infty(z)$, where $\pi_0(z) = \prod_{j=1}^{n} \pi_j^0(z)$ and $\pi_\infty(z) = \prod_{j=1}^{n} \pi_j^\infty(z)$, we see

$$\begin{aligned}T(r, g) - T(r, \pi_0) - T(r, \pi_\infty) - O(\log r) &\leqq T(r, f)\\ &\leqq T(r, g) + T(r, \pi_0) + T(r, \pi_\infty) + O(\log r).\end{aligned}$$

From (2), we see $\max(\lambda_u(N(r, 0, f)), \lambda_u(N(r, \infty, f))) < \lambda_u(g) = \mu_u(g)$, whence

$$(1 - o(1))T(r, g) \leqq T(r, f) \leqq (1 + o(1))T(r, g).$$

Thus we obtain $\lambda_u(f) = \mu_u(f) = \max_{1\leqq j\leqq n}(k_j/p_j)$.

Therefore we have

$$\lambda = \mu = \inf_u \mu_u(f) = \inf_{p_j} \max_{1\leqq j\leqq n}\left(\frac{k_j}{p_j}\right).$$

This infimum is attained by taking $(k_{j_1}/p_{j_1}) = \cdots = (k_{j_q}/p_{j_q})$, $(k_{j_s} \neq 0)$. By noting $0 < p_j \leqq 1$, $\sum_{j=1}^{n} p_j = 1$ and by taking $\sum_{i=q+1}^{n} p_{j_i} \to 0$, we see that $\mu = \sum_{j=1}^{n} k_j$ is an integer.

REMARK. In Theorem 2, if we replace the condition "for every Evans' function $u(z)$ w.r.t. E" by "for some Evans' function $u(z)$ w.r.t. E", then the order λ (= the lower order μ) of $f(z)$ need not be an integer.

REFERENCES

[1] A. EDREI and W. H. J. FUCHS, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc., 128 (1959), 292-328.

[2] G. HÄLLSTRÖM, Über meromorphe Funktionen mit mehrfach zusammenhängenden Existenzgebieten, Acta Acad. Aboensis, Math. et Phys., 12 (1939), pp. 100.

[3] M. TSUJI, Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero, Japan. J. Math., 19 (1944), 139-154.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY