

INJECTIVE ENVELOPES OF BANACH MODULES

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1. Introduction. As in homology theory, the notion of injectivity was introduced in the category whose objects are Banach spaces and whose morphisms are contractive (i.e., of norm ≤ 1) linear maps, and the existence and uniqueness of the injective envelope of a Banach space was proved by H. B. Cohen [1] (cf. also [6]).

In the present paper we show that the corresponding statements are valid in the category whose objects are Banach modules over a Banach algebra and whose morphisms are contractive module homomorphisms, and that a flow (i.e., a compact Hausdorff space with a discrete group acting on it as onto homeomorphisms) has a projective cover. The latter, which seems to be, in a certain sense, a natural generalization of a result of A. M. Gleason [3; Theorem 3.2] (cf. 1° and Lemma 5 (i), (ii) below), is used to give a characterization of injective Banach modules over a discrete group algebra (Theorem 2 below). In the last section we are concerned with self-injective C^* -algebras (i.e., C^* -algebras which, considered as Banach modules over themselves, are injective).

Let A be a fixed Banach algebra with unit 1. We shall always assume that $\|1\| = 1$. A unital left A -module X is called a left Banach A -module if its underlying vector space is a Banach space with the norm satisfying the condition:

$$\|a \cdot x\| \leq \|a\| \|x\| \quad \text{for } a \in A \text{ and } x \in X.$$

Similarly a right or two-sided Banach A -module is defined. But throughout this paper we shall exclusively treat left Banach A -modules unless otherwise specified, and abbreviate them to Banach A -modules. The letter X will denote a fixed but arbitrary Banach A -module.

DEFINITIONS. An extension of X is a pair (Y, κ) of a Banach A -module Y and an isometric module homomorphism $\kappa: X \rightarrow Y$. A Banach A -module X is injective if for each Banach A -module Y and each extension (Z, κ) of Y , any continuous module homomorphism $\alpha: Y \rightarrow X$ extends to a continuous module homomorphism $\hat{\alpha}: Z \rightarrow X$, i.e., $\hat{\alpha} \circ \kappa = \alpha$, with $\|\hat{\alpha}\| = \|\alpha\|$. An extension (Y, κ) of X is injective if Y is an injective

Banach A -module, and it is essential if for each Banach A -module Z and each contractive module homomorphism $\alpha: Y \rightarrow Z$, α is an isometry whenever $\alpha \circ \kappa$ is so. If the pair (Y, κ) is both injective and essential, we call it an injective envelope of X . (See Theorem 1 below for the existence and uniqueness of the injective envelope.) This definition is also equivalent to the following: An extension (Y, κ) of X is an injective envelope of X if and only if Y itself is the only injective submodule of Y containing $\text{Im } \kappa = \kappa(X)$. (See Remark 2 below for this equivalence.)

We recall a few known results on Banach spaces. An injective Banach space (=an injective Banach C - or R -module, according as the coefficient field is complex or real) is usually called a P_1 space and completely characterized as follows:

1°. (L. Nachbin, D. B. Goodner, J. L. Kelley, M. Hasumi) A Banach space is a P_1 space if and only if it is linearly isometric to a Banach space of continuous functions $C(P)$ with P some stonean (i.e., extremally disconnected compact Hausdorff) space.

The above-mentioned result of H. B. Cohen is stated as follows:

2°. Every Banach space X has a unique (up to a linear isometry) injective envelope (Y, κ) , i.e., $\kappa: X \rightarrow Y$ is a linear isometry into and Y itself is the only injective linear subspace of Y which contains $\text{Im } \kappa$. In particular the injective envelope of a Banach space $C(K)$ with K a compact Hausdorff space is of the form $(C(P), \varphi^\circ)$, where φ is a minimal continuous map of a stonean space P onto K (i.e., $\varphi(P) = K$ but for each closed subset $F \subsetneq P$, $\varphi(F) \subsetneq K$) and $\varphi^\circ: C(K) \rightarrow C(P)$ is defined by $\varphi^\circ(f)(p) = f(\varphi(p))$ for $f \in C(K)$ and $p \in P$. Moreover P and φ are uniquely determined by the properties that P is stonean and that φ is minimal.

2. Construction of the envelope. The construction of the injective envelope of a Banach A -module which we present here is more or less analogous to the one given in homological algebra (cf. e.g., S. MacLane [8; pp. 92-94]).

The following lemma assures the existence of sufficiently many injectives in our category.

LEMMA 1. *Every Banach A -module X is a closed submodule of an injective Banach A -module.*

PROOF. As is well-known, X is, as a Banach space, a closed linear subspace of a P_1 space Y . [E.g., take a subset Γ of B_{X^*} (the closed unit ball of X^*) such that the weak* closed convex circled hull of Γ is B_{X^*} . Then $x \mapsto (\langle x, x^* \rangle)_{x^* \in \Gamma}$ is an isometric embedding of X into a P_1 space

$l^\infty(\Gamma)$, the Banach space of all bounded functions on the (discrete) set Γ .] Let $Z = L(A, Y)$ be the Banach space of all continuous linear maps of A into Y , which is made into a Banach A -module by setting

$$(a \cdot z)(b) = z(ba) \quad \text{for } a, b \in A \text{ and } z \in Z.$$

The map $\kappa: X \rightarrow Z$ given by $\kappa(x)(a) = a \cdot x$, $a \in A$, $x \in X$ is an isometric module homomorphism. By identifying X with $\text{Im } \kappa$, we need only prove that Z is injective. Let V be a Banach A -module, (W, λ) an extension of V and $\alpha: V \rightarrow Z$ a continuous module homomorphism. Define a continuous linear map $\beta: V \rightarrow Y$ by $\beta(v) = \alpha(v)(1)$, $v \in V$. Since Y is a P_1 space, there exists a continuous linear map $\hat{\beta}: W \rightarrow Y$ such that $\hat{\beta} \circ \lambda = \beta$ and $\|\hat{\beta}\| = \|\beta\|$. Then if we note that A has a unit 1 with $\|1\| = 1$, it is readily checked that the map $\hat{\alpha}: W \rightarrow Z$ defined by

$$\hat{\alpha}(w)(a) = \hat{\beta}(a \cdot w), \quad a \in A, w \in W$$

is the desired module homomorphism extending α and such that $\|\hat{\alpha}\| = \|\alpha\|$.
 q.e.d.

REMARK 1. In the above proof if we take $l^\infty(\Gamma)$ as Y , then we see that $Z = L(A, l^\infty(\Gamma)) \cong l^\infty(\Gamma, A^*)$, the l^∞ -sum of Γ copies of the Banach A -module A^* with the module operation given by

$$\langle b, a \cdot f \rangle = \langle ba, f \rangle, \quad a, b \in A, f \in A^*,$$

is an injective Banach A -module and contains X as a closed submodule. This fact will be used in §3.

To state the following lemma, we give a definition: Let (Y, κ) be an extension of X . A seminorm p on Y is admissible (relative to X) if it satisfies the conditions:

$$\begin{aligned} p(y) &\leq \|y\|, \quad p(a \cdot y) \leq \|a\|p(y) \quad \text{and} \\ p(\kappa(x)) &= \|x\| \quad \text{for } a \in A, x \in X \text{ and } y \in Y. \end{aligned}$$

Clearly the original norm on Y is an admissible seminorm.

LEMMA 2. *With notations as above, (Y, κ) is essential if and only if there exists no admissible seminorm on Y except for the original norm on Y .*

PROOF. Necessity: Let p be an admissible seminorm on Y , Z the completion of $Y/p^{-1}(0)$ in the norm induced by p , which is a Banach A -module with the natural module operation, and $\pi: Y \rightarrow Y/p^{-1}(0) \subset Z$ the canonical projection. Then π is a contractive module homomorphism with $\pi \circ \kappa$ an isometry. Hence by assumption π is an isometry, so that

p is the norm on Y .

Sufficiency: Let Z be a Banach A -module and $\alpha: Y \rightarrow Z$ a contractive module homomorphism such that $\alpha \circ \kappa$ is an isometry. Then the seminorm p on Y defined by $p(y) = \|\alpha(y)\|$, $y \in Y$ is admissible, so that $\|\alpha(y)\| = \|y\|$, i.e., α is an isometry. q.e.d.

We introduce a relation (resp. an equivalence relation) in the family of all extensions of a Banach A -module X by the rule

$$(Y, \kappa) \leq (Y_1, \kappa_1) \quad [\text{resp. } (Y, \kappa) \cong (Y_1, \kappa_1)]$$

if and only if there exists an isometric module homomorphism (resp. isomorphism) $\iota: Y \rightarrow Y_1$ such that $\iota \circ \kappa = \kappa_1$. We remark that the conditions $(Y, \kappa) \leq (Y_1, \kappa_1)$ and $(Y, \kappa) \geq (Y_1, \kappa_1)$ need not imply $(Y, \kappa) \cong (Y_1, \kappa_1)$. With these relations essential extensions and injective extensions are related as follows:

LEMMA 3. (i) For an essential extension (Y, κ) of X and an injective extension (Z, λ) of X , we have

$$(Y, \kappa) \leq (Z, \lambda).$$

(ii) Let (Y, κ) be an extension of X and (Z, λ) an extension of Y . Then $(Z, \lambda \circ \kappa)$ is an essential extension of X if and only if (Y, κ) and (Z, λ) are both essential.

(iii) In the family of all essential extensions of X , each increasing net has an upper bound.

(iv) A Banach A -module Y is injective if and only if it has no proper essential extension [i.e., if (Z, κ) is an essential extension of Y , then κ is an isomorphism].

PROOF. (i), (ii) and necessity of (iv) follow immediately from the definitions.

(iii) Let $\{(Y_\gamma, \kappa_\gamma)\}_{\gamma \in I}$ be an increasing net of essential extensions of X and $\iota_{\gamma', \gamma}: Y_\gamma \rightarrow Y_{\gamma'}$ the isometric module homomorphism such that $\iota_{\gamma', \gamma} \circ \kappa_\gamma = \kappa_{\gamma'} (\gamma < \gamma')$. Then $(Y_\infty, \kappa_\infty)$ is an upper bound of the increasing net, where $Y_\infty = \lim_{\rightarrow} \iota_{\gamma', \gamma}(Y_\gamma)$, the inductive limit of the Banach spaces Y_γ , is naturally equipped with the module structure and $\kappa_\infty: X \rightarrow Y_\infty$ is the canonical embedding.

Sufficiency of (iv): Noting Lemma 1 we need only prove that any extension (Z, κ) of Y splits, i.e., there exists a contractive module homomorphism $\sigma: Z \rightarrow Y$ with $\sigma \circ \kappa = \text{id}_Y$ (the identity map on Y). Let \mathcal{F} be the family of all admissible seminorms on Z relative to Y . Obviously \mathcal{F} is partially and inductively ordered under the relation $p \leq q$ defined

by $p(z) \leq q(z)$ for all $z \in Z$. Hence by Zorn's lemma \mathcal{F} has a minimal element p_0 . As in the proof of Lemma 2, let Z_1 be the completion, with the natural Banach module structure, of $Z/p_0^{-1}(0)$ in the norm induced by p_0 and let $\kappa_1 = \pi \circ \kappa: Y \rightarrow Z_1$ be the isometric embedding, where $\pi: Z \rightarrow Z/p_0^{-1}(0) \subset Z_1$ is the canonical projection. By minimality of p_0 the norm on Z_1 is a unique admissible seminorm on Z_1 relative to Y , so that the extension (Z_1, κ_1) of Y is essential (Lemma 2). Hence, κ_1 being onto by assumption, the map

$$\sigma = \kappa_1^{-1} \circ \pi: Z \rightarrow Z/p_0^{-1}(0) = Z_1 \rightarrow Y$$

is the desired splitting.

q.e.d.

THEOREM 1. *Every Banach A -module X has a unique (up to the equivalence relation \cong) injective envelope (Y, κ) .*

PROOF. By Lemma 3 (iii) and Zorn's lemma, there exists a maximal essential extension (Y, κ) of X . Then Lemma 3 (ii) and (iv) imply that Y is injective, so by definition (Y, κ) is an injective envelope of X .

To show the uniqueness let (Y_1, κ_1) be another injective envelope of X . Then $(Y, \kappa) \leq (Y_1, \kappa_1)$ by Lemma 3 (i), i.e., there exists an isometric module homomorphism $\iota: Y \rightarrow Y_1$ such that $\iota \circ \kappa = \kappa_1$. Since Y is injective and the extension (Y_1, ι) of Y is essential (Lemma 3 (ii)), we obtain $\text{Im } \iota = Y_1$. Thus $(Y, \kappa) \cong (Y_1, \kappa_1)$.

q.e.d.

REMARK 2. Let (Y, κ) be an injective extension of X . Then (Y, κ) is essential (i.e., it is an injective envelope of X) if and only if Y itself is the only injective submodule of Y containing $\text{Im } \kappa$. Indeed necessity follows from Lemma 3 (ii), (iv). Sufficiency: By Theorem 1 and Lemma 3 (i) there exists an injective submodule Y_1 of Y such that $\text{Im } \kappa \subset Y_1$ and (Y_1, κ) is an injective envelope of X , so that if Y is the only injective submodule of Y containing $\text{Im } \kappa$, we have $Y_1 = Y$, i.e., (Y, κ) is essential.

3. Injective Banach G -modules and projective flows. Let G be a discrete group. A Banach space X is said to be a Banach G -module if there exists a group homomorphism θ of G into $\text{Aut } X$ (=the group of all linear isometries of X onto itself). For simplicity we write $g \cdot x$ instead of $\theta(g)x$ ($g \in G, x \in X$). Any Banach G -module can be viewed as a Banach module over the discrete group algebra $l^1(G)$ with the module operation defined by

$$a \cdot x = \sum_{g \in G} a(g)g \cdot x, \quad a \in l^1(G), x \in X,$$

and vice versa. So injectivity of a Banach G -module can be defined (or

more directly) as that of the corresponding Banach $l^1(G)$ -module.

As seen in § 1, to each P_1 space X there corresponds a stonian space P so that X is linearly isometric to $C(P)$. On the other hand, A. M. Gleason [3] characterized stonian spaces as 'projective' objects in the category of compact Hausdorff spaces and continuous maps between them. We shall show results analogous to these: To each injective Banach G -module X there correspond a flow (P, G) and a 1 cocycle u on it so that X is isomorphic (as a Banach G -module) to the Banach G -module constructed from the flow and the 1 cocycle. Further, such a flow is characterized in the category of flows with the discrete group G acting on them and homomorphisms between them (Theorem 2 below). The existence and uniqueness of the injective envelope of a Banach module established in § 2 is used to show an analogue of [3; Theorem 3.2] (Theorem 3 below).

The precise definitions of the terminologies in the preceding paragraph and next theorems will be given later.

THEOREM 2. *Given an injective Banach G -module X , there exist a unique projective flow (P, G) and a 1 cocycle u (unique except for a 1 coboundary) on it such that X is, as a Banach G -module, isomorphic to the Banach G -module $(C(P), u)$ associated with the flow (P, G) and the 1 cocycle u .*

Conversely if (P, G) is a projective flow and u is any 1 cocycle on it, the Banach G -module $(C(P), u)$ is injective.

THEOREM 3. *A flow (S, G) has a unique projective cover $(P, G; \varphi)$, and the injective envelope of the Banach G -module $(C(S), 1)$ associated with the flow (S, G) and the trivial 1 cocycle 1 is of the form $((C(P), 1), \varphi^\circ)$.*

Before going into the proofs, we prepare some notations and definitions. A Banach G -module $l^\infty(\Gamma \times G)$ (Γ , an index set) with the module operation:

$$(g \cdot x)(\gamma, h) = x(\gamma, hg), \quad g, h \in G, x \in l^\infty(\Gamma \times G), \gamma \in \Gamma$$

is a typical example of an injective Banach G -module; moreover each Banach G -module is a closed submodule of an injective Banach G -module of this type (cf. Remark 1). This observation and the fact that $l^\infty(\Gamma \times G)$ is a P_1 space as a Banach space imply that each injective Banach G -module is a P_1 space, hence linearly isometric to the space $C(P)$ with P a stonian space (§ 1, 1°).

DEFINITIONS. A pair (S, G) of a compact Hausdorff space S and the discrete group G is called a flow if there exists a group homomorphism $\pi: G \rightarrow \text{Aut } S$ (=the group of all homeomorphisms of S onto itself),

which will be denoted simply by $\pi(g)(s) = s \cdot g$ ($s \in S, g \in G$). Let a continuous function $u: S \times G \rightarrow T = \{\lambda \in C: |\lambda| = 1\}$ satisfy the condition:

$$u(s, g_1)u(s \cdot g_1, g_2) = u(s, g_1g_2), \quad s \in S, g_1, g_2 \in G.$$

We call such a function u a 1 cocycle on the flow (S, G) . If the function u is trivial, i.e., $u(s, g) = 1$ for all $s \in S$ and $g \in G$, we denote it by 1. A 1 cocycle u of the form $u(s, g) = v(s)^{-1}v(s \cdot g)$ for some unitary element v in the commutative C^* -algebra $C(S)$ is called a 1 coboundary on (S, G) . Given a flow (S, G) and a 1 cocycle u , we can make the Banach space $C(S)$ into a Banach G -module by setting

$$(g \cdot x)(s) = u(s, g)x(s \cdot g), \quad g \in G, x \in X.$$

We refer to it as the Banach G -module associated with the flow (S, G) and the 1 cocycle u and denote it by $(C(S), u)$.

Conversely, it is immediate to see that given a Banach G -module X which is linearly isometric to a space $C(T)$ with T a compact Hausdorff space, the dual action of G on X^* induces an action of G on T to define a flow (T, G) and a 1 cocycle u on it so that X is isomorphic to the Banach G -module $(C(T), u)$ constructed above. In fact, identifying X with $C(T)$, the dual action of G on $X^* = C(T)^*: x^* \mapsto x^* \cdot g, \langle x, x^* \cdot g \rangle = \langle g \cdot x, x^* \rangle$ induces onto homeomorphisms of the set $\{\lambda \delta_t: \lambda \in T, t \in T\}$ of extreme points of the unit ball of $C(T)^*$, where δ_t is the point mass at t and $T = \{\lambda \in C: |\lambda| = 1\}$. Then denote $(\delta_t) \cdot g$ by $u(t, g)\delta_{t \cdot g}$ with $u(t, g) \in T$ and $t \cdot g \in T$.

REMARK 3. It is easily shown that given flows (S, G) and (T, G) and 1 cocycles u and v on them, respectively, the Banach G -modules $(C(S), u)$ and $(C(T), v)$ are isomorphic

\Leftrightarrow there exists an isomorphism of flows $\varphi: (S, G) \rightarrow (T, G)$ (i.e., homeomorphism of S onto T with $\varphi(s \cdot g) = \varphi(s) \cdot g$) and a unitary element $w \in C(S)$ such that

$$u(s, g)^{-1}v(\varphi(s), g) = w(s)^{-1}w(s \cdot g), \quad s \in S, g \in G$$

\Leftrightarrow identifying S and T by φ , u and v are cohomologous.

By the above argument an injective Banach G -module X is of the form $(C(P), u)$, where (P, G) is a flow and u is a 1 cocycle on it. Thus Remark 3 implies uniqueness of (P, G) up to an isomorphism and of u except for a 1 coboundary. We shall show that injectivity of X depends only on the flow (P, G) but not on the 1 cocycle u (Lemma 4 below), and characterize, in the category of flows and homomorphisms, a flow (S, G) such that the Banach G -module $(C(S), u)$ associated with the flow

(S, G) and any 1 cocycle u on it is injective (Lemma 5 (ii) below).

LEMMA 4. Let $X = (C(S), u)$ [resp. $X_1 = (C(S), 1)$] be a Banach G -module associated with a flow (S, G) and a 1 cocycle u (resp. a trivial 1 cocycle 1). Then X is injective if and only if X_1 is so.

PROOF. Let $Y = l^\infty(S \times G)$ be the injective Banach G -module considered above. Define an isometric module homomorphism κ (resp. κ_1) of X (resp. X_1) into Y by

$$\begin{aligned} \kappa(x)(s, g) &= (g \cdot x)(s) = u(s, g)x(s \cdot g), & x \in X \\ \text{(resp. } \kappa_1(x)(s, g) &= (g \cdot x)(s) = x(s \cdot g), & x \in X_1). \end{aligned}$$

We note that $\text{Im } \kappa_1$ is a C^* -subalgebra of the commutative C^* -algebra Y with the same unit as Y and that $\tilde{u} \text{Im } \kappa_1 = \text{Im } \kappa$, where \tilde{u} is the unitary element of Y such that $\tilde{u}(s, g) = u(s, g)$. Since Y is injective, X (resp. X_1) is injective if and only if there exists a contractive idempotent module homomorphism π (resp. π_1) of Y onto $\text{Im } \kappa$ (resp. $\text{Im } \kappa_1$).

Suppose that X is injective, hence that such a π exists. Then the map $\pi_1: Y \rightarrow Y$ defined by

$$\pi_1(y) = \tilde{u}^{-1}\pi(\tilde{u}y), \quad y \in Y$$

is clearly a contractive idempotent linear map of Y onto its C^* -subalgebra $\text{Im } \kappa_1$. Since $u_g \in C(S) = X_1$, where $u_g(s) = u(s, g)$, and $\tilde{u}\kappa_1(u_g) = g \cdot \tilde{u}$, we get by [11; Theorem 1]

$$\begin{aligned} \pi_1(g \cdot y) &= \kappa_1(u_g)^{-1}\pi_1(\kappa_1(u_g)g \cdot y) \\ &= (g \cdot \tilde{u})^{-1}\pi(\tilde{u}\kappa_1(u_g)g \cdot y) \\ &= g \cdot (\tilde{u}^{-1})\pi((g \cdot \tilde{u})(g \cdot y)) \\ &= g \cdot (\tilde{u}^{-1})g \cdot \pi(\tilde{u}y) = g \cdot \pi_1(y) \end{aligned}$$

for all $g \in G$ and $y \in Y$,

so that π_1 is a module homomorphism. Thus X_1 is injective. The converse implication is shown similarly. q.e.d.

Let (S, G) be a flow and $E(C(S))$ the set of all $\mu \in C(S)^*$ such that $\|\mu\| = 1$ and $\mu(1) = 1$. Then the set $E(C(S))$ equipped with the weak* topology is made into a flow by setting

$$(\mu \cdot g)(x) = \mu(g \cdot x), \quad \mu \in E(C(S)), g \in G, x \in C(S).$$

The set of point masses on S is a subflow (i.e., a closed G -invariant subset) of the flow $E(C(S))$, which can be identified with the flow (S, G) .

DEFINITIONS. Given two flows (S, G) and (T, G) , continuous map

$\varphi: T \rightarrow S$ is said to be a homomorphism if

$$\varphi(t \cdot g) = \varphi(t) \cdot g \quad \text{for } t \in T \text{ and } g \in G .$$

(Of course φ is an isomorphism if it is one-to-one and onto.) The homomorphism φ induces a continuous module homomorphism between Banach G -modules

$$\begin{aligned} \varphi^\circ: (C(T), 1) &\leftarrow (C(S), 1) , \\ \varphi^\circ(x)(t) &= x(\varphi(t)) , \quad x \in C(S), t \in T \end{aligned}$$

and a homomorphism of flows

$$\tilde{\varphi} = (\varphi^\circ)^*|_{E(C(T))}: E(C(T)) \rightarrow E(C(S))$$

which extends φ . With this duality between flows and Banach G -modules in mind, we define an (essential) extension of a flow as follows: The triple $(T, G; \varphi)$ is an (essential) extension of a flow (S, G) if φ is a homomorphism of a flow (T, G) onto (S, G) [and moreover $\tilde{\varphi}$ is minimal in the sense that $\tilde{\varphi}(E(C(T))) = E(C(S))$, but for each weak* closed G -invariant convex subset $K \subseteq E(C(T))$, $\tilde{\varphi}(K) \subseteq E(C(S))$]. We note that an extension $(T, G; \varphi)$ of a flow (S, G) induces an extension $((C(T), 1), \varphi^\circ)$ of the Banach G -module $(C(S), 1)$. A flow (P, G) is called projective if it has no proper essential extension (i.e., if $(Q, G; \varphi)$ is an essential extension of (P, G) , then φ is an isomorphism). Given a flow (S, G) , an extension $(P, G; \varphi)$ is a projective cover of (S, G) if it is both projective and essential.

The next lemma, combined with the preceding argument and Lemma 4, will complete the proofs of Theorems 2 and 3.

LEMMA 5. *Let (S, G) be a flow.*

(i) *There exists a unique projective cover $(P, G; \varphi)$ of the flow (S, G) , and the injective envelope of the Banach G -module $(C(S), 1)$ associated with the flow and the trivial 1 cocycle 1 is of the form $((C(P), 1), \varphi^\circ)$.*

(ii) *The flow (S, G) is projective if and only if the Banach G -module $(C(S), 1)$ is injective.*

(iii) *$(T, G; \varphi)$ is an essential extension of (S, G) if and only if the extension $((C(T), 1), \varphi^\circ)$ of the Banach G -module $(C(S), 1)$ is essential.*

PROOF. (i) Let (Y, κ) be an injective envelope of the Banach G -module $X = (C(S), 1)$ (Theorem 1). We construct the projective cover $(P, G; \varphi)$ from this (Y, κ) . As noted above, $Y = C(T)$ as a Banach space, where T is a stonean space. Define the dual action of G on Y^* by

$$\langle y, y^* \cdot g \rangle = \langle g \cdot y, y^* \rangle , \quad y \in Y, y^* \in Y^*, g \in G .$$

Then the set $E = \{y^* \in Y^*: \langle \kappa(1), y^* \rangle = 1, \|y^*\| \leq 1\}$ is weak* compact convex and G -invariant, where $1 \in X$ (resp. $\kappa(1) \in Y$) is the constant function on S (resp. T). We have by the Hahn-Banach theorem $\kappa^*(E) = E(C(S))$. Lemma 2 and essentiality of the extension (Y, κ) imply that the weak* closed convex circled hull of $E = B_{Y^*}$ (the unit ball of Y^*) and (*) for each weak* closed convex G -invariant subset $F \subseteq E, \kappa^*(F) \subseteq E(C(S))$. In fact, suppose that $F \subset E$ is weak* closed convex G -invariant and $\kappa^*(F) = E(C(S))$. Then the seminorm p on Y defined by $p(y) = \sup\{\langle y, y^* \rangle : y^* \in F\}$ is admissible, so (Lemma 2) $p(y) = \|y\|$ for all $y \in Y$, i.e., the weak* closed convex circled hull of $F = B_{Y^*}$. (In particular the weak* closed convex circled hull of $E = B_{Y^*}$.) From this and the Krein-Milman theorem, we have $\{\lambda y^* : \lambda \in T, y^* \in F\} \supset \{\lambda \delta_t : \lambda \in T, t \in T\}$. Hence for each $t \in T$, $\delta_t = \lambda y^*$ for some $\lambda \in T$ and $y^* \in F$, so $\kappa(1)(t) = \langle \kappa(1), \lambda y^* \rangle = \lambda \in T$. Thus $\kappa(1)$ is a unitary element of the commutative C^* -algebra $Y = C(T)$ and $\kappa(1)^{-1} \cdot \delta_t = \langle \cdot, \kappa(1)^{-1} \cdot \delta_t \rangle = \kappa(1)(t)^{-1} \delta_t \in F$ for all $t \in T$. The former assertion implies that $E = \kappa(1)^{-1} \cdot E(C(T))$ and the latter implies that $\kappa(1)^{-1} \cdot E(C(T)) \subset F$; hence $F = E$.

It follows from (*) and the Krein-Milman theorem that $\kappa^*(\overline{\text{ext } E}) = \overline{\text{ext } E(C(S))} = \{\delta_s : s \in S\}$, and from $E = \kappa(1)^{-1} \cdot E(C(T))$ that $\text{ext } E = \kappa(1)^{-1} \cdot \text{ext } E(C(T)) = \kappa(1)^{-1} \cdot \{\delta_t : t \in T\}$, where $\text{ext } \{ \}$ (resp. $\overline{\{ \}}$) means the set of extreme points (resp. weak* closure) of $\{ \}$. Thus $\text{ext } E$ is weak* closed and G -invariant; so putting $P = \text{ext } E$ and $\varphi = \kappa^*|_P : P \rightarrow \{\delta_s : s \in S\}$ and identifying $\{\delta_s : s \in S\}$ with S , we get an extension $(P, G; \varphi)$ of the flow (S, G) . Then it is immediate to see that $Y = (C(P), 1)$ and $\kappa = \varphi^\circ$. Moreover, essentiality of the extension $(P, G; \varphi)$ follows from (*) if we note that $E = E(C(P))$.

We show that (P, G) is projective, i.e., it has no proper essential extension. In fact let $(Q, G; \chi)$ be an essential extension of (P, G) . This induces an extension of Banach G -modules

$$\chi^\circ : (C(P), 1) \rightarrow (C(Q), 1).$$

Since $(C(P), 1) = Y$ is injective, there exists a contractive module homomorphism

$$\sigma : (C(P), 1) \leftarrow (C(Q), 1)$$

such that $\sigma \chi^\circ = \text{id}_{(C(P), 1)}$. Hence $\tilde{\chi}(\sigma^*(E(C(P)))) = E(C(P))$ and $\sigma^*(E(C(P))) \subset E(C(Q))$ is weak* closed convex G -invariant. Thus by essentiality of the extension $(Q, G; \chi)$, $\sigma^*(E(C(P))) = E(C(Q))$, so it follows that χ is an isomorphism; i.e., (P, G) is projective.

Uniqueness of a projective cover of (S, G) will be shown after the proofs of (ii) and (iii).

Sufficiencies of (ii) and (iii) follow from the argument in the proof of (i) (with (Y, κ) replaced by $((C(T), 1), \varphi^\circ)$ for the case (iii)).

Necessity of (ii): For a projective flow (S, G) let (P, G) and φ be taken as in the proof of (i). Then by essentiality of $(P, G; \varphi)$, φ is an isomorphism. So $(C(S), 1)$, which is isomorphic to its injective envelope, is injective.

Necessity of (iii): Let p be an admissible seminorm on $(C(T), 1)$ relative to $(C(S), 1)$. Then $K = \{\mu \in E(C(T)): |\mu(y)| \leq p(y), y \in C(T)\}$ is weak* closed convex G -invariant, and by the Hahn-Banach theorem $\tilde{\psi}(K) = E(C(S))$; hence essentiality of the extension $(T, G; \psi)$ implies that $K = E(C(T))$. Thus $p(y) = \|y\|$ for all $y \in Y$, which shows by Lemma 2 essentiality of the extension $((C(T), 1), \psi^\circ)$.

Uniqueness of the projective cover of (S, G) : Let (P, G) be as in (i) and let $(Q, G; \chi)$ be another projective cover of (S, G) . Then (ii) and (iii) show that the extension $((C(Q), 1), \chi^\circ)$ is an injective envelope of $(C(S), 1)$, so by uniqueness of the injective envelope of a Banach module, there exists an isomorphism κ of $(C(P), 1)$ onto $(C(Q), 1)$ such that $\kappa\varphi^\circ = \chi^\circ$, which induces an isomorphism ι of (Q, G) onto (P, G) such that $\varphi = \iota\chi$. q.e.d.

4. The projective cover of the trivial flow. Let G be, as in § 3, a discrete group. By a trivial flow we mean a flow $(\{s_0\}, G)$, consisting of a one point set $\{s_0\}$ and the group G , with the trivial action: $s_0 \cdot g = s_0, g \in G$. In this section we examine the projective cover of the trivial flow in some detail.

The Banach G -module X associated with the flow $(\{s_0\}, G)$ and the trivial 1 cocycle 1 is no other than the scalar field C with the trivial action of G , and it is embedded in the injective Banach G -module $l^\infty(G)$ as constant functions on G . If we note that a splitting of this embedding (i.e., a contractive linear map $\sigma: l^\infty(G) \rightarrow X$ with $\sigma(g \cdot y) = g \cdot \sigma(y) = \sigma(y)$ and $\sigma(1) = 1, g \in G, y \in l^\infty(G)$) corresponds bijectively to a (right) invariant mean on $l^\infty(G)$, we see that injectivity of X , hence projectivity of the trivial flow, is equivalent to amenability of G . So we are interested in the case where G is non-amenable.

Let (P_0, G) be the projective cover of $(\{s_0\}, G)$. (The homomorphism of (P_0, G) onto $(\{s_0\}, G)$, being trivial, is omitted.) Identifying X with the set of constant functions on G , we embed the injective envelope $(C(P_0), 1)$ of X in $l^\infty(G)$ (Lemma 3 (i)). Let $(E(l^\infty(G)), G)$ be the flow with the action of G defined as in § 3, where $E(l^\infty(G)) = \{f \in l^\infty(G): f(1) = 1, \|f\| = 1\}$. If K is a minimal weak* closed convex G -invariant subset of $E(l^\infty(G))$, then the argument similar to the one in the proof of Lemma

5 shows that $\text{ext } K$ (the set of extreme points of K), with the action of G induced on it, becomes a subflow of $(E(l^\infty(G)), G)$ (i.e., $\text{ext } K$ is weak* closed and G -invariant) which is isomorphic to (P_0, G) .

DEFINITION. We call such a set K a minimal subset of $E(l^\infty(G))$. An element of some minimal subset is called a minimal state on $l^\infty(G)$.

For $f \in E(l^\infty(G))$ define a map $\varphi_f: l^\infty(G) \rightarrow l^\infty(G)$ by

$$\varphi_f(x) = \langle \langle g \cdot x, f \rangle \rangle_{g \in G}, \quad x \in l^\infty(G).$$

The map φ_f is a contractive module homomorphism with $\varphi_f(1) = 1$ and it is clear that conversely, a contractive module homomorphism $\varphi: l^\infty(G) \rightarrow l^\infty(G)$ with $\varphi(1) = 1$ is of the form φ_f for some $f \in E(l^\infty(G))$.

DEFINITION. A map $\varphi: l^\infty(G) \rightarrow l^\infty(G)$ is called a minimal projection on $l^\infty(G)$ if it is a contractive idempotent module homomorphism with $\varphi(1) = 1$ and $(\text{Im } \varphi, \kappa)$ is an injective envelope of X , where κ is the embedding of X into $l^\infty(G)$ as constant functions on G . A closed submodule Y of $l^\infty(G)$ is called a minimal injective submodule if $Y = \text{Im } \varphi$ for some minimal projection φ .

It is immediately seen that the map $f \mapsto \varphi_f$ is a one-to-one map of the set of all minimal states onto the set of all minimal projections and that for a fixed minimal subset K , the map $f \mapsto \text{Im } \varphi_f$ is a one-to-one map of K onto the set of all minimal injective submodules of $l^\infty(G)$. Moreover, using the isomorphism between the flows (P_0, G) and $(\text{ext } K, G)$ (K , a minimal subset), it is shown that the subgroup

$$G_0 = \{g \in G: p \cdot g = p \text{ for all } p \in P_0\}$$

is the largest amenable normal subgroup of G .

Summing up the above assertions, we get the following

PROPOSITION 1. (i) *The subgroup $G_0 = \{g \in G: p \cdot g = p \text{ for all } p \in P_0\}$ is the largest amenable normal subgroup of G .*

(ii) *If K is a minimal subset of $E(l^\infty(G))$, $(\text{ext } K, G)$ is a subflow of $(E(l^\infty(G)), G)$ which is isomorphic to (P_0, G) . Hence the flow (K, G) is isomorphic to the flow $(E(C(P_0)), G)$.*

(iii) *The map $f \mapsto \varphi_f$ is a one-to-one map of the set of all minimal states of $l^\infty(G)$ onto the set of all minimal projections on $l^\infty(G)$.*

(iv) *For a fixed minimal subset K of $E(l^\infty(G))$, the map $f \mapsto \text{Im } \varphi_f$ is a one-to-one map of K onto the set of all minimal injective submodules of $l^\infty(G)$. Moreover, $\text{Im } \varphi_f$ is a C^* -subalgebra of the commutative C^* -algebra $l^\infty(G)$ if and only if $f \in \text{ext } K$.*

PROOF. We prove only (iii) and the second part of (iv). The other

statements will be easily checked.

(iii) We have only to show idempotentness of φ_f for a minimal state f . To see this, we use a technique of R. Kaufman [6; the proof of Theorem 1]. Minimality of f implies that the seminorm p on $l^\infty(G)$ defined by $p(y) = \|\varphi_f(y)\| = \sup_{g \in G} |\langle g \cdot y, f \rangle|$ is a minimal admissible seminorm relative to X . Since the seminorm p_1 on $l^\infty(G)$ defined by

$$p_1(y) = \limsup_{n \rightarrow \infty} \|(\varphi_f + \varphi_f^2 + \dots + \varphi_f^n)(y)/n\|$$

is also an admissible seminorm relative to X with $p_1 \leq p$, we have $p_1 = p$; hence for each $y \in l^\infty(G)$,

$$\begin{aligned} \|\varphi_f(y) - \varphi_f^2(y)\| &= p(y - \varphi_f(y)) = p_1(y - \varphi_f(y)) \\ &= \limsup_{n \rightarrow \infty} \|(\varphi_f(y) - \varphi_f^{n+1}(y))/n\| = 0, \end{aligned}$$

i.e., $\varphi_f^2 = \varphi_f$.

The second part of (iv): Let K be a minimal subset of $E(l^\infty(G))$ and $f \in K$. Put $\Omega = \text{ext } K$ and $\Omega_1 =$ the weak* closure of $\{f \cdot g : g \in G\}$. By minimality of K , we have $\Omega \subset \Omega_1$. Regarding Ω (resp. Ω_1) as a flow, we get an isometric module isomorphism (resp. homomorphism)

$$\begin{aligned} \kappa: \text{Im } \varphi_f &\rightarrow C(\Omega), \kappa(y)(\omega) = \langle y, \omega \rangle, y \in \text{Im } \varphi_f, \omega \in \Omega \\ (\text{resp. } \kappa_1: \text{Im } \varphi_f &\rightarrow C(\Omega_1), \kappa_1(y)(\omega) = \langle y, \omega \rangle, y \in \text{Im } \varphi_f, \omega \in \Omega_1). \end{aligned}$$

Let A be the C^* -subalgebra of $l^\infty(G)$ generated by $\text{Im } \varphi_f$. Then κ_1 is extended to an algebra-isomorphism $\hat{\kappa}_1$ of A onto $C(\Omega_1)$. Therefore, since $\hat{\kappa}_1 \kappa_1^{-1}: C(\Omega) \rightarrow \text{Im } \varphi_f \subset A \rightarrow C(\Omega_1)$ is a map such that $(\hat{\kappa}_1 \kappa_1^{-1})(y)(\omega) = y(\omega)$ for $y \in C(\Omega)$ and $\omega \in \Omega \subset \Omega_1$, $\text{Im } \varphi_f$ is a C^* -subalgebra of $l^\infty(G)$, i.e., $\text{Im } \varphi_f = A$ if and only if $\Omega = \Omega_1$, i.e., $f \in \Omega = \text{ext } K$. q.e.d.

REMARK 4. It is noted that the flow (P_0, G) coincides with the ‘universal minimal strongly proximal flow’ for the discrete group G defined by S. Glasner [2; Chapter III]. Although he defined the flow for any topological group and proved its uniqueness up to a flow-isomorphism, some of the above results seem to be new even for a discrete group.

5. Self-injective C^* -algebras. We say that a unital C^* -algebra is self-injective if it is injective as a left Banach module over itself. M. Takesaki [10] showed that a commutative AW^* -algebra is self-injective in this sense. (Although only two-sided Banach modules were treated in [10], the reasoning in it can be applied also to the one-sided case.) We see in the following that the converse is true, i.e., a self-injective C^* -algebra is a commutative AW^* -algebra. These results seem to show that for a C^* -algebra A , the category of Banach A -modules and contrac-

tive module homomorphisms is too large.

PROPOSITION 2. *A self-injective C^* -algebra A is a commutative AW^* -algebra.*

PROOF. We first show that A is commutative. Suppose the contrary, i.e., A , hence the enveloping von Neumann algebra A^{**} of A , is not commutative. Then there exist minimal projections e_1, e_2 in A^{**} such that

$$e_1e_2 = 0, \quad e_1 = u^*u \quad \text{and} \quad e_2 = uu^* \quad \text{for some } u \text{ in } A^{**}.$$

By the transitivity theorem, there exist a_1, a_2 in A such that

$$\begin{aligned} \|a_1\| = \|a_2\| = 1, \quad a_1e_1 = e_1, \quad a_1e_2 = 0, \\ a_2e_1 = 0, \quad a_2e_2 = u^* \quad \text{and} \quad a_1a_2^* = 0. \end{aligned}$$

Let $X = (\sum^\oplus Ap)_\infty$ be the l^∞ -sum of the Banach A -modules $Ap = \{ap : a \in A\}$, where p runs through the set of all minimal projections in A^{**} and the module operation in X is defined by componentwise left multiplication in A^{**} . The map

$$\kappa : A \rightarrow X, \quad \kappa(a) = (ap), \quad a \in A$$

is an isometric module homomorphism. Since A is self-injective, we have a splitting $\sigma : A \leftarrow X$, i.e., a contractive module homomorphism such that $\sigma \circ \kappa = \text{id}_A$. Let f_1 be the pure state of A with support $f_1 = e_1$. For each x in X , we have

$$\begin{aligned} \|x\|^2 &\geq \|e_1\sigma(x)e_1 + e_1\sigma(x)(1 - e_1)\|^2 \\ &= \|e_1\sigma(x)e_1\sigma(x)^*e_1 + e_1\sigma(x)(1 - e_1)\sigma(x)^*e_1\| \\ &\geq \| |e_1\sigma(x)e_1|^2 e_1 + e_1\sigma(x)(ue_1)(ue_1)^*\sigma(x)^*e_1 \| \\ &= \| |f_1(\sigma(x))|^2 e_1 + |f_1(\sigma(x)u)|^2 e_1 \| \\ &= |f_1(\sigma(x))|^2 + |f_1(\sigma(x)u)|^2. \quad (*) \end{aligned}$$

Given $0 < \epsilon \leq 1 - 1/\sqrt{2}$, define $x_i = ((x_i)_p)$ in X by $(x_i)_p = a_i p$ if $\|a_i p\| > 1 - \epsilon$, $= 0$ otherwise ($i = 1, 2$). Then we have $\|x_1 + e^{i\theta}x_2\| = 1$ for each θ in $[0, 2\pi]$, so that

$$\begin{aligned} f_1(\sigma(x_1)) &= 1, \quad f_1(\sigma(x_1)u) = 0, \\ f_1(\sigma(x_2)) &= 0 \quad \text{and} \quad f_1(\sigma(x_2)u) = 1. \end{aligned}$$

For we have

$$\begin{aligned} 1 &= f_1(a_1) = f_1(\sigma \circ \kappa(a_1)) \\ &= f_1(\sigma(x_1)) + f_1(\sigma(\kappa(a_1) - x_1)) \\ &= f_1(\sigma(x_1)) + \gamma, \quad \text{say,} \end{aligned}$$

and $|f_1(\sigma(x_1))| \leq 1$. Since $\|\kappa(a_1) - x_1\| \leq 1 - \varepsilon$, by the construction of x_1 , we have

$$\|x_1 + (\kappa(a_1) - x_1)/(1 - \varepsilon)\| \leq 1,$$

so

$$1 \geq |f_1(\sigma(x_1 + (\kappa(a_1) - x_1)/(1 - \varepsilon)))| = |f_1(\sigma(x_1)) + \gamma/(1 - \varepsilon)|.$$

Hence $\gamma = 0$, $f_1(\sigma(x_1)) = 1$. On the other hand, the inequality

$$1 \geq |f_1(\sigma(x_1 + e^{i\theta}x_2))| = |1 + e^{i\theta}f_1(\sigma(x_2))|$$

($\theta \in [0, 2\pi]$) implies that $f_1(\sigma(x_2)) = 0$. The other equalities follow similarly. Thus (*) yields $1 = \|x_1 + x_2\|^2 \geq |f_1(\sigma(x_1 + x_2))|^2 + |f_1(\sigma(x_1 + x_2)u)|^2 = 2$, a contradiction.

Let Ω be the spectrum of the commutative C^* -algebra A . Then the Banach A -module X constructed above coincides with the P_1 space $l^\infty(\Omega)$, so that A is also a P_1 space. Hence it follows from §1, 1° that Ω is stonean and consequently that A is a commutative AW^* -algebra. q.e.d.

REFERENCES

- [1] H. B. COHEN, Injective envelopes of Banach spaces, *Bull. Amer. Math. Soc.*, 70 (1964), 723-726.
- [2] S. GLASNER, Proximal flows, *Lecture Notes in Math.* 517, Springer-Verlag, Berlin, 1976.
- [3] A. M. GLEASON, Projective topological spaces, *Illinois J. Math.*, 2 (1958), 482-489.
- [4] D. B. GOODNER, Projections in normed linear spaces, *Trans. Amer. Math. Soc.*, 35 (1950), 89-108.
- [5] M. HASUMI, The extension property of complex Banach spaces, *Tôhoku Math. J.*, 10 (1958), 135-142.
- [6] R. KAUFMAN, A type of extension of Banach spaces, *Acta Sci. Math. (Szeged)*, 27 (1966), 163-166.
- [7] J. L. KELLEY, Banach spaces with the extension property, *Trans. Amer. Math. Soc.*, 72 (1952), 323-326.
- [8] S. MACLANE, *Homology*, Springer-Verlag, Berlin, 1967.
- [9] L. NACHBIN, A theorem of the Hahn-Banach type for linear transformations, *Trans. Amer. Math. Soc.*, 68 (1950), 28-46.
- [10] M. TAKESAKI, On the Hahn-Banach type theorem and the Jordan decomposition of module linear mapping over some operator algebras, *Kôdai Math. Sem. Rep.*, 12 (1960), 1-10.
- [11] J. TOMIYAMA, On the projection of norm one in W^* -algebras, *Proc. Japan Acad.*, 33 (1957), 608-612.

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