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ON THE DISTRIBUTIONS OF BOUNDED FUNCTIONS IN THE ABSTRACT HARDY SPACE THEORY AND SOME OF THEIR APPLICATIONS

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Let $H = H(X, \Sigma, m)$ be an abstract H^∞ space; that is, (X, Σ, m) is a probability measure space and H is a nontrivial $\sigma(L^\infty(m), L^1(m))$ closed (i.e. weak* closed) complex subalgebra of $L^\infty(m)$ such that $1 \in H$ and $\int uvdm = \int udm \int vdm$ for all $u, v \in H$. We fix arbitrarily an abstract H^∞ space H throughout this paper. In our former works [13, 15] we have shown how the function values of an $f \in H$ distribute on the outer boundary of its essential range. In this note we shall show how the function values of an $f \in H$ distribute on the interior of its essential range (Proposition 2.1 and Theorem 2.3). We apply it to the distributions of conjugate functions of conjugable bounded functions and those of functions of class H^+ (definition will be given later). And then we shall see how our results can be applied to the classical case and the function theory. We give new proofs of all the theorems in Davis [1] without use of Brownian motion and some theorems of Stein-Weiss [10]. In Section 1 we give some preliminaries from the abstract Hardy space theory and definition of conjugation operation and some of its properties [6, 12]. We state there three key lemmas for this paper, which we already know [15]. Our main results on the distributions of functions in H are treated in Section 2. Proposition 2.1 is general but weak one and Theorem 2.3 is given under assumption of Jensen measure. In Section 3 we treat the distributions of conjugate functions of bounded functions and functions of class H^+. Our results are generalizations of those of B. Davis. In Section 4 we shall see that a theorem of Stein-Weiss is valid for our case. We compute precisely how function values of conjugate functions of characteristic functions distribute. In Section 5 we remark that results of Zygmund and Pichorides are also valid for our setting. In Section 6 we apply our results in former sections to the classical case. We shall also see how our method is applied to the distributions of conjugate functions (Hilbert transforms) of characteristic functions on the real line.

1. Preliminaries and Notations. We write $u_n \rightarrow u$ if a sequence of

m-measurable functions u_n converges m-almost everywhere to an m-measurable function u as n tends to infinity. $L = L(m)$ is the set of all m-measurable functions and $L^\# = L^\#(m)$ is the set of all functions $f \in L(m)$ such that there exist $u_n \in H$ with $|u_n| < 1$, $u_n \to 1$ and $u_n f \in L^\infty(m)$. $H^\# = H^\#(m)$ is the set of all functions $f \in L(m)$ for which there exist $u_n \in H$ and $F \in L^\#(m)$ such that $|u_n| < F$ and $u_n \to f$. $L^\#$ and $H^\#$ are algebras and we have $L^\infty(m) \subset L^\#$, $H \subset H^\# \subset L^\#$ and $H = H^\# \cap L^\infty(m)$. Let us denote by ϕ the multiplicative linear functional on H defined by $\phi(u) = \int u dm$ for $u \in H$. Then there exists a unique extension of ϕ to a multiplicative linear functional $\phi: H^\# \to \boldsymbol{C}$ such that if $u_n, u \in H^\#$, $F \in L^\#$ and if $|u_n| < F$ and $u_n \to u$, one has $\phi(u_n) \to \phi(u)$. We recall further the function class H^+. H^+ consists of all m-measurable functions f such that $\operatorname{Re} f \geqq 0$ and $e^{-tf} \in H$ for all $t > 0$. We have $H^+ \subset H^\#$ and for non-constant $f \in H^+$ $\operatorname{Re}\phi(f) > 0$ and $f^{-1} \in H^+$. As is easily seen, if $f \in H$ and $\operatorname{Re} f \geqq 0$, $f \in H^+$. If $f_n \in H^+$ and $f_n \to f$, then $f \in H^+$ [12, p. 165]. Next we recall the definition of conjugate functions.

DEFINITION 1.1. A real-valued function $f \in L$ is said to be conjugable if there exists $g \in L$ such that

$$\exp t(f + ig) \in H^\# \quad \text{for all} \quad t \in \boldsymbol{R} = (-\infty, \infty) . \tag{1}$$

In this case g is unique up to an additive real constant and there exist a unique $g \in L$ and a real number $\lambda(f)$ such that

$$\phi(\exp t(f + ig)) = e^{t\lambda(f)} \quad \text{for all} \quad t \in \boldsymbol{R} . \tag{2}$$

This unique g is denoted by $\tilde{f}$.

It is known that if f is bounded and conjugable, $\lambda(f) = \int f dm$. Note also that if f is bounded, (1) is equivalent to

$$\exp t(f + ig) \in H \quad \text{for all} \quad t \in \boldsymbol{R} , \tag{3}$$

since $H = H^\# \cap L^\infty$.

REMARK. The following are equivalent.

(i) All $f \in L^\infty(m)$ are conjugable.

(ii) All characteristic functions are conjugable.

(iii) m is a Szegö measure, i.e., if $f \in L^1(m)$, $f \geqq 0$ and $\int ufdm = \int udm$ for all $u \in H$, then $f = 1$.

An approximation theorem holds for bounded conjugable functions, which we learned in a lecture of König.

LEMMA 1.2. *Let f be bounded and conjugable. Then there exist*

$h_n \in H$ such that

i) $$h_n \to f + i\tilde{f} ,$$

ii) $$|h_n| \leqq |f + i\tilde{f}| ,$$

iii) $$|\mathrm{Re}\, h_n| \leqq |f| ,$$

iv) $$\int \mathrm{Im}\, h_n\, dm = 0 ,$$

v) $$\int h_n dm \to \int f dm .$$

We rewrite the proof by König, since he gave it only in his lectures. Let $s = u + iv \in \boldsymbol{C}$, $\alpha > 0$ with $\alpha|u| < 1$. Then

$$\left|\frac{s}{2}\left(\frac{1}{1+\alpha s} + \frac{1}{1-\alpha s}\right)\right| \leqq \frac{|s|}{1-(\alpha u)^2} , \tag{4}$$

$$\left|\mathrm{Re}\,\frac{s}{2}\left(\frac{1}{1+\alpha s} + \frac{1}{1-\alpha s}\right)\right| \leqq \frac{|u|}{1-(\alpha u)^2} .$$

Let $c = ||f||_\infty$, and $\alpha > 0$ with $\alpha c < 1$. Then, if we write $h = f + i\tilde{f}$, by assumption we have $\exp t(1 \pm \alpha h) \in H$ for all $t \in \boldsymbol{R}$. It is easily seen that

$$(1 \pm \alpha h)^{-1} = \int_0^\infty \exp - t(1 \pm \alpha h) dt \in H .$$

Let

$$h_\alpha = (1 - (\alpha c)^2) h((1 + \alpha h)^{-1} + (1 - \alpha h)^{-1})/2 .$$

Then, since $h_\alpha = (1 - (\alpha c)^2)((1 - \alpha h)^{-1} - (1 + \alpha h)^{-1})/2\alpha$, we have $h_\alpha \in H$. Further by (4) we get $|h_\alpha| \leqq |h|$ and $|\mathrm{Re}\, h_\alpha| \leqq |f|$. Since

$$\int \exp - t(1 \pm \alpha h) dm = \exp - t\left(1 \pm \alpha \int f dm\right) ,$$

$\int (1 \pm \alpha h)^{-1} dm$ are also real. Hence $\int \mathrm{Im}\, h_\alpha dm = 0$. Therefore, if $\alpha_n > 0$, $c\alpha_n < 1$ and if $\alpha_n \to 0$, the sequence of functions $h_n = h_{\alpha_n}$ satisfies i), ii), iii), iv). Further, since $|\exp th_n| \leqq \exp t|f| \in L^\infty \subset L^\#$, by the continuity of ϕ

$$\phi(\exp th_n) = \exp t\phi(h_n) \to \phi(\exp th) = \exp t \int f dm .$$

Hence $\int h_n dm = \phi(h_n) \to \int f dm$. This completes the proof of the lemma.

We next recall some results of our previous work.

LEMMA 1.3 (An extension of Löwner's lemma). *Let $u \in H$ with $|u| \leqq 1$*

and $\int u dm = b, |b| < 1$. *Then*

$$\int \frac{1 - |ru|^2}{|e^{i\theta} - ru|^2} dm = \frac{1 - |rb|^2}{|e^{i\theta} - rb|^2}$$

for all $0 < r < 1$ *and* $e^{i\theta} \in \boldsymbol{T} = \{|z| = 1\}$.

And for any Lebesgue measurable set E *on* $\boldsymbol{T}$

$$\int_E d\theta \int_{\{|u(x)|<1\}} \frac{1 - |u|^2}{|e^{i\theta} - u|^2} dm = \int_E \frac{1 - |b|^2}{|e^{i\theta} - b|^2} d\theta - 2\pi m\{x \in X: u(x) \in E\} .$$

Further, if $|u| = 1$ *and* $\int u dm = 0$,

$$m\{x: u(x) \in E\} = L(E) ,$$

where L *is the normalized Lebesgue measure on* $\boldsymbol{T}$. [13, p. 90].

LEMMA 1.4. *Let* u, b *be the same as in Lemma* 1.3. *Let* $1 \leqq p < \infty$, $f(e^{i\theta}) \in L^p(\boldsymbol{T})$ *and* $f(re^{i\theta})$ *be the Poisson integral of* f. *Then the composed function* $f \circ u = f(u)$ *is well-defined and*

i) $$\lim_{r \to 1} f(ru) = f(u) \qquad m\text{-a.e. and in } L^p(m) ,$$

ii) $$\|f(u)\|_p \leqq \left(\frac{1 + |b|}{1 - |b|}\right)^{1/p} \|f\|_p ,$$

iii) $$\int f(u) dm = f\left(\int u dm\right) . \qquad [15, \text{ p. } 521] .$$

Finally in this section we recall the definitions of Jordan domain and Carathéodory domain. A Jordan domain is a bounded domain in the complex plane $\boldsymbol{C}$ bounded by a closed Jordan curve. A bounded domain D in $\boldsymbol{C}$ is said to be a Carathéodory domain if the boundary ∂D of D coincides with that of the unbounded component of the complement of the closure of D. Naturally D is simply connected. If D is a domain in $\boldsymbol{C}$ and $a \in D$, $\mu_{a,D}$ will denote the harmonic measure on the boundary of D with respect to a and D. If f is a mapping of a set E into another set F and G is a subset of E, we denote by $f(G)$, as usual, the set $\{f(g): g \in G\}$. We use these notations throughout this paper.

2. Distributions of functions in abstract H^∞ spaces. As a consequence of Lemma 1.4 we give first

PROPOSITION 2.1. *Let* D *be a Carathéodory domain in* $\boldsymbol{C}$ *and* $a \in D$ *be fixed. Let* G *be a Jordan domain such that* $a \in G \subset D$. *Suppose further* *there exists a bounded harmonic function on D such that*

$$\begin{aligned} \alpha(z) &= 1 \quad on \quad \partial G \cap D \\ &= 0 \quad on \quad \partial G \cap \partial D \\ &\geqq 1 \quad on \quad D \backslash \bar{G} . \end{aligned}$$

Then, if $u \in H$, $\int u dm = a$, $m\{x: u(x) \in \bar{D}\} = 1$, it holds

$$m\{x: u(x) \notin G \cup (\partial G \cap \partial D)\} \leqq \mu_{a,G}(\partial G \cap D) ,$$

or equivalently

$$m\{x: u(x) \in G \cup (\partial G \cap \partial D)\} \geqq \mu_{a,G}(\partial G \cap \partial D) .$$

PROOF. Let g be a conformal mapping of the unit disc U onto the domain D such that $g(0) = a$. Then the composed function $\alpha \circ g$ is bounded and harmonic in U. We know by Theorem 3.1 in [15] that $g^{-1} \circ u \in H$, $\int g^{-1} \circ u dm = g^{-1}\left(\int u dm\right) = g^{-1}(a) = 0$ and $|g^{-1} \circ u| \leqq 1$. Hence by Lemma 1.4 we have

$$(5) \quad \int \alpha \circ g \circ g^{-1} \circ u dm = \alpha \circ g\left(\int g^{-1} \circ u dm\right) = \alpha(g(0)) = \alpha(a) = \mu_{a,G}(\partial G \cap D) .$$

On the other hand, since $\alpha(z) \geqq 1$ on $D \backslash \{G \cup (\partial G \cap \partial D)\}$, using Lemma 1.4 $\alpha \circ g \circ g^{-1} \circ u(x) \geqq 1$ on the set $\{u(x) \notin G \cup (\partial G \cap \partial D)\}$. Hence by (5) we get the desired inequality.

If m is a Jensen measure for H, i.e., $\log\left|\int u dm\right| \leqq \int \log |u| dm$ for all $u \in H$, then one can get the best result in this direction. Before stating it we give a lemma, which we need only for the case of bounded subharmonic functions.

LEMMA 2.2. *Suppose m is a Jensen measure for H. Let D be a Carathéodory domain, and $u \in H$, $\int u dm \in D$ and $m\{x: u(x) \in \bar{D}\} = 1$. Then, if f is a subharmonic function in D such that for a $p: 1 < p < \infty$ there exists a harmonic majorant $g(z)$ of $|f(z)|^p$, the composed function $f(u)$ is well-defined and*

$$\int f(u) dm \geqq f\left(\int u dm\right) .$$

PROOF. It is sufficient to show in the case where D is the unit disc U by Theorem 3.1 in [15]. By an easy variant of a theorem of Littlewood (see for example Tsuji [11, p. 173]), $f(z)$ has the representation of the form

$$(6) \qquad f(z) = h(z) - \int \log\left|\frac{1 - \bar{a}z}{a - z}\right| d\mu(a) ,$$

where h is harmonic in U, $\lim_{r\to 1} h(re^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. and

$$\sup_{0<r<1}\int |h(re^{i\theta})|^p d\theta < \infty$$

and μ is a σ-finite nonnegative measure on U such that $\int_U (1-|a|)d\mu(a) < \infty$. Hence by Lemmas 1.3 and 1.4 $f(u)$ is well-defined. Since $u \in H$, $|u| \leqq 1$, we have $(a - ru)/(1 - \bar{a}ru) \in H$ for all $0 < r < 1$. Hence by assumption

$$\int \log\left|\frac{1 - \bar{a}ru}{a - ru}\right| dm \leqq - \log\left|\int \frac{a - ru}{1 - \bar{a}ru} dm\right| = \log\left|\frac{1 - \bar{a}rb}{a - rb}\right| ,$$

where $b = \int u dm$. Since $|(1 - \bar{a}ru)/(a - ru)| \geqq 1$, we have via Fubini's theorem

$$\iint \log\left|\frac{1 - \bar{a}ru}{a - ru}\right| d\mu(a) dm = \iint \log\left|\frac{1 - \bar{a}ru}{a - ru}\right| dm d\mu(a) \tag{7}$$
$$\leqq \int \log\left|\frac{1 - \bar{a}rb}{a - rb}\right| d\mu(a) .$$

For $h(z)$ we have by Lemma 1.4

$$\int h(u) dm = h(b) . \tag{8}$$

Combining (6), (7) and (8) we get

$$\int f(ru) dm \geqq f(rb) \quad (0 < r < 1) . \tag{9}$$

Let $G(z)$ be the Poisson integral of $g^{1/p}(e^{i\theta})$. Then, since by Lemma 1.4 $G(ru)$ converges in $L^p(m)$ to $G(u)$, there exist a sequence r_j and an $F \in L^p(m)$ such that $|G(r_j u)| \leqq F$. Since $|f(ru)| \leqq |g(ru)|^{1/p} \leqq G(ru)$, by Lebesgue's dominated convergence theorem we get

$$\lim_{j\to\infty} \int f(r_j u) dm = \int f(u) dm ,$$

and hence combining this with (9)

$$\int f(u) dm \geqq f(b) .$$

This completes the proof.

REMARK. If the conclusion in Lemma 2.2 is valid for all bounded subharmonic functions in D and for all $u \in H$ with $\int u dm \in D$ and $m\{x: u(x) \in D\} = 1$, then m is a Jensen measure for H. In fact, let $v \in H$.

Then there exist $M > 0$ and $a \in C$ such that $\int(Mv(x) + a)dm(x) \in D$ and $m\{x: Mv(x) + a \in D\} = 1$. Hence by assumption, for any $0 < p < \infty$ we have $\left|\int Mvdm\right|^p \leqq \int|Mv|^p dm$, since $|z - a|^p$ is bounded and subharmonic in D. Hence $\left|\int vdm\right| \leqq \left(\int|v|^p dm\right)^{1/p}$, $0 < p < \infty$. Letting $p \to 0$ we have $\left|\int vdm\right| \leqq \exp\int\log|v|dm$, the Jensen inequality for v.

Now we state our main result.

THEOREM 2.3. *Suppose m is a Jensen measure for H. Let D be a Carathéodory domain and $a \in D$ be fixed. Let G be a Jordan domain such that $a \in G \subset D$. Then if $u \in H$, $\int udm = a$ and $m\{x: u(x) \in \bar{D}\} = 1$,*

$$m\{x: u(x) \notin G \cup (\partial G \cap \partial D)\} \leqq \mu_{a,G}(\partial G \cap D) .$$

PROOF. Let $\alpha(z)$ be the harmonic function on G such that $\alpha(z) = 1$ on $\partial G \cap D$, $\alpha(z) = 0$ on $\partial G \cap \partial D$. Let

$$\begin{aligned}\beta(z) &= \alpha(z) \quad \text{on} \quad \bar{G} \\ &= 1 \quad\quad \text{on} \quad \bar{D}\backslash\bar{G} .\end{aligned}$$

Then $-\beta(z)$ is bounded and subharmonic in D. Hence by Lemma 2.2

$$\int\beta(u)dm \leqq \beta\left(\int udm\right) = \beta(a) = \alpha(a) = \mu_{a,G}(\partial G \cap D) .$$

Since $\beta(u) = 1$ on $\{x: u(x) \notin G \cup (\partial G \cap \partial D)\}$, we have

$$m\{x: u(x) \notin G \cup (\partial G \cap \partial D)\} \leqq \mu_{a,G}(\partial G \cap D) .$$

This completes the proof.

REMARK 1. If m is not Jensen, the above theorem is in general false. In fact, let $0 < a < 1$ and $X = \{|z| = 1\} \cup \{a\}$ and

$$m = \frac{1-a}{1+a}\delta_a + \frac{1}{2\pi}\left(1 - \frac{1-a}{1+a}\,\frac{1-a^2}{1+a^2-2a\cos\theta}\right)d\theta ,$$

where δ_a is the Dirac measure at $\{a\}$. Let $H = H^\infty(U)|_X$. Then $H = H(X, m)$ is an abstract H^∞ space. Let

$$f(re^{i\theta}) = \frac{1-a}{1+a}\,\frac{1-r^2}{1+r^2-2r\cos\theta} .$$

Then f is harmonic in $U\backslash[a, 1]$, and $f(e^{i\theta}) = 0$ for $0 < \theta < 2\pi$ and $f(r) > 1$ for $a < r < 1$. Hence

$$\mu_{0,U\setminus[a,1]}([a, 1]) < \frac{1-a}{1+a}\,.$$

Hence there exists θ_0, $0 < \theta_0 < \pi/2$ such that

$$\mu_0(L_+ \cup L_-) < \frac{1-a}{1+a}\,,$$

where $L_\pm$ are the line segments joining a and $e^{\pm\theta_0}$ respectively, and μ_0 is the harmonic measure with respect to the origin and the domain D bounded by the curves $C_0 = \{e^{i\theta}: \theta_0 < \theta < 2\pi - \theta_0\}$ and L_+, L_-. Let $u(z) = z$ for $z \in X$. Then $u \in H(dm)$, $|u(z)| \leqq 1$, $\int u(z)dm = u(0) = 0$. Now

$$m\{z \in X: u(z) \in \bar{U}\setminus(D \cup C_0)\} = \frac{1-a}{1+a} + \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} \frac{2a(1-\cos\theta)}{1+a^2-2a\cos\theta}d\theta$$
$$> \frac{1-a}{1+a} > \mu_0(L_+ \cup L_-) = \mu_0(\partial D \cap U)\,.$$

This implies the theorem is false in this case.

REMARK 2. The proofs of Lemma 2.2 and Theorem 2.3 show that they are valid if m is Jensen only for the linear span of $\{1, u, u^2, \cdots\}$. In particular, they always hold for the functions of the form $f(u)$, where $u \in H$, $|u| = 1$ and f is a bounded holomorphic function on the unit disc U. In fact if P is a polynomial in z, then by Lemma 1.3 we get

$$\int \log|P(f(u))|dm = \int \log|P(f(e^{i\theta}))|d\mu_{b,U} \geqq \log|P(f(b))|$$
$$= \log\left|\int P(f(u))dm\right|\,,$$

where $b = \int udm$. That is m is Jensen for the linear span of $\{1, f(u), (f(u))^2, \cdots\}$.

REMARK 3. In Theorem 2.3 the constant in the inequality is the smallest possible one if there exists a nonconstant $u \in H$ with $|u| = 1$. In fact, there exists $v \in H$ with $|v| = 1$ and $\int vdm = 0$ in this case. Let $g(z)$ be a conformal mapping of U onto G such that $g(0) = a$. Then for the composed function $g(v)$ we have

$$m\{x: g(v)(x) \in \partial G\} = 1\,, \quad \int g(v)dm = g\left(\int vdm\right) = a\,,$$

and by Lemma 1.3

$$m\{x: g(v)(x) \notin G \cup (\partial G \cap \partial D)\} = m\{x: v(x) \in g^{-1}(\partial G \cap D)\}$$
$$= \mu_{0,U}(g^{-1}(\partial G \cap D)) = \mu_{a,G}(\partial G \cap D) .$$

3. Distributions of conjugate functions of bounded functions and of functions of class H^+. In this section we apply our results in the previous sections to the distributions of conjugate functions of conjugable bounded functions and of functions of class H^+. As consequences of Lemma 1.3 or Proposition 2.1 we have two results. $f \in L^\infty_R(m)$ means that f is a bounded real-valued m-measurable function.

THEOREM 3.1. *Let $f \in L^\infty_R(m)$, $0 \leqq f \leqq 1$ and be conjugable. Let g be a conformal mapping of $U = \{|z| < 1\}$ onto $\{w \in C: 0 < \operatorname{Re} w < 1\}$ such that $g(0) = \int f dm$. Then*

i) $$m\{x: \tilde{f}(x) \geqq y\} \leqq 2L\{e^{i\theta}: \operatorname{Im} g(e^{i\theta}) \geqq y\} \qquad y > 0 .$$

ii) $$m\{x: \tilde{f}(x) \leqq -y\} \leqq 2L\{e^{i\theta}: \operatorname{Im} g(e^{i\theta}) \geqq y\} \qquad y > 0 .$$

In particular,

iii) $$m\{x: |\tilde{f}(x)| \geqq y\} \leqq Const.\ e^{-\pi y} , \qquad y > 0 .$$

We give here two proofs. The first one uses Lemma 1.3 and the second one uses Proposition 2.1. The first one gives somewhat better estimate than the second, but the second is clearer than the first.

FIRST PROOF. Let $v(x) = f(x) + i\tilde{f}(x)$ and $v_n = f_n + ih_n \in H$ be a sequence guaranteed by Lemma 1.2, i.e., $v_n \to v$, $\phi(v_n) = \int f_n dm \to \int f dm = g(0)$. For each fixed n there exists an $R > 0$ such that $m\{x: v_n(x) \in \{z \in C: 0 \leqq \operatorname{Re} z \leqq 1, -R \leqq \operatorname{Im} z \leqq R\}\} = 1$ and $\int v_n dm$ lies in that rectangle. Since g^{-1} can be approximated uniformly on that rectangle by polynomials in z (by virtue of Walsh theorem), $g^{-1}(v_n)$ clearly belongs to H and $\int g^{-1}(v_n)dm = g^{-1}\left(\int v_n dm\right)$ and $|g^{-1}(v_n)| \leqq 1$. Letting $n \to \infty$, we get $g^{-1}(v) \in H$, $\int g^{-1}(v)dm = g^{-1}\left(\int v dm\right) = g^{-1}\left(\int f dm\right) = 0$, and $|g^{-1}(v)| \leqq 1$. Let $D = \{z \in C: 0 < \operatorname{Re} z < 1\}$, $\lambda > 0$ and $C_1 = \{iy: y \geqq \lambda\}$, $C_2 = \{1 + iy: y \geqq \lambda\}$. Then if $\zeta \in D$ and $0 < \lambda \leqq \operatorname{Im} \zeta$, by symmetry we see that

$$\mu_{\zeta,D}(C_1 \cup C_2) \geqq \frac{1}{2} .$$

Since harmonic measures are invariant under conformal mappings, we have for each $x \in X$ with $\operatorname{Im} v(x) = \tilde{f}(x) \geqq \lambda$, $0 < \operatorname{Re} v(x) = f(x) < 1$,

$$\frac{1}{2\pi}\int_{\{\operatorname{Im} g(e^{i\theta})\geqq\lambda\}}\frac{1-|g^{-1}(v(x))|^2}{|e^{i\theta}-g^{-1}(v(x))|^2}d\theta \geqq \frac{1}{2}\,.$$

Integrating the both sides with respect to m on the set $\{x\in X: \tilde{f}(x)\geqq\lambda, 0<f(x)<1\}$ and using Lemma 1.3, we have

$$\begin{aligned}L\{e^{i\theta}: \operatorname{Im} g(e^{i\theta})\geqq\lambda\} - m\{x: g^{-1}(v(x))\in\{e^{i\theta}: \operatorname{Im} g(e^{i\theta})\geqq\lambda\}\}\\ \geqq \frac{1}{2}m\{x: \tilde{f}(x)\geqq\lambda,\ 0<f(x)<1\}\,.\end{aligned}$$

Hence

$$\begin{aligned}m\{x: \tilde{f}(x)\geqq\lambda\} + m\{x: \tilde{f}(x)\geqq\lambda,\ f(x)=0 \quad\text{or}\quad 1\}\\ \leqq 2L\{e^{i\theta}: \operatorname{Im} g(e^{i\theta})\geqq\lambda\}\,,\end{aligned}$$

which implies the desired inequality i). The same argument yields the inequality ii), since $L\{e^{i\theta}: \operatorname{Im} g(e^{i\theta})\geqq y\} = L\{e^{i\theta}: \operatorname{Im} g(e^{i\theta})\leqq -y\}$. iii) is gained by direct calculation or by a theorem of Stein-Weiss, which we shall prove later by our method.

SECOND PROOF. As in the first proof we have $g^{-1}(v)\in H$, $\int g^{-1}(v)dm=0$ and $|g^{-1}(v)|\leqq 1$. Let $\lambda>0$ and C_1, C_2, D be as before. Let $C_3=\{iy: y\leqq\lambda\}$, $C_4=\{1+iy: y\leqq\lambda\}$, $C_5=\{x+i\lambda: 0<x<1\}$ and G be the domain in D bounded by curves C_3, C_4, C_5. Let $\alpha(z)$ be the harmonic function on G satisfying $\alpha(z)=1$ on C_5, $\alpha(z)=0$ on $C_3\cup C_4$. Then by the principle of reflection, $\alpha(z)$ can be continued harmonically into $D\backslash G$. If we denote by $\alpha(z)$ this continued function, we have $\alpha(z)=2$ on $C_1\cup C_2$ and $|\alpha(z)|\geqq 1$ on $D\backslash G$. Now applying Proposition 2.1 we have

$$\begin{aligned}m\{x: \tilde{f}(x)\geqq\lambda\} = m\{x: g^{-1}(v(x))\notin g^{-1}(\bar{G}\backslash C_5)\} \leqq \mu_{0,g^{-1}(G)}(g^{-1}(C_5))\\ = \mu_{g(0),G}(C_5)\,.\end{aligned}$$

Since $\alpha(z)$ is harmonic on D and $\alpha(z)=1$ on C_5, $\alpha(z)=0$ on $C_3\cup C_4$, $\alpha(z)=2$ on $C_1\cup C_2$, we have

$$\mu_{g(0),G}(C_5) = 2\mu_{g(0),D}(C_1\cup C_2)\,,$$

which proves i). The same argument yields ii). iii) is showed as before.

Next we state similar estimates for functions in class H^+.

THEOREM 3.2. *If $v\in H^+$ is nonconstant and $y>0$, then*

i) $$m\{x: \operatorname{Im} v(x)\geqq y+\operatorname{Im}\phi(v)\} < \frac{2\operatorname{Re}\phi(v)}{\pi y}\,,$$

ii) $$m\{x: \operatorname{Im} v(x)\leqq -y+\operatorname{Im}\phi(v)\} < \frac{2\operatorname{Re}\phi(v)}{\pi y}\,.$$

PROOF. We may assume $\phi(v) = 1$ without loss of generality. Let $g(z) = (z - 1)/(z + 1)$. Then g maps the half plane $S = \{z \in \boldsymbol{C}: \operatorname{Re} z > 0\}$ onto the unit disc U and $g(1) = 0$. We know $g(v) \in H$, $|g(v)| \leqq 1$ and $\phi(g(v)) = g(\phi(v)) = g(1) = 0$ [12, p. 165-166]. Hence quite in the same way as in the proofs of Theorem 3.1 we have

$$\begin{aligned} m\{x: \operatorname{Im} v(x) \geqq y\} &\leqq \mu_{1,\{\operatorname{Re} z>0, \operatorname{Im} z<y\}}(\{x + iy: x > 0\}) \\ &= 2\mu_{1,\{\operatorname{Re} z>0\}}(\{ix, x \geqq y\}) \\ &= \frac{2}{\pi}\int_y^\infty \frac{dx}{1 + x^2} < \frac{2}{\pi} y . \end{aligned}$$

The same argument yields the second inequality.

REMARK 1. In Theorems 3.1, 3.2 the bounds are the smallest possible ones respectively, if there exists a nonconstant $u \in H$ with $|u| = 1$, as is shown similarly to Remark 3 in Section 2.

REMARK 2. Let h be a characteristic function of an arc of the unit circle $\boldsymbol{T}$ such that $(1/2\pi)\int_0^{2\pi} h(e^{i\theta})d\theta = \int f dm$, where f is given in Theorem 3.1. Then the Poisson integral k of $h + i\tilde{h}$ is a conformal mapping of the unit disc onto the strip $\{z \in \boldsymbol{C}: 0 < \operatorname{Re} z < 1\}$ satisfying $k(0) = \int f dm$ and hence a candidate for g in Theorem 3.1.

If m is a Jensen measure, we can say more.

THEOREM 3.3. *Suppose m is a Jensen measure. Let $f \in L_{\boldsymbol{R}}^\infty(m)$, $0 \leqq f \leqq 1$ and be conjugable. Let $y > 0$ and $h(z)$ be a conformal mapping of the unit disc U onto the rectangle $R = \{0 < \operatorname{Re} w < 1, -y < \operatorname{Im} w < y\}$ such that $h(0) = \int f dm$. Then*

$$m\{x: |\tilde{f}(x)| \geqq y\} \leqq L\{e^{i\theta}: |\operatorname{Im} h(e^{i\theta})| = y\} .$$

PROOF. Let $g(z)$ be a conformal mapping of U onto $D = \{0 < \operatorname{Re} w < 1\}$ with $g(0) = \int f dm$, and $J = \{x \pm iy: 0 < x < 1\}$ and $v = f + i\tilde{f}$. Then as in Theorem 3.1, $g^{-1}(v) \in H$, $|g^{-1}(v)| \leqq 1$, $\int g^{-1}(v) dm = 0$. Applying Theorem 2.3 to $g^{-1}(v)$, we have

$$\begin{aligned} m\{x: |\tilde{f}(x)| \geqq y\} &= m\{x: g^{-1}(v(x)) \notin g^{-1}(R) \cup g^{-1}(\partial R \cap \partial D)\} \\ &\leqq \mu_{0,g^{-1}(R)}(g^{-1}(J)) . \end{aligned}$$

On the other hand

$$\begin{aligned} \mu_{0,g^{-1}(R)}(g^{-1}(J)) &= \mu_{g(0),R}(J) = \mu_{h(0),R}(J) \\ &= \mu_{0,U}(h^{-1}(J)) = L\{e^{i\theta}: |\operatorname{Im} h(e^{i\theta})| = y\} , \end{aligned}$$

which completes the proof.

THEOREM 3.4. *Suppose m is a Jensen measure. If $v \in H^+$ is nonconstant and $y > 0$, then*

$$m\{x: |\operatorname{Im} v(x) - \operatorname{Im} \phi(v)| \geqq y\} < \frac{\operatorname{Re} \phi(v)}{y} .$$

PROOF. We may assume $\phi(v) = 1$ without loss of generality. A similar argument to the proof of Theorem 3.3 yields

$$m\{x: |\operatorname{Im} v(x)| \geqq y\} \leqq \mu_{1,\{\operatorname{Re} z>0,-y<\operatorname{Im} z<y\}}(\{x \pm iy: x>0\}) .$$

The right side is equal to $C(y) = (2/\pi)\int_\alpha^\infty (1 + x^2)^{-1}dx$, where $1/\alpha = \sinh \pi/2y$. Let $k(y) = y^{-1} - C(y)$. Then $\lim_{y\to\infty} k(y) = 0$ and

$$\begin{aligned} k'(y) &= -y^{-2} + y^{-2}(1 + \sinh^2 \pi/2y)^{-1} \cosh \pi/2y \\ &= y^{-2}((\cosh \pi/2y)^{-1} - 1) < 0 \quad \text{for} \quad y > 0 . \end{aligned}$$

Hence we get $k(y) > 0$ for $y > 0$, which completes the proof.

REMARK. Also in the above two theorems, the same remarks as those to Theorems 3.1 and 3.2 hold.

Combining Theorem 3.4 with the remark to Definition 1.1 we have the following result whose proof we omit.

COROLLARY 3.5. *If m is a Szegö measure, the conjugation operator is a linear operator of weak type* (1.1), *i.e.*

$$m\{x: |\tilde{f}(x)| \geqq y\} \leqq \frac{2}{y}\int |f(x)|dm$$

for all $y > 0$ and $f \in L^1(m)$.

One more result is the following one of strong type which is valid without additional assumption for m.

THEOREM 3.6. *Let $f \in L_R^\infty(m)$, $0 \leqq f \leqq 1$ and be conjugable. Let g be a conformal mapping of the unit disc U onto $\{w \in C: 0 < \operatorname{Re} w < 1\}$ such that $g(0) = \int f dm$. Then, if Φ is a nonnegative convex function on $(-\infty, \infty)$,*

$$\int \Phi(\tilde{f})dm \leqq \frac{1}{2\pi}\int_0^{2\pi} \Phi(\operatorname{Im} g(e^{i\theta}))d\theta .$$

PROOF. Let $u(x) = f(x) + i\tilde{f}(x)$. Then as in the proof of Theorem 3.1 we get $g^{-1}(u) \in H$, $|g^{-1}(u)| \leqq 1$ and $\int g^{-1}(u)dm = 0$. For each $0 < r < 1$ let

$$k_r(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |rg^{-1}(u)|^2}{|e^{i\theta} - rg^{-1}(u)|^2} \operatorname{Im} g(e^{i\theta})d\theta .$$

Then, since $(1/2\pi)\int_0^{2\pi}(1 - |a|^2)/|e^{i\theta} - a|^2 d\theta = 1$ for all $a\colon |a| < 1$, we have by Jensen's inequality

$$\Phi(k_r(x)) \leqq \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |rg^{-1}(u)|^2}{|e^{i\theta} - rg^{-1}(u)|^2}\Phi(\operatorname{Im} g(e^{i\theta}))d\theta .$$

Integrating the both sides with respect to m and using the first part of Lemma 1.3 and Fubini's theorem

$$\int \Phi(k_r(x))dm \leqq \frac{1}{2\pi}\int_0^{2\pi} \Phi(\operatorname{Im} g(e^{i\theta}))d\theta .$$

Letting $r \to 1$, we have by Lemma 1.4 the desired inequality. The proof is thus complete.

4. Distributions of conjugate functions of characteristic functions. Here we show that if f is a conjugable characteristic function one can compute precisely the distribution function of the conjugate function of f. For an m-measurable set E we denote by χ_E the characteristic function of E.

THEOREM 4.1. *Let E be an m-measurable set in X such that χ_E is conjugable. Then for any Lebesgue measurable subset of the set $\{i\boldsymbol{R}\} \cup \{1 + i\boldsymbol{R}\}$ it holds*

$$(*) \qquad m\{x\colon (\chi_E + i\tilde{\chi}_E)(x) \in F\} = L(h(F)) = \frac{1}{2\pi}\int_{g(F)} \frac{1 - |g(m(E))|^2}{|e^{i\theta} - g(m(E))|^2}d\theta ,$$

where g, h are the conformal mappings of the strip $\{0 < \operatorname{Re} z < 1\}$ onto the unit disc given by

$$g(z) = \tan\frac{\pi}{2}\left(z - \frac{1}{2}\right),$$

$$h(z) = (g(z) - g(m(E)))/(1 - g(m(E))g(z)) .$$

In particular, the distribution of $\chi_E + i\tilde{\chi}_E$ depends only on $m(E)$.

PROOF. Let $u(x) = \chi_E(x) + i\tilde{\chi}_E(x)$. Then by assumption for h we have as in the proof of Theorem 3.1 $h(u) \in H$, $|h(u)| = 1$ and $\int h(u)dm = h\left(\int udm\right) = h(m(E)) = 0$. Hence by Lemma 1.3

$$m\{x\colon u(x) \in F\} = m\{x\colon h(u(x)) \in h(F)\} = L(h(F)) .$$

The second equality follows immediately from the invariance of harmonic measures under conformal mappings, or one can get it in a similar way to the first one. The proof is complete.

As a consequence we have a theorem of Stein-Weiss for our setting.

COROLLARY 4.2 (Stein-Weiss). *Let E be as above and $\lambda(y)$ be the distribution function of $\tilde{\chi}_E$, i.e., $\lambda(y) = m\{|\tilde{\chi}_E(x)| \geqq y\}$. Then*

$$\exp \pi i \lambda(y) = \frac{\sinh \pi y + i \sin \pi m(E)}{\sinh \pi y - i \sin \pi m(E)} .$$

PROOF. Let $\alpha = \tan(\pi/2)(m(E) - 1/2)$. Then by simple computation we have $(1 - \alpha^2)/(1 + \alpha^2) = \sin \pi m(E)$. Further if we let $\lambda^+(y) = m\{\tilde{\chi}_E(x) \geqq y\}$, then we have by Theorem 4.1

$$\begin{aligned} \exp 2\pi i \lambda^+(y) &= h(iy)/h(1 + iy) \\ &= [(1 + \alpha^2) \sinh \pi y + (1 - \alpha^2)i]/[(1 + \alpha^2) \sinh \pi y - (1 - \alpha^2)i] . \end{aligned}$$

Since $(1 - \alpha^2)/(1 + \alpha^2) = \sin \pi m(E)$, we have

$$\exp 2\pi i \lambda^+(y) = \frac{\sinh \pi y + i \sin \pi m(E)}{\sinh \pi y - i \sin \pi m(E)} .$$

In a similar way we have the same equality for $\lambda^-(y) = m\{\tilde{\chi}_E(x) \leqq -y\}$. Hence we obtain the desired equality.

5. Results of Zygmund-Pichorides type. Here we give inequalities of strong type which are deduced easily from Lemma 1.2.

THEOREM 5.1 (Pichorides). *Let f be a real-valued, bounded and conjugable function on X such that $|f| \leqq k < \pi/2$. Then*

$$\|\sinh(\tilde{f}/2)\|_2 \leqq (\cos k)^{-1/2} \|\sin(f/2)\|_2 .$$

PROOF. Let $h = f + i\tilde{f}$ and $h_n = u_n + iv_n$ be a sequence in H satisfying the approximation property of Lemma 1.2. Then one easily sees that $\cos h_n \in H$ and $\int \cos h_n dm = \cos \int h_n dm \in \boldsymbol{R}$. Hence

$$\begin{aligned} \int \cosh v_n \cos u_n dm = \int \operatorname{Re} \cos h_n dm &= \int \cos h_n dm \\ &= \cos \int h_n dm = \cos \int u_n dm . \end{aligned}$$

Therefore, since $\cos u_n \geqq \cos |f| \geqq \cos k$, we have

$$\cos k \int \sinh^2 (v_n/2) dm \leqq \int (\cosh v_n - 1) \cos u_n dm$$

$$= \cos \int u_n dm - \int \cos u_n dm$$
$$= \int \sin^2 (u_n/2) - \sin^2 \int (u_n/2) dm \,.$$

Letting $n \to \infty$ we have

$$\|\sinh(\tilde{f}/2)\|_2^2 \leqq (\cos k)^{-1}[\|\sin(f/2)\|_2^2 - \sin^2 \int (f/2) dm] \,,$$

which implies the desired inequality. This completes the proof.

A similar argument, applied to $\exp \pm ih$, yields the Zygmund type inequality, which will be also proved by Theorem 3.5.

THEOREM 5.2. *Let f be a real-valued bounded and conjugable function on X such that $|f| \leqq 1$ and $0 < k < \pi/2$. Then*

$$\int \exp k|f| dm \leqq 2(\cos k)^{-1} \,.$$

6. Applications. Let $T = \{z \in C : |z| = 1\}$ and $U = \{z \in C : |z| < 1\}$. Let $H^\infty(T) = \{f \in L^\infty(T) : \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0,\ n = 1, 2, \cdots\}$ with essential supremum norm and $H^\infty(U)$ be the set of all bounded holomorphic functions in U with supremum norm. Then boundary functions of functions in $H^\infty(U)$ are in $H^\infty(T)$ and this correspondence is an isometrical isomorphism, and $H^\infty(T)$ is weak* closed in $L^\infty(T)$. For each $f \in H^\infty(U)$, $(1/2\pi)\int_0^{2\pi} f(e^{i\theta}) d\theta = f(0)$. Hence $\phi : f \in H^\infty(T) \to (1/2\pi)\int_0^{2\pi} f(e^{i\theta}) d\theta$ is a multiplicative linear functional on $H^\infty(T)$ and hence $H^\infty(T)$ with Lebesgue measurable sets and $d\theta/2\pi$ satisfies the assumptions for abstract H^∞ space, and $d\theta/2\pi$ is a Jensen measure as is well-known. The classical conjugation operation coincides with our one in Definition 1.1. If μ is a finite nonnegative measure on T, $\tilde{\mu}$ is its conjugate function and g is the absolutely continuous part of μ with respect to Lebesgue measure, f is defined on T by $g + i\tilde{\mu}$ and $F(z)$ is defined on U by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) d\mu(e^{i\theta}) + i \frac{1}{2\pi} \int_0^{2\pi} Q(z, e^{i\theta}) d\mu(e^{i\theta}) \,,$$

where P and Q are the Poisson and conjugate Poisson kernels respectively. F is holomorphic in U and $\operatorname{Re} F(z) > 0$ there. Hence $F(re^{i\theta}) \in H^+(d\theta/2\pi)$ for all $0<r<1$. Since $\lim_{r\to 1} F(re^{i\theta}) = f(e^{i\theta})$ a.e., by the property of H^+ as is noted in Introduction, we have $f \in H^+(d\theta/2\pi)$ and $\phi(f) = \lim_{r\to 1} \phi(F(re^{i\theta})) = F(0) = \mu(T)$. Hence we can apply all the results in the previous sections to this classical case. These applications give new proofs of known results due to Stein-Weiss [10] and B. Davis [1]. Especially, for results by Davis we can give proofs without use of Brownian motion. Unfortunately we could not prove the Davis' result on the best possible constant in weak type (1.1) inequality for conjugate functions. We also note that our method can be applied for any domain in $\boldsymbol{C}$ and $\boldsymbol{C}^n$. Finally we see how our method can be applied to conjugate functions on the real line $\boldsymbol{R}$. Here we prove only an analogue of Theorem 4.1 which can be deduced from that theorem or directly from Lemma 1.3.

THEOREM 6.1. *Let E be a Lebesgue measurable set on $\boldsymbol{R}$ such that $|E| < \infty$ and $\tilde{\chi}_E$ be the conjugate function of the characteristic function χ_E of E. Then for any Lebesgue measurable set F on the set $J = \{i\boldsymbol{R}\} \cup \{1 + i\boldsymbol{R}\}$,*

$$(*) \qquad |\{t \in \boldsymbol{R}: (\chi_E + i\tilde{\chi}_E)(t) \in F\}| = \frac{\pi}{4}|E|\left(\int_{\{it\in F\}} \frac{dt}{\sinh^2 \frac{\pi}{2}t} + \int_{\{1+it\in F\}} \frac{dt}{\cosh^2 \frac{\pi}{2}t}\right),$$

where $|\ |$ denotes the usual Lebesgue measure on $\boldsymbol{R}$. We understand the equality $(*)$ as follows; if the left side is infinite, the right side is also infinite and vice versa.

PROOF. Let $f(it) = \chi_E(t) + i\tilde{\chi}_E(t)$ and

$$f(x + iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2 + (y - t)^2} f(it)dt .$$

Then $f(z)$ is holomorphic on the right half plane $S = \{z \in \boldsymbol{C}: \operatorname{Re} z > 0\}$ and $0 < \operatorname{Re} f(z) < 1$ there and $f(iy) \in \{i\boldsymbol{R}\} \cup \{1 + i\boldsymbol{R}\}$ for almost all $y \in \boldsymbol{R}$. Let $g(z) = \tan(\pi/2)(z - 1/2)$, $h(z) = \exp \pi i(z - 1/2)$ and

$$k(z) = -i(z - 1)(z + 1)^{-1} .$$

Then $g(z) = k\circ h(z)$ and h maps $D = \{z \in \boldsymbol{C}: 0 < \operatorname{Re} z < 1\}$ conformally onto S and k maps S conformally onto U. Now for $\zeta = \sigma + i\tau \in S$ let

$$d\mu_\zeta(t) = \frac{1}{\pi} \frac{\sigma}{\sigma^2 + (\tau - t)^2} dt .$$

Let $H^\infty(S)$ be the set of all bounded holomorphic functions on S. Then for any $b \in H^\infty(S)$ $\lim_{x\to 0} b(x + it) = b(it)$ exists a.e. and $b(it) \in L^\infty(\boldsymbol{R})$. Let $H^\infty(\boldsymbol{R})$ be the set of all boundary functions of $H^\infty(S)$ with essential supremum norm. Then $H^\infty(\boldsymbol{R})$ is a weak* closed subalgebra of $L^\infty(\boldsymbol{R})$, which is isometrically isomorphic to $H^\infty(S)$, and

$$\int b(it)dm_\zeta(t) = b(\zeta) \qquad \text{for all} \quad b \in H^\infty(S)\,.$$

Hence $\phi_\zeta\colon b(it) \in H^\infty(\boldsymbol{R}) \longrightarrow \int b(it)dm_\zeta(t)$ is a multiplicative linear functional on $H^\infty(\boldsymbol{R})$. Hence $H^\infty(\boldsymbol{R})$ with dm_ζ is an abstract H^∞ space. The classical conjugation operation coincides with that of Definition 1.1. Now for our $f(it)$ we have $m_\zeta(E) = f(\zeta)$. Hence by Theorem 4.1

$$m_\zeta(\{f(it) \in F\}) = \frac{1}{2\pi}\int_{g(F)} \frac{1 - |g\circ f(\zeta)|^2}{|e^{i\theta} - g\circ f(\zeta)|^2}d\theta\,.$$

Changing the integration variable by $e^{i\theta} = k(it)$ we get

$$(\ast) \qquad m_\zeta(\{f(it) \in F\}) = \frac{1}{\pi}\int_{h(F)} \frac{\operatorname{Re} h\circ f(\zeta)}{(\operatorname{Re} h\circ f(\zeta))^2 + (\operatorname{Im} h\circ f(\zeta) - t)^2}dt\,.$$

As is easily seen,

$$\begin{aligned}\lim_{x\to\infty} x \operatorname{Re} h\circ f(x+iy) &= \lim_{x\to\infty} x \exp(\pi \operatorname{Im} f(x+iy)) \sin(\pi \operatorname{Re} f(x+iy)) \\ &= \lim_{x\to\infty} x \int_E \frac{x}{x^2 + (y-t)^2}dt = |E|\,,\end{aligned}$$

and

$$\begin{aligned}&\lim_{x\to\infty} x^2(\operatorname{Im} h\circ f(x+iy) + 1) \\ &\quad = \lim_{x\to\infty} x^2(1 - \exp(\pi \operatorname{Im} f(x+iy)) \cos(\pi \operatorname{Re} f(x+iy))) \\ &\quad = \lim_{x\to\infty} \frac{x^2}{2}\left(\int_E \frac{x}{x^2+(y-t)^2}dt\right)^2 - \lim_{x\to\infty} x^2(\pi \operatorname{Im} f(x+iy))^2 \\ &\quad = \frac{|E|^2}{2}\,.\end{aligned}$$

Hence multiplying the both sides of $(\ast)$ by σ and then letting $\sigma \longrightarrow \infty$ we get

$$|\{f(it) \in F\}| = |E|\int_{h(F)} \frac{dt}{1+t^2}\,.$$

Here if the left side is infinite, the right side is infinite and vice versa. Next changing the variables by $-e^{-\pi y} = t$ for $t < 0$ and $e^{-\pi y} = t$ for $t > 0$, we have

$$\int_{h(F)} \frac{dt}{1+t^2} = \frac{\pi}{4}\left(\int_{\{iy\in F\}} \frac{dy}{\sinh^2 \frac{\pi}{2}y} + \int_{\{1+iy\in F\}} \frac{dy}{\cosh^2 \frac{\pi}{2}y}\right),$$

which completes the proof.

As a consequence we have

COROLLARY 6.2 (Stein-Weiss). *Let E be as in Theorem* 6.1. *Then for each $y>0$*

$$|\{t\in \boldsymbol{R}: |\tilde{\chi}_E(t)|\geqq y\}| = \frac{2|E|}{\sinh \pi y}.$$

PROOF. Applying Theorem 6.1 we have

$$|\{t\in \boldsymbol{R}: |\tilde{\chi}_E(t)|\geqq y\}| = \frac{\pi|E|}{2}\left(\int_y^\infty \frac{ds}{\sinh^2 \frac{\pi}{2}s} + \int_y^\infty \frac{ds}{\cosh^2 \frac{\pi}{2}s}\right)$$

$$= \frac{2|E|}{\sinh \pi y}.$$

The proof is complete.

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