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REMARKS CONCERNING CONTACT MANIFOLDS

Dedicated to Professor Shigeo Sasaki on the occasion of his retirement from Tôhoku University

HAJIME SATO

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This paper will consist of two unrelated remarks concerning contact manifolds. In § 1, we study normal almost contact 3-manifolds. Let M be a closed normal almost contact 3-manifold. We show that $\pi_2(M) = 0$ unless M is homotopy equivalent to $S^1 \times S^2$, by using the results of Kodaira [7] and Inoue [5] on the classification of compact complex analytic surfaces. Especially it follows that the connected sum of two non-simply-connected closed 3-manifolds has never normal almost contact structure. In § 2, we study the contact structures on Brieskorn manifolds defined by Sasaki-Hsu [15]. Using the result of Morita [12], we show that there are many essentially different contact structures on odd dimensional spheres. We also have non-deformable strongly pseudo-convex structures (see e.g., Tanaka [17]) on odd dimensional spheres.

1. Normal almost contact 3-manifolds.

An almost contact structure on a $(2n+1)$-dimensional manifold M is a reduction of the structure group of the tangent bundle of M to the unitary group $U(n)$. If a manifold M has a contact structure, i.e., there exists a globally defined 1-form η on M such that $\eta\wedge(d\eta)^n$ never vanishes, then η induces a unique (up to homotopy) almost contact structure on M (see Gray [4], or Sasaki [14]). An almost contact structure on M naturally gives an almost complex structure on the product manifold $M \times \boldsymbol{R}$. If this almost complex structure is integrable, i.e., the Nijenhuis tensor vanishes, then we call the given almost contact structure on M normal. The normal almost contact structure is also called almost Sasakian structure [14]. If a normal almost contact structure comes from a contact form, we call the structure normal contact or Sasakian.

By a result of Sasaki-Hsu [15] (see also Abe-Erbacher [2], or Abe [1]), every Brieskorn manifold has a normal contact structure.

By a result of Martinet [10], every orientable closed 3-manifold has a contact structure.

Our theorem is as follows.

THEOREM 1. *Let M be a 3-dimensional closed normal almost contact manifold. If M is not homotopy equivalent to $S^1 \times S^2$, then*

$$\pi_2(M) = 0 ,$$

where $\pi_2(M)$ denotes the second homotopy group of M.

Let N be an orientable closed 3-manifold. Before the proof, we need the following lemmas. The coefficient of the homology groups are assumed to be the integer $\boldsymbol{Z}$.

Let j be the inclusion of N into $N \times S^1$ defined by $j(x) = x \times \{0\}$, for $x \in N$ and 0 is a base point of S^1.

LEMMA 1. *Let x be an element of $H_2(N \times S^1)$, then the self intersection number $x \circ x$ is an even integer. If one of the following conditions is satisfied,*

(i) *x is represented by a 2-sphere,*

(ii) *x lies in the image of $H_2(N)$ under j_*,*

then $x \circ x = 0$.

PROOF. Corresponding to the isomorphism

$$H_2(N \times S^1) \cong H_2(N) \otimes H_0(S^1) \oplus H_1(N) \otimes H_1(S^1) ,$$

we have the direct sum decomposition $x = x_1 \oplus x_2$. We have $x \circ x = x_1 \circ x_1 + 2x_1 \circ x_2 + x_2 \circ x_2 = 2x_1 \circ x_2$. Hence $x \circ x$ is an even integer. If x lies in the image of j_*, then x belongs to the first direct summand and $x \circ x = 0$. We know that $\pi_2(S^1) = 0$. Hence if x is represented by a 2-sphere, x lies in the image of j_*. The proof finishes.

LEMMA 2. *If $N \times S^1$ is diffeomorphic to the total space of a differentiable fiber bundle over an orientable 2-manifold V with fiber diffeomorphic to S^2, then N is homotopy equivalent to $S^2 \times S^1$.*

PROOF. Since Diff S^2 is homotopy equivalent to $SO(3)$ (Smale [16]), there are two non-equivalent fiber bundle over V. Both have cross sections. Let $[V]$ denote the homology class of the image of a cross section. If the bundle is non-trivial, then $[V] \circ [V]$ is an odd integer. By Lemma 1, we infer that $N \times S^1$ is diffeomorphic to the product $S^2 \times V$. Since $H_2(S^2 \times V) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$, we have $H_2(N) \otimes H_0(S^1) \oplus H_1(N) \otimes H_1(S^1) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$. The Poincaré duality theorem shows that rank $H_1(N) =$ rank $H_2(N) = 1$. Thus we have $H_1(V) \cong H_1(S^2 \times V) \cong H_1(N \times S^1) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$, which shows that V is diffeomorphic to the torus $S^1 \times S^1$. We have a diffeomorphism $f: S^2 \times S^1 \times S^1 \to N \times S^1$. Let $N \times \boldsymbol{R} \to N \times S^1$ be the covering of $N \times S^1$ and let $p: P = f^*(N \times \boldsymbol{R}) \rightarrow S^2 \times S^1 \times S^1$ be the induced covering. We can define a projection $q: S^2 \times S^1 \times S^1 \rightarrow S^2 \times S^1$ so that the composition $q \circ p: P \rightarrow S^2 \times S^1$ is a homotopy equivalence. Consequently we have

$$N \simeq N \times \boldsymbol{R} \simeq P \simeq S^2 \times S^1 ,$$

where $\simeq$ means homotopy equivalence. The proof finishes.

PROOF OF THEOREM 1. Since M has a normal almost contact structure, it follows easily that $M \times S^1$ has a structure of a complex analytic manifold. Let c_1 be the first Chern class of the complex manifold $W = M \times S^1$. By Lemma 1, it follows that $c_1^2 = 0$ and that W has no exceptional curves. Obviously the topological Euler Number of W is zero. By using the Kodaira's results on the classification of compact complex manifolds ([7], Theorem 55), we infer that W belongs to the class I_0, III_0, IV_0, VI_0, or VII_0. The surfaces in class I_0 are projective plane or rulled surfaces. The surfaces in class III_0 are complex tori. Rulled surfaces are S^2-bundles over orientable 2-manifolds. By Lemma 2, it follows that surfaces in class I_0 or III_0 which are diffeomorphic to the product $M \times S^1$ are given only when M is homotopy equivalent to $S^2 \times S^1$ or $S^1 \times S^1 \times S^1$. The classes IV_0 and VI_0 are elliptic surfaces. Since the Euler number is zero, they are obtained from elliptic surfaces free from singular fibers by means of logarithmic transformations ([7], § 4). This shows that W is diffeomorphic to the product $M' \times S^1$, where M' is an orientable Seifert bundle over orientable 2-manifold with typical fiber S^1. Such Seifert bundles are classified by Seifert (see Orlik [13]). Seifert bundles are devided into two types—large and small. Large Seifert manifolds are known to be $K(\pi, 1)$ spaces, i.e., their i-th homotopy groups vanish for all $i > 1$. Small orientable Seifert bundles over orientable 2-manifolds are diffeomorphic to torus bundles over S^1, $S^2 \times S^1$, or quotient spaces of S^3 by finite linear groups. Since

$$\pi_2(M) \cong \pi_2(W) \cong \pi_2(M') ,$$

the theorem is proved, except the case where $W = M \times S^1$ belongs to the class VII_0.

The complex manifold $W = M \times S^1$ belongs to class VII_0 only when M is a rational homology 3-sphere. Let b_i be the i-th Betti number of a surface $W = M \times S^1$ in class VII_0. Then $b_1 = 1$, and $b_2 = 0$. Further suppose that W contains a curve, then W is an elliptic surface or a Hopf surface. If it is elliptic, then the same discussion as for the classes IV_0 and VI_0 holds and the conclusion of the theorem holds. The universal covering of a Hopf surface is $S^3 \times \boldsymbol{R}^1$.

For other surfaces in class VII_0, we need the following result due to Inoue. Let Π be the fundamental group of W and let $[\Pi, \Pi]$ be the commutater subgroup.

THEOREM (INOUE [6]).*) *Let W be a compact analytic surface in class* VII_0 *with* $b_1 = 1, b_2 = 0$. *Suppose W contains no curve. Then there exists a holomorphic line bundle F_0 on W such that* $\dim H^0(\Omega^1(F_0)) \neq 0$ *if and only if $[\Pi, \Pi]$ is finitely generated.*

Let π be the fundamental group of the manifold M. Since $W = M \times S^1$, then $\Pi = \pi_1(W) \cong \pi \oplus \boldsymbol{Z}$. Hence we have $[\Pi, \Pi] = [\pi, \pi]$.

LEMMA 3. *Let π be the fundamental group of a compact 3-manifold M. If $H_1(M; \boldsymbol{Z})$ is a torsion group T, then $[\pi, \pi]$ is finitely generated.*

PROOF. We have a covering manifold $\tilde{M}$ such that $\pi_1(\tilde{M}) = [\pi, \pi]$. The covering index is equal to $\sharp \pi/[\pi, \pi] = \sharp T$, and so, is finite. The manifold $\tilde{M}$ is compact and $[\pi, \pi]$ is finitely generated.

PROOF OF THEOREM 1 (CONTINUED). Suppose $W = M \times S^1$ is in class VII_0 and suppose W contains no curve. Let $\Pi = \pi_1(W)$. By Lemma 3, $[\Pi, \Pi]$ is finitely generated. By the theorem of Inoue, there exists a holomorphic line bundle F_0 over W such that $\dim H^0(\Omega^1(F_0)) \neq 0$. But such manifolds are completely classified by Inoue [5]. Their universal covering spaces are all diffeomorphic to $\boldsymbol{R}^4$. Especially $\pi_2(M) \cong \pi_2(W) = 0$, which finishes the proof of Theorem 1.

Let M_1, M_2 be two closed 3-manifolds such that $\pi_1(M_j) \neq 0$ for $j = 1, 2$. The argument in Milnor ([11], p. 5) shows that $\pi_2(M_1 \sharp M_2) \neq 0$, where $M_1 \sharp M_2$ denotes the connected sum of M_1 and M_2.

COROLLARY. *Let M_1, M_2 be two closed 3-manifolds. If $\pi_1(M_j) \neq 0$, for $j = 1, 2$, then the manifold $M_1 \sharp M_2$ has no normal almost contact structure.*

2. Essentially different contact structures on spheres.

Let η_0 and η_1 be two contact forms on a $(2n+1)$-dimensional manifold M. Let T^*M be the cotangent bundle of M.

DEFINITION. η_1 is deformable to η_0 if there exists a differential homotopy of cross sections $F: M \times I \to T^*M$, with $F_0 \equiv F|M \times \{0\} = \eta_0$, $F_1 \equiv F|M \times \{1\} = \eta_1$ and $F_t = F|M \times \{t\}$ is a contact form of M for all $0 \leqq t \leqq 1$.

*) The author thanks M. Inoue for showing him this result with proof. The author had only a partial result.

In the paper [4], Gray has shown the rigidity of deformation of contact structure.

THEOREM (GRAY). *If η_1 is deformable to η_0, then there exists a diffeomorphism f of M diffeotopic to the identity such that $f^*\eta_1 = \tau\eta_0$, where τ is a positive function on M.*

DEFINITION. Two contact forms η_0 and η_1 are essentialy equivalent if there exists a diffeomorphism f of M such that $f^*\eta_1 = \tau\eta_0$, where τ is a non-zero function on M. We say they are essentially different if no such diffeomorphism exists.

Remark that if $f^*\eta_1 = \tau\eta_0$ for some positive function τ, and if f is diffeotopic to the identity, then η_1 is deformable to η_0. In fact, let f_t be the diffeotopy of f such that $f_1 =$ identity and $f_0 = f$. We define a one-parameter family of contact forms F_t on M by

$$F_t = \Big(t + (1 - t)\frac{1}{\tau}\Big)f_t^*\eta_1 .$$

Then $F_0 = (1/\tau)f^*\eta_1 = \eta_0$ and $F_1 = \eta_1$. F is a deformation connecting η_0 and η_1.

DEFINITION. Two almost contact structure λ_0 and λ_1 are said to be homotopic if there exists a one-parameter family of the reduction of the structure group of the tangent bundle of M to $U(n)$ connecting λ_0 and λ_1.

If there exists a deformation between two contact structures η_0 and η_1, then the induced almost contact structures λ_0 and λ_1 are homotopic.

Let $\tau(M)$: $M \to BSO(2n + 1)$ denote the classifying map of the tangent bundle TM of M. The homotopy classes of almost contact structures on M correspond bijectively to the homotopy classes of liftings λ: $M \to BU(n)$ so that the diagram

$$\begin{tikzcd} & BU(n) \arrow[d, "Bj"] \\ M \arrow[ur, "\lambda"] \arrow[r, "\tau(M)"'] & BSO(2n+1) \end{tikzcd}$$

commutes, where Bj: $BU(n) \to BSO(2n + 1)$ is induced by the inclusion j: $U(n) \to SO(2n + 1)$.

The inclusion maps i: $SO(2n + 1) \to SO(2n + 2)$ and i: $U(n) \to U(n + 1)$ induce the maps Bi: $BSO(2n + 1) \to BSO(2n + 2)$ and Bi: $BU(n) \to BU(n + 1)$ respectively. The composition $Bi \circ \lambda$: $M \to BU(n + 1)$ is a lifting of $Bi \circ \tau(M)$: $M \to BSO(2n + 2)$, i.e., the diagram

$$\begin{tikzcd} & BU(n) \arrow[r, "Bi"] \arrow[d, "Bj"] & BU(n+1) \arrow[d, "Bj"] \\ M \arrow[ur, "\lambda"] \arrow[r, "\tau(M)"] & BSO(2n+1) \arrow[r, "Bi"] & BSO(2n+2) \end{tikzcd}$$

commutes, where $Bj: BU(n+1) \to BSO(2n+2)$ also denotes the map induced by the inclusion $j: U(n+1) \to SO(2n+2)$.

Let $p: M \times \boldsymbol{R} \to M$ be the projection. Then the composition $Bi \circ \tau(M) \circ p: M \times \boldsymbol{R} \to BSO(2n+2)$ is a classifying map of the tangent bundle of $M \times \boldsymbol{R}$. A lifting of $Bi \circ \tau(M) \circ p$ to $BU(n+1)$ is an almost complex structure of $M \times \boldsymbol{R}$. Since $M \times \boldsymbol{R}$ is homotopy equivalent to M, the homotopy classes of almost complex structures on $M \times \boldsymbol{R}$ correspond bijectively to the liftigs of $Bi \circ \tau(M)$ to $BU(n+1)$.

The fibers of the fiberings $BU(n) \to BSO(2n+1)$ and $BU(n+1) \to BSO(2n+2)$ are $SO(2n+1)/U(n)$ and $SO(2n+2)/U(n+1)$ respectively. But we know the following. (See e.g., [4], Corollary 3.1.3.)

LEMMA 4. *The inclusion* $i: SO(2n+1) \to SO(2n+2)$ *induces a homeomorphism*

$$SO(2n+1)/U(n) \cong SO(2n+2)/U(n+1)\,.$$

Hence we obtain the following.

PROPOSITION 1. *The set of homotopy classes of almost contact structures on M corresponds bijectively to the set of homotopy classes of almost complex structures on $M \times \boldsymbol{R}$.*

In the paper [15] (see also [1], [2]), Sasaki and Hsu have proved that any Brieskorn manifold has a normal contact structure. It induces the complex structure on $M \times \boldsymbol{R}$. This complex structure is just equal to the complex structure of the hypersurface punctured at the origin defining the Brieskorn manifold.

If a Brieskorn manifold is homeomorphic to the sphere S^{2n+1}, then we have a complex structure on $S^{2n+1} \times \boldsymbol{R}$, which is homeomorphic to $\boldsymbol{R}^{2n+2} - \{0\}$. We can express homotopy spheres which bound parallelizable manifolds as Brieskorn manifolds in many different ways. At this point, we can apply Morita's work [12]. He investigates the homotopy classes of almost complex structures on $\boldsymbol{R}^{2n+2} - \{0\}$ which are induced from complex analytic structures.

Let us define the number $a(n)$ by

$$a(n) = \begin{cases} \infty & \text{if } n \text{ is odd} \\ n! & \text{if } n \equiv 0 \pmod 4 \\ n!/2 & \text{if } n \equiv 2 \pmod 4\,. \end{cases}$$

Direct application of Morita's work to Proposition 1 shows that;

THEOREM 2. *Let* Σ^{2n+1} $(n > 1)$ *be a homotopy sphere which bounds a parallelizable manifold. Then there are at least* $a(n)$ *contact structures on* Σ^{2n+1} *any two of which cannot be connected by deformations.*

For S^3, similar results have been obtained by Lutz [9].

If n is even, then the contact forms η and $-\eta$ define the different orientations. The contact forms can never be homotopic. If n is odd, the conjugation from η to $-\eta$ defines an action of order 2 on the set of homotopy classes of almost contact structures.

To know the classification by the "essentially equivalence", we must consider the action of the homotopy classes of diffeomorphisms of the manifold on the set of homotopy classes of almost contact structures.

Let θ^{2n+1} be the abelian group of the h-cobordim classes of the homotopy $(2n+1)$-spheres $(n > 1)$.

LEMMA 5. *The order of the homotopy classes of diffeomorphisms of a homotopy sphere* Σ^{2n+1} *is one or two according as* $\Sigma^{2n+1}\# \Sigma^{2n+1}$ *is non-trivial in* θ^{2n+1} *or not.*

The non-zero homotopy class of diffeomorphisms of Σ^{2n+1} is (if it exists) given by an orientation reversing diffeomorphism, say f. Let η_0 and η_1 be two contact forms on Σ^{2n+1}, and let λ_0 and λ_1 be their induced almost contact structures. Denote by $-\lambda_1$ the almost contact structure induced from $-\eta_1$, and by $\bar{\lambda}_1$ the almost contact structure corresponding to the almost complex structure induced by $f \times \mathrm{id.}$ on $\Sigma^{2n+1} \times \boldsymbol{R}$ from the almost complex structure corresponding to λ_1.

PROPOSITION 2. *Let* Σ^{2n+1} $(n > 1)$ *be a homotopy sphere which bounds a parallelizable manifold. Then* η_0 *is essentially different to* η_1 *if* λ_0 *is not homotopic to*

$$\begin{cases} \lambda_1 & \textit{when } n \textit{ is even and } \Sigma^{2n+1}\#\Sigma^{2n+1} \neq 0 \textit{ in } \theta^{2n+1}, \\ \lambda_1, \textit{ nor } \bar{\lambda}_1 & \textit{when } n \textit{ is odd and } \Sigma^{2n+1}\#\Sigma^{2n+1} = 0 \textit{ in } \theta^{2n+1}, \\ \lambda_1, \textit{ nor } \bar{\lambda}_1, \textit{ nor } -\lambda_1 & \textit{when } n \textit{ is odd.} \end{cases}$$

We define the number $b(n)$ for $n > 2$ by

$$b(n) = \begin{cases} \infty & \text{if } n \text{ is odd,} \\ n!/2 & \text{if } n \equiv 0 \pmod 4, \\ n!/4 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Combining Proposition 2 with Theorem 2, we have;

THEOREM 3. *Let Σ^{2n+1} $(n > 2)$ be a homotopy sphere which bounds a parallelizable manifold. If $\Sigma^{2n+1} \# \Sigma^{2n+1}$ is non-zero in θ^{2n+1}, there are at least $a(n)$ essentially different contact structures on Σ^{2n+1}. If $\Sigma^{2n+1} \# \Sigma^{2n+1}$ is zero in θ^{2n+1}, there are at least $b(n)$ essentially different contact structures on Σ^{2n+1}.*

Having as the model real strongly pseudo-convex hypersurfaces in C^{n+1}, we know the abstract definition of strongly pseudo-convex manifolds (see e.g., Folland-Kohn [3] etc.). Universal family of deformations of strongly pseudo-convex manifolds has been constructed by Kuranishi [8]. A strongly pseudo-convex manifold has a contact structure. The almost strongly pseudo-convex structure is just equal to the almost contact structure. Indeed Tanaka [17] has shown that any Brieskorn manifold has a strongly pseudo-convex structure. Hence we have non-deformable strongly pseudo-convex structures on homotopy spheres.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN