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PERTURBATION OF NONLINEAR HYPERCONTRACTIVE SEMIGROUPS

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We present an extension to nonlinear operators of some results of I. Segal. Let $S_A(t)$ generated by $-A$ be a semigroup of nonlinear contractions in L^p, and take L^p to $L^{p+\varepsilon(t)}$. This strong condition allows us to perturb $-A$ by $-F$, with weak conditions on F, so that $-A-F$ has closure generating a semigroup in L^p. F is a nonlinear Nemytskii operator.

Introduction. This work extends some ideas of I. Segal [10, 11], following also B. Simon and R. Hoegh-Krohn [12, Section 2]. In their work, $-A$ is self adjoint and generates a hypercontractive semigroup, while F is given by multiplication by the function V. They approximate V by $V_n \in L^\infty$, giving semigroups $S_{A+V_n}(t)$. In the linear case the convergence of $S_{A+V_n}(t)$ follows Du Hamel's formula; If A is m-accretive, B and C bounded, then

$$S_{A+B}(t) = S_{A+C}(t) + \int_0^t S_{A+B}(t-u)(C-B)S_{A+C}(u)du \, .$$

In the nonlinear case we do not have this formula, but we can show convergence of $S_{A+F_n}(t)$. Also we do not have their results [12, Lemma 2.15] on self adjoint operators. As in the linear case we do have the Trotter product formula for giving bounds on $S_{A+V_n}(t)$.

In this paper there are three sections: one on convergence of $A+F_n$, one on almost accretive Nemytskii operators, and one on hypercontractive semigroups.

The following comments raise a problem for further work. The section on hypercontractive semigroups $S_A(t)$ gives

$$|S_A(t)u - S_A(t)v|_{(p^{-1}-a(t))^{-1}} \leqq C^t |u-v|_p$$

with $a(t)$ linear only when $S(z)$ is contractive for $\operatorname{Re}(z) \geqq 0$, making it affine. This clashes with the result on Nemytskii operators F, where we have

$$|S_F(t)u - S_F(t)v|_p \leqq K^t |u-v|_{(p^{-1}-a(t))^{-1}}$$

with $a(t)$ nonlinear only when F satisfies very strong conditions.

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1. Convergence of $A + F_n$. Let $(X, |\ |)$ be a Banach space over $\boldsymbol{C}$ or $\boldsymbol{R}$ with dual X^*, and pairing denoted by parentheses. Let $J: X \to P(X^*)$ be the duality map defined by $f \in Jx$ when $(x, f) = \|x\|^2 = \|f\|^2$. An operator A in X is a function from X to $P(X)$. A is single-valued if Ax never contains more than one point. The domain $D(A)$ of A is the set of x with Ax nonempty, and the range $R(A)$ of A is the union of the sets Ax. We identify A with its graph in $X \times X$. We add operators, multiply by scalars and take inverses. Let $(\ ,\ )_s: X \times X \to \boldsymbol{R}$ be defined by

$$(f, g)_s = \lim_{d \downarrow 0} d^{-1}(\|g + df\|^2 - \|g\|^2) .$$

If A is an operator in X, the following are equivalent by Bénilan [1] or Kato [7].

(1) If $\lambda > 0$, $x_1 \in Ax$, $y_1 \in Ay$, then $\|(x + \lambda x_1) - (y + \lambda y_1)\| \geqq \|x - y\|$.

(2) If $x_1 \in Ax$, $y_1 \in Ay$, then $(x_1 - y_1, x - y)_s \geqq 0$.

(3) If $x_1 \in Ax$, $y_1 \in Ay$, then there is $f \in J(x - y)$ with $\operatorname{Re}(x_1 - y_1, f) \geqq 0$.

A is called accretive iff. any of these hold. If there are several Banach spaces we will index the norms, duality map, functions $(\ ,\ )_s$, closure operations, etc, by the space, as $|x|_X$, $J_X(x)$, and $(x, y)_{X,s}$, and $\operatorname{cl}_X(A)$. Supposing A accretive, A is called m-accretive iff. $R(I + \lambda A) = X$ for $\lambda > 0$, and A is called maximal iff. it is maximal with respect to inclusion among accretive sets with domain contained in $\operatorname{cl}(D(A))$.

We write $A \in A(w)$ to mean $A + wI$ is accretive, in which case A is maximal means $A + wI$ is maximal.

Let (Ω, B, μ) be a measure space. For $p \in [1, \infty]$, let $L^p = (L^p(M; X), |\ |_p)$ denote the space of (equivalence classes of) measurable functions $f: M \to X$, with $|f|_p^p = \int |f|^p d\mu < \infty$, and the usual modification for $p = \infty$.

THEOREM 1.1. *Let $(Y, |\ |_Y)$ and $(Z, |\ |_Z)$ be Banach spaces over $\boldsymbol{R}$ or $\boldsymbol{C}$, with Z continuously contained in Y. Let F_n be a sequence of single valued operators in Z, with $D(F_n) = D(F)$ for all n. Suppose $F_n: D(F) \to Y$ converges to $F: D(F) \to Y$ uniformly on bounded subsets of Z. Let A be an operator in Z. Suppose $w \in \boldsymbol{R}^+$ and $A + F_n + wI$ is accretive in Y for all n. Let C and D be subsets of Z.*

Suppose one of the following hold. (1) *C is bounded.* (2) *There is*

$x_0 \in D(A)$ with $F_n x_0$ bounded in Z and also $A + F_n + wI$ is accretive in Z for all n.

Suppose $(I + \lambda(A + F_n))C \supseteqq D$ for $\lambda \in (0, w^{-1})$, and all n. Then the closure $\mathrm{cl}_Y(A + F)$ of $A + F$ in $Y \times Y$ satisfies $(I + \lambda\, \mathrm{cl}_Y(A + F))\, \mathrm{cl}_Y(C) \supseteqq \mathrm{cl}_Y(D)$ for $\lambda \in (0, w^{-1})$, and $wI + \mathrm{cl}_Y(A + F)$ is accretive in Y.

PROOF. Let $(1 + \lambda w)y_i + \lambda(a_i + Fy_i) = x_i$, $i = 1, 2$, with $w^{-1} > \lambda > 0$, and $a_i \in Ay_i$. Then $(1 + \lambda w)y_i + \lambda(a_i + F_n y_i) \to x_i$ in Y. Since $A + F_n + wI$ are accretive in Y, taking limits gives $|y_1 - y_2|_Y \leqq |x_1 - x_2|_Y$. That is, $A + F + wI$ is accretive in Y, and consequently $\mathrm{cl}_Y(A + F) + wI$ is accretive in Y.

Since $\mathrm{cl}_Y(A + F) + wI$ is accretive it is enough to show

$$(I + \lambda\, \mathrm{cl}_Y(A + F))\, \mathrm{cl}_Y(C) \supseteqq D$$

for $\lambda \in (0, w^{-1})$. Given x in D, let $y_n = (I + \lambda(A + F_n))^{-1}x$. We claim y_n are bounded. If C is not bounded, take $a \in Az_0$. Since $A + F_n + wI$ are accretive in Z,

$$\begin{aligned}(1 - \lambda w)|y_n - x_0|_Z^2 &\leqq (x - (x_0 + \lambda a + \lambda F_n x_0),\, y_n - x_0)_{Z,s} \\ &\leqq K|y_n - x_0|_Z\end{aligned}$$

for some $K \in \boldsymbol{R}$, proving the claim. Take $a_n \in Ay_n$ with $y_n + \lambda a_n + \lambda F_n y_n = x$. Then

$$\begin{aligned}(1 - \lambda w)|y_n - y_m|_Y^2 &\leqq ((y_n + \lambda a_n + \lambda F_n y_n) - (y_m + \lambda a_m + \lambda F_n y_m),\, y_n - y_m)_{Y,s} \\ &\leqq \lambda(F_m y_m - F_n y_m,\, y_n - y_m)_{Y,s}\,.\end{aligned}$$

Now $F_n \to F$ uniformly on the bounded set $\{y_m\}$ of Z, giving $|y_n - y_m|_Y \to 0$. Hence, there is $y \in Y$ with $y_n \to y$ in Y. Since $y_n + \lambda a_n + \lambda F y_n = x + \lambda(F - F_n)y_n \to x$ in Y, we have $x \in (I + \lambda\, \mathrm{cl}_Y(A + F)\, \mathrm{cl}_Y(C)$. q.e.d.

LEMMA 1.1. *Let X be a Banach space, with X and X^* uniformly convex. Suppose A and B are single-valued, and $A \in A(w_A)$, $B \in A(w_B)$. Let C be a closed convex subset of X such $\mathrm{cl}\,(C \cap D(A + B)) = C$. Let A be maximal and B closed. Suppose that for λ small, $\lambda > 0$, $R(I + \lambda A) \supset D(A)$, $(I + \lambda A)^{-1}C \subset C$, $(I + \lambda B)C \supset C$ and $(I + \lambda(A + B))C \supset C$. Let S_A be generated on $\overline{D(A)} (= \mathrm{cl}(D(A)))$ by $-A$, and let S_B and S_{A+B} be generated on C by $-B_{|C}$ and $-(A + B)_{|C}$, to use the terminology of Brezis and Pazy* [3], i.e., *$S_A(t)x = \lim_{n\to\infty} (I + (t/n)A)^{-n}$ for $x \in \overline{D(A)}$, and S_B and S_{A+B} likewise. Then for $x \in C$, $S_{A+B}(t)x = \lim_{n\to\infty} (S_A(t/n)S_B(t/n))^n x$, and the limit is uniform in t on every finite interval.*

PROOF. Since $A \in A(w_A)$ and A is maximal, we have $R(I + \lambda A) \supset \overline{D(A)}$

for small $\lambda > 0$ and $-A$ is the infinitesimal generator of the semigroup S_A defined by $S_A(t)u = \lim_{n\to\infty}(I + (t/n)A)^{-n}u$ for $u \in \overline{D(A)}$ and $t \geqq 0$ (see [9, Theorem 3]). It follows from $(I + \lambda A)^{-1}C \subset C$ that each $S_A(t)$ maps C into itself. We next consider $B_1 = B_{|C}$ (the restriction of B to $C \cap D(B)$). Clearly $B_1 \in A(w_B)$ and $R(I + \lambda B_1)(=(I + \lambda B)C) \supset C = \overline{C \cap D(B)} = \overline{D(B_1)}$ for small $\lambda > 0$. Note that the closedness of B implies that B_1 is also closed. Therefore $-B_1$ is the infinitesimal generator of the semigroup S_B on C defined by $S_B(t)u = \lim_{n\to\infty}(I+(t/n)B)^{-n}u (= \lim_{n\to\infty}(I+(t/n)B_1)^{-n}u)$ for $u \in C$ and $t \geqq 0$, i.e., $\lim_{t\to 0} t^{-1}(u - S_B(t)u) = B_1 u = Bu$ for $u \in D(B_1) = D(B) \cap C$. (See [9. Cor. 2].) Also, $-(A + B)_{|C}(= -(A + B_1))$ generates a semigroup S_{A+B} on C, because $A + B_1 \in A(w_A + w_B)$ and $R(I + \lambda(A + B_1)) \supset C = \overline{D(A + B) \cap C} = \overline{D(A + B_1)}$ for small $\lambda > 0$.

We use the following result from [2, Cor. 4.3]. For $t > 0$, let $T(t)$ be Lipschitz with constant $M(t)$ mapping a closed convex subset C of X into itself. Let $\tilde{A} \in A(w)$ be single-valued, $\mathrm{cl}\, D(\tilde{A}) = C$, $\mathrm{cl}\,(R(I + \lambda\tilde{A})) \supset C$ for $\lambda \in (0, w^{-1})$. Then $-\mathrm{cl}\,(\tilde{A})$ generates a semigroup $S(t)$ on C. If (i) $M(t) = 1 + wt + o(t)$ as $t \to 0$ and (ii) $t^{-1}(x - T(t)x) \to \tilde{A}x$ as $t \to 0$ for $x \in D(\tilde{A})$, then $\lim_{n\to\infty}(T(t/n))^n x = S(t)x$ for $x \in C$, and the limit is uniform on bounded t intervals.

We now use the above results by putting $T(t) = S_A(t)S_B(t)$ and $\tilde{A} = A + B_1$. For each $t \to 0$, $T(t)\colon C \to C$ is Lipschitz with constant $e^{w_A t}e^{w_B t} = 1 + (w_A + w_B)t + o(t)$ as $t \to 0$. Thus, to prove the lemma, it suffices to show that

$$(*) \qquad \lim_{t\to 0} t^{-1}(u - T(t)u) = (A + B_1)u \quad \text{for} \quad u \in D(A + B_1)\,.$$

For $u \in D(A + B_1)(= D(A + B) \cap C)$, $t^{-1}(u - T(t)u) = t^{-1}(u - S_A(t)u) + y_t$, where $y_t = t^{-1}(S_A(t)u - S_A(t)S_B(t)u)$. Now $|y_t| \leqq e^{(w_A+w_B)t}|Bu|$. Apply $I - S_A(t)$ at $v \in D(A)$ and $S_B(t)u$, noting $I - S_A(t)$ is $A(w_A(t) + o(t))$.

$$\begin{aligned} &\mathrm{Re}\,((v - S_A(t)v) - (u - S_A(t)u) + (u - S_B(t)u) - ty_t, J(v - S_B(t)u)) \\ &\qquad \geqq -(w_A t + o(t))|v - S_B(t)u|^2. \end{aligned}$$

Suppose $t(n) \to 0$ and $y_{t(n)}$ converges weakly to y. Putting $t = t(n)$, dividing by $t(n)$ and letting $n \to \infty$, we obtain

$$\mathrm{Re}\,(Av - Au + Bu - y, J(v - u)) \geqq -w_A|v - u|^2\,.$$

Since A is maximal and $u \in \mathrm{cl}\,(D(A))$, $Au + y - Bu = Au$. Then $y_{t(n)} \to Bu$, and consequently $y_t \to Bu$ as $t \to 0$. q.e.d.

LEMMA 1.2. *Let X be a Banach space, X and X^* uniformly convex. Let C be a closed convex subset of $L^p = L^p(M; X)$, $p \in (1, \infty)$. Suppose A, F are single-valued operators in L^p, A and $F \in A(w)$, and*

$$\mathrm{cl}\,(C \cap D(F + A)) = C\,.$$

Let F be maximal and A closed. Suppose that for λ small, $\lambda > 0$, we have $R(I + \lambda F) \supseteqq D(F)$, $(I + \lambda F)^{-1}C \subseteqq C$, $(I + \lambda A)C \supseteqq C$ and

$$(I + \lambda(F + A))C \supseteqq C\,.$$

Let S_F be generated on $\mathrm{cl}\,(D(F))$ by $-F$, and let S_A and S_{F+A} be generated on C by $-A|_C$ and $-(F + A)|_C$. For $u, v \in C$, and $t \in (0, 1)$, suppose $|S_A(t)u - S_A(t)v|_{(p^{-1}-a(t))^{-1}} \leqq H^t|u - v|_p$ and

$$|S_F(t)u - S_F(t)v|_p \leqq K^t|u - v|_{(p^{-1}-a(t))^{-1}}\,,$$

where $K, H \in \boldsymbol{R}$ and $a\colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ are given. Then $S_{A+F}(t)$ is of type HK, i.e., $|S_{A+F}(t)u - S_{A+F}(t)v|_p \leqq (HK)^t|u - v|_p$.

PROOF. By Day [5, 6], $L^p(M; X)$ and $L^q(M; X^*)$ are uniformly convex. By Lemma 1.1,

$$|S_{A+F}(t)u - S_{A+F}(t)v|_p = \left|\lim_{n\to\infty}\left(S_F\left(\frac{t}{n}\right)S_A\left(\frac{t}{n}\right)\right)^n u - \left(S_F\left(\frac{t}{n}\right)S_A\left(\frac{t}{n}\right)\right)^n v\right|_p\,.$$

Since

$$\left|S_F\left(\frac{t}{n}\right)S_A\left(\frac{t}{n}\right)x - S_F\left(\frac{t}{n}\right)S_A\left(\frac{t}{n}\right)y\right|_p \leqq H^{t/n}K^{t/n}|x - y|_p\,,$$

for x, y in C, the result follows.

COROLLARY 1.1. *If F_n is sequence of operators satisfying the above for all n, $F_n \in A(w_n)$, $A + F_n$ closed, then the restriction of $A + F_n$ to C is in $A(\log(HK))$.*

PROOF. By Miyadera [9, Corollary 2], since $A + F_n$ is single-valued, for $x \in D(A + F_n)$, the right derivative of $S_{A+F_n}(t)x$ exists and is equal to $-(A + F_n)x$. q.e.d.

2. Almost accretive Nemytskii operators. Let (M, B, μ) be a measure space. Let $(X, |\ \ |)$ be a separable Banach space over $\boldsymbol{C}$. For $p \in [1, \infty]$, let $L^p = L^p(M; X)$. We also put L^p for $L^p(M; \boldsymbol{R})$ as in (2), (3) when there is no confusion. Given $f\colon M \times X \to X$, we define $F\colon U \to U$, where $U = \{u\colon M \to X\}$, by $(Fu)(x) = f(x, u(x))$. F also denotes the mapping on equivalence classes of functions equal a.e. F is called a Nemytskii operator. We will use the following conditions.

(1) f satisfies the Carathéodory conditions, i.e., f is measurable in x for $u \in X$, and continuous in u for x a.e.

(2) $W\colon M \to (-\infty, 0]$ is measurable, $e^{-tW} \in L^1$ for $t \geqq 0$, and for $s \in M$, $u \to f(s, u) - W(s)u$ is accretive in X.

(3) $|f(x, u)| \leqq \sum_{i=1}^{m} T_i(x)|u|^{\beta_i}$, where $p_2\beta_i < p_1$, $p_1, p_2 \in [1, \infty)$, and $T_i \in L^{p_1p_2/(p_1-p_2\beta_i)}$.

(4) $W_1: M \to [0, \infty)$ is measurable, and for $s \in M$, $u \in X$, $|f(s, u)| \leqq W_1(s)(1 + |u|)$.

(5) $E_n = \{s \in M: u \to f(s, u) + nu$ is accretive in X and $|f(s, u)| \leqq n(1 + |u|)$ for $u \in X\}$. Let $f_n(s, u) = f(s, u)$ if $s \in E_n$, and $f_n(s, u) = 0$ if $s \notin E_n$.

THEOREM 2.1. *Let f satisfy* (1), (2), (3), *and* (4). *Defining f_n by* (5), *and F_n and F from f_n and f, F_n and F are bounded continuous operators from L^{p_1} to L^{p_2}. F_n converges to F uniformly on bounded subsets of L^{p_1}. $F_n + nI$ is bounded, continuous, and accretive in L^p for $p \in [1, \infty)$. Letting S_n be generated by $-F_n$ in L^p, for all r,*

$$|S_n(t)u - S_n(t)v|_{1/(1/p+t/r)} \leqq |e^{-W}|_r^t |u - v|_p .$$

PROOF. Since X is separable, the sets E_n are measurable. Hence, f_n satisfy (1). Also, f_n satisfy (3). Since f_n and f satisfy (1) and (3), F_n and F are bounded and continuous from L^{p_1} to L^{p_2} by Krasnoselskii [8]. By (2) and (4), $M = \bigcup_{n=1}^{\infty} E_n$. Consequently, $\int_{E_n'} T_i^{p_1p_2/(p_1-p_2\beta_i)} \to 0$ as $n \to \infty$ for $1 \leqq i \leqq m$. By (3), F_n converges to F uniformly on bounded sets.

The definition of E_n gives $|f_n(s, u)| \leqq n(1 + |u|)$, and so F_n is bounded and continuous from L^p to L^p for all p since (1) is satisfied by f_n [8]. Since $u \to f_n(s, u) + nu$ is accretive in X, it follows that $F_n + nI$ is accretive in L^p.

Let S_n be generated in L^p by $-F_n$. Then for $u, v \in L^p$, s a.e. in M, by (2),

$$|(S_n(t)u)(s) - (S_n(t)v)(s)| \leqq e^{-tW(s)}|u(s) - v(s)| .$$

Hence, we have $|S_n(t)u - S_n(t)v|_{1/(1/p+t/r)} \leqq |e^{-W}|_r^t |u - v|_p$. q.e.d.

3. Hypercontractive semigroups. Let (M, B, μ) be a measure space. Let $\Sigma = \{z \in C: |\arg(z)| < \theta\pi\}$ where $\theta \in (0, 1/2]$. Let $(X, |\ |)$ be a reflexive Banach space over C, and for $p \in [1, \infty]$ let $L^p = L^p(M; X)$. Let C be a closed convex nonempty subset of L^p, $p \in (1, \infty)$.

DEFINITION. We say $\{U(z): z \in \mathrm{cl}\,\Sigma\}$ is a hypercontractive semigroup on C if the following are satisfied. For $z \in \mathrm{cl}\,\Sigma$, $U(z): C \to C$ is nonexpansive, i.e., $|U(z)u - U(z)v|_p \leqq |u - v|_p$. Also $U(0)u = u$ and $U(z)U(w)u = U(z + w)u$ for $u \in C$ and $z, w \in \mathrm{cl}\,\Sigma$. $U(z)u \to u$ for $x \in C$ as $z \to 0$. There is $\varepsilon \neq 0$, $K \in \boldsymbol{R}$, such that for $u, v \in C$, $|U(1)u - U(1)v|_{p+\varepsilon} \leqq K|u - v|_p$. For u in C, $z \to U(z)u$ is holomorphic on Σ.

LEMMA 3.1. *Let Ω denote the bent strip* $\{z \in \Sigma: |\arg(z - 1)| > \theta\pi\}$.

Let $f: \mathrm{cl}(\Omega) \to \boldsymbol{C}$ be continuous, and analytic on the region Ω. Suppose $|f(z)| \leqq M_0$ on bdy Σ and $|f(z)| \leqq M_1$ on bdy $\Sigma + 1$. Then there is a continuous function $a(t): [0, 1] \to \boldsymbol{R}$, analytic on $(0, 1)$, $a(t) \leqq Ct$ for some C, satisfying $|f(t)| \leqq M_1^{a(t)} M_0^{1-a(t)}$.

PROOF. We use the standard three lines theorem and a Schwartz-Christoffel transform. We map

$$\{z: \mathrm{Im}(z) > 0,\ 0 < \mathrm{Re}(z) < 1\} \text{ into } \{z \in \Omega: \mathrm{Im}(z) > 0\}$$

by the composition of $z(s) = -\cos \pi s$ and

$$w(z) = K\int_0^z (1 + p)^{\theta-1}(1 - p)^\theta dp + H$$

where $K = 1/\int_{-1}^1 (1 + p)^{\theta-1}(1 - p)^\theta dp$ and $H = -K\int_0^{-1} (1 + p)^{\theta-1}(1 - p)^\theta dp$. The map $s \to w(s)$ takes $[0, 1]$ to $[0, 1]$, and is continuous on the boundary. The derivative of $s \to w(s)$ at $s \in (0, 1)$ is a constant times

$$\sin \pi s(1 - \cos \pi s)^{\theta-1}(1 + \cos \pi s)^\theta .$$

Let the inverse mapping be a. Since $a'(w) = 1/w'(a(w))$, by expanding the trig. functions we see there is $C \in \boldsymbol{R}$ such that for $a(w)$ small, $a'(w) \leqq Ca(w)^{2(1-\theta)}/a(w)$. Since $2\theta \leqq 1$, $a'(w)$ is bounded near 0. Thus there is C such that $a(w) \leqq Cw$ for $w \in [0, 1]$. Extend a to Ω by reflection. Putting $g(s) = (M_0/M_1)^s f(a^{-1}(s))$ and applying the three lines theorem [13] to g gives the result. q.e.d.

LEMMA 3.2. *Let $\{U(z): z \in \mathrm{cl}\,\Sigma\}$ be a hypercontractive semigroup. Let $a(t)$ be as in Lemma 3.1. Then there exists $\delta \neq 0$, with the same sign as ε, and $M > 0$ such that for $t \in [0, 1]$,*

$$|U(t)f - U(t)h|_{(p^{-1}-a(t)\delta)^{-1}} \leqq H^t |f - h|_p .$$

Also, if C has nonempty interior, and there is q with $|U(z)f - U(z)h|_q \leqq |f - h|_q$, then given a closed interval I contained in the open interval between p and q, S and C may be taken so that the inequality above holds with p replaced by any $r \in I$.

PROOF. Given $f, h \in C$, $t \in [0, 1]$, put $s^{-1} = (1 - a(t))/p + a(t)/(p + \varepsilon)$, where $a(t)$ is as in Lemma 3.1. Take g, a measurable simple function from M to X^* with support of finite measure, $g(s) = G(s)e(s)$ where $G: M \to \boldsymbol{R}^+$, $|e(s)|_{X^*} = 1$, and $|g|_{s'} = 1$. Define

$$\varphi(z) = \int (U(z)f - U(z)h, g(z))d\mu$$

where $g(z) = eG^{((1-a(z))/p'+a(z)/(p+\varepsilon)')s'}$. Then φ is analytic in the interior of the bent strip of Lemma 3.1, and bounded and continuous on the closure. On bdy Σ, $|\varphi(z)| \leqq |U(z)f - U(z)h|_{L^p(M;X)}|g(z)|_{L^{p'}(M;X^*)} \leqq |f-h|_p$. And on bdy $\Sigma + 1$, $|\varphi(z)| \leqq |U(z)f - U(z)h|_{L^p(M;X)}|g(z)|_{L^{p'}(M;X^*)} \leqq k|f-h|_p$. By Lemma 3.1, we have $|U(t)f - U(t)h|_s \leqq K^{a(t)}|f-h|_p$. Since $a(t) \leqq Ct$ it follows that $|U(t)f - U(t)h|_{(p^{-1}-a(t)(p^{-1}-(p+\varepsilon)^{-1}))^{-1}} \leqq (K^C)^t|f-h|_p$. The second result follows from the nonlinear Riesz-Thorin theorem of Browder [4, Theorem 1, Proposition 1, and Remark in Section 3]. Note that $U(t)$ may not take L^q to L^q, but a translate $U(t) + z$ does take L^q to L^q, where $z \in L^p$. Hence, we may assume $U(t)$ takes $X_0 = L^r \cap L^p \cap L^q$ (in the terminology of [4]) to itself. q.e.d.

THEOREM 3.2. *Let X be a separable Banach space with X and X^* uniformly convex. Let f satisfy* (1), (2), (3) *and* (4) *of Section* 2, *and let $F: L^{p_1} \rightarrow L^p$, $p_1, p \in (1, \infty)$, be given by f. Let C be a closed convex subset of L^p. Suppose $A \in A(0)$, single-valued, $(I + \lambda A)C \supseteqq C$ for $\lambda > 0$ small, and $C \cap D(A)$ dense in C. Let $S_A(t)$ generated by $-A_{|C}$ on C have an extension to a hypercontractive semigroup $\{S_A(z): \operatorname{Re} z \geqq 0\}$ on C, with $\varepsilon > 0$. Suppose for all n, $(I + \lambda F_n)C \supseteqq C$ and $(I + \lambda(F_n + A))C \supseteqq C$ for $\lambda > 0$ small, where F_n is defined by* (5), *Section* 2. *Suppose A is closed in L^p. Suppose $C \cap L^{p_1}$ is bounded in L^{p_1} and L^p, or C has non-empty interior and $S_A(z)$ is nonexpansive in the L^q norm, with p_1 strictly between p and q.*

Then $A + F \in A(w)$ in L^p for some w, and $(I + \lambda \operatorname{cl}(A + F))C \supseteqq C$ for $\lambda > 0$, small.

PROOF. Take $Z = L^p \cap L^{p_1}$. To apply Theorem 1.1, we need $A + F_n + wI$ accretive in L^p(and in L^{p_1} if $C \cap L^{p_1}$ not bounded). This follows by Lemma 3.2, Corollary 1.1, and Theorem 2.1. q.e.d.

COROLLARY 3.1. *Let X be a separable Banach space with X and X^* uniformly convex. Let f satisfy* (1), (2), (3), *and* (4) *of Section* 2, *giving $F: L^{p_1} \rightarrow L^p$, $p_1, p \in (1, \infty)$. Let A be m-accretive in L^p, with dense domain, and single valued. Let $S_A(t)$ generated on L^p by $-A$ have an extension to a hypercontractive semigroup $\{S_A(z): \operatorname{Re}(z) \geqq 0\}$, with $\varepsilon > 0$. Suppose $S_A(z)$ is nonexpansive in the L^q norm, p_1 strictly between p and q.*

Then $\operatorname{cl}(A + F) + wI$ is m-accretive for some w.

PROOF. F_n is continuous we have $(I + \lambda F_n)$ and $I + \lambda(F_n + A)$ surjective for $\lambda > 0$, small, and $F_n + A$ is closed. The result follows by Theorem 3.2. q.e.d.

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