Tôhoku Math. Journ.
30 (1978), 423-438.

ON INTERTWINING DILATIONS. IV

GRIGORE ARSENE AND ZOIA CEAUȘESCU

(Received March 8, 1977, revised September 27, 1977)

Abstract. We give a generalization of the theorems of the existence (see [9]) and the uniqueness (see [3]) of the contractive intertwining dilations in the presence of some representations of a C^*-algebra.

1. Let H_j $(j=1,2)$ be some (complex) Hilbert spaces and let $\mathscr{L}(H_1, H_2)$ denote the set of all (linear bounded) operators from H_1 into H_2. For a Hilbert space H, $\mathscr{L}(H)$ will stand for $\mathscr{L}(H, H)$. If $T \in \mathscr{L}(H_1, H_2)$ is a contraction, then we denote $D_T = (I - T^*T)^{1/2}$ and $\mathscr{D}_T = D_T(H_1)^-$. For a contraction $T \in \mathscr{L}(H)$, $U \in \mathscr{L}(K)$ will be the minimal isometric dilation of T; in other words:

$$K = H \oplus \mathscr{D}_T \oplus \mathscr{D}_T \oplus \cdots$$

and

$$U = \begin{pmatrix} T & 0 & 0 & \cdots \\ D_T & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ 0 & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(For this and for any fact connected with the geometry of isometric dilations of contractions see [9], ch. I and II).

If $T_j \in \mathscr{L}(H_j)$ $(j=1,2)$ are two contractions, $I(T_1, T_2)$ will be the set of all operators $A \in \mathscr{L}(H_2, H_1)$ such that $T_1 A = A T_2$. Let $U_j \in \mathscr{L}(K_j)$ be the minimal isometric dilation of T_j and P_j the (orthogonal) projection of K_j onto H_j $(j=1,2)$. For a contraction $A \in I(T_1, T_2)$, a *contractive intertwining dilation* ((T_1, T_2)-CID) of A will be a contraction $B \in I(U_1, U_2)$, such that $P_1 B = A P_2$.

The existence of a (T_1, T_2)-CID for every contraction of $I(T_1, T_2)$ was proved by B. Sz.-Nagy and C. Foiaș in 1968 (see [9], ch. II, th. 2.3); recently T. Ando, Z. Ceaușescu and C. Foiaș proved in [3] that the uniqueness of the (T_1, T_2)-CID is equivalent to the fact that one of the factorizations $T_1 \cdot A$ or $A \cdot T_2$ be regular (in the sense of [9], ch. VII, §3). A generalization of this criterion is used in [6] for the uniqueness problem of the lifting of operators which commute with shifts (see [4] for the existence problem). In [5] it is given a generalization of the existence theorem of [4] for isometries (instead of shifts); the uniqueness in this case asks for a uniqueness theorem of lifting involving representations of C^*-algebras.

In this note we formulate such a theorem (section 2 below) and use it for a generalization of the uniqueness criterion of [3], in the presence of representations of a C^*-algebra.

We take this oppertunity to express our gratitude to Professor C. Foiaş for posing the problem and for helpful discussions concerning this matter. We also thank D. Voiculescu for discussions concerning Theorem 2.1.

In the sequel let $\mathfrak{a}$ be a C^*-algebra and $\rho: \mathfrak{a} \mapsto \mathscr{L}(H)$ a representation of $\mathfrak{a}$. We use the terminology of [7] concerning representations of C^*-algebras. So, for any set $\mathfrak{M} \subset \mathscr{L}(H)$, $\mathfrak{M}'$ will be the commutant of $\mathfrak{M}$ and for a projection $P = P_{H_0} \in [\rho(\mathfrak{a})]'$ we denote by ρ_P (or ρ_{H_0}) the subrepresentation of ρ given by P. If $\rho_j: \mathfrak{a} \mapsto \mathscr{L}(H_j)$ $(j = 1, 2)$ are representations of $\mathfrak{a}$, we denote by $I(\rho_1, \rho_2)$ the set of operators $A: H_2 \mapsto H_1$ such that $A \in I(\rho_1(x), \rho_2(x))$, for every $x \in \mathfrak{a}$; ρ_1 and ρ_2 are disjoint ($\rho_1 \mathbin{\delta} \rho_2$) if $I(\rho_1, \rho_2) = \{0\}$. We use without quotations the properties of $I(\rho_1, \rho_2)$, of disjointness or of equivalence of representations as they are presented in [7], §§2 and 5.

The typical situation in this note is the following: for $j = 1, 2$, $\rho_j: \mathfrak{a} \mapsto \mathscr{L}(H_j)$ are representations of $\mathfrak{a}$ and $T_j \in [\rho_j(\mathfrak{a})]'$ are contractions. Note that $P_{\mathscr{D}_{T_j}} \in [\rho_j(\mathfrak{a})]'$ $(j = 1, 2)$; we consider for every $n = 1, 2, \cdots, \infty$ the representation

$$\rho_j^{(n)} = \rho_j \oplus \left(\bigoplus_{i=1}^{n} \tau_{ji} \right) \qquad (j = 1, 2)\,, \tag{1.1}$$

where $\tau_{ji} = (\rho_j)_{\mathscr{D}_{T_j}}$ for every i.

An easy computation proves that

$$U_j \in [\rho_j^{(\infty)}(\mathfrak{a})]' \qquad (j = 1, 2)\,, \tag{1.2}$$

where $U_j \in \mathscr{L}(K_j)$ is the minimal isometric dilation of T_j. Let $A \in I(T_1, T_2)$ be a contraction such that $A \in I(\rho_1, \rho_2)$.

DEFINITION 1.1. If B is a (T_1, T_2)-CID for A, we say that B is a $(\rho_1, \rho_2; T_1, T_2)$-CID for A if $B \in I(\rho_1^{(\infty)}, \rho_2^{(\infty)})$.

Recall that for $S \in \mathscr{L}(H_1, H_2)$ and $R \in \mathscr{L}(H_2, H_3)$, the product $R \cdot S$ is called a regular factorization of RS, if $\mathscr{R}(R \cdot S) = \{0\}$, where

$$\mathscr{R}(R \cdot S) = \mathscr{D}_R \oplus \mathscr{D}_S \ominus \{D_R S h_1 \oplus D_S h_1 : h_1 \in H_1\}^- \,. \tag{1.3}$$

With our notations we infer that

$$P_{\mathscr{R}(T_1\cdot A)} \in [(\rho_1 \oplus \rho_2)(\mathfrak{a})]' \quad \text{and} \quad P_{\mathscr{R}(A\cdot T_2)} \in [\rho_2^{(1)}(\mathfrak{a})]' . \tag{1.4}$$

By (1.4) the following definition makes sense:

DEFINITION 1.2. With previous notations, we say that A is $(\rho_1, \rho_2; T_1, T_2)$-*regular* if

$$(\rho_1 \oplus \rho_2)_{\mathscr{R}(T_1\cdot A)} \,\mathring{\smile}\, (\rho_2^{(1)})_{\mathscr{R}(A\cdot T_2)} . \tag{1.5}$$

REMARK 1.1. If the representations ρ_1 and ρ_2 are non-disjoint and factorial, the condition (1.5) is equivalent to the condition that one of the factorizations $T_1\cdot A$ or $A\cdot T_2$ be regular.

The main result of this note is the following:

THEOREM 1.1. *Let $\mathfrak{a}$ be a C^*-algebra, $\rho_j: \mathfrak{a} \mapsto \mathscr{L}(H_j)$ a representation of $\mathfrak{a}$, $T_j \in [\rho_j(\mathfrak{a})]'$ a contraction $(j = 1, 2)$ and $A \in I(T_1, T_2) \cap I(\rho_1, \rho_2)$ a contraction. Then:*

(1) *A has always a $(\rho_1, \rho_2; T_1, T_2)$-*CID.

(2) *A has a unique $(\rho_1, \rho_2; T_1, T_2)$-*CID *iff A is $(\rho_1, \rho_2; T_1, T_2)$-regular.*

In the last section we give an application concerning a recent result of T. Ando [2].

2. In this section we analyze the following situation: $\mathfrak{a}$ is a C^*-algebra, $\rho_j: \mathfrak{a} \mapsto \mathscr{L}(H_j)$ $(j = 1, 2)$ are representations of $\mathfrak{a}$, $H_0 \subset H_1$ is an invariant subspace for ρ_1 and $P = P_{H_0}$. Then $P \in [\rho_1(\mathfrak{a})]'$. Let also $T_0 \in I(\rho_2, (\rho_1)_P)$ be a contraction.

DEFINITION 2.1. A contraction $T \in I(\rho_2, \rho_1)$ is called a *contractive intertwining lifting* of T_0 (shortly a CIL for T_0) if $T|_{H_0} = T_0$.

Note that T_0P is always a CIL for T_0. Since $T_0 \in I(\rho_2, (\rho_1)_P)$, we infer that $P_{\mathscr{D}_{T_0^*}} \in [\rho_2(\mathfrak{a})]'$. Let $1 - E$ be the central support of $1 - P$ (in $[\rho_1(\mathfrak{a})]'$) and $1 - F$ be the central support of $P_{\mathscr{D}_{T_0^*}}$ (in $[\rho_2(\mathfrak{a})]'$).

THEOREM 2.1. *The following conditions are equivalent:*

(I) *T_0 has a unique* CIL.

(II) $(\rho_1)_{1-P} \,\mathring{\smile}\, (\rho_2)_{\mathscr{D}_{T_0^*}}$.

(III) a) $(\rho_1)_{1-P} \,\mathring{\smile}\, (\rho_2)_{\ker T_0^*}$.

b) *T_0 is a partial isometry* on $(1 - E)P(H_1)$.

(IV) a) $(\rho_1)_{1-P} \,\mathring{\smile}\, (\rho_2)_{\ker T_0^*}$.

b) $T_0(\mathscr{D}_{T_0}) \subset T_0E(H_1)$.

Moreover if R is the projection on $T_0(1 - E)P(H_1)^-$, then R is central in $[\rho_2(\mathfrak{a})]'$.

PROOF. The theorem is trivial when $H_0 = H_1$. Let us suppose that $H_0 \neq H_1$.

(I) $\Rightarrow$ (II). Let us suppose that T_0 has an unique CIL and though there exists $Y \in I((\rho_2)_{\mathscr{D}_{T_0^*}}, (\rho_1)_{1-P})$, $Y \neq 0$. We can choose Y such that $\|Y\| \leqq 1$. Define $S: H_1 \mapsto H_2$ by

$$S = T_0 P + D_{T_0^*} Y(1 - P) . \tag{2.1}$$

From (2.1) it is clear that

$$S|_{H_0} = T_0 . \tag{2.2}$$

Because $D_{T_0^*}$ is a positive selfadjoint operator and Y takes values in $\mathscr{D}_{T_0^*}$, we have that $D_{T_0^*}Y \neq 0$. So:

$$S \neq T_0 P . \tag{2.3}$$

From $T_0 \in I(\rho_2, (\rho_1)_P)$ we infer that $D_{T_0^*} \in I(\rho_2, \rho_2)$, whence $D_{T_0^*} Y \in I((\rho_2)_{\mathscr{D}_{T_0^*}}, (\rho_1)_{1-P})$. Using (2.1) we obtain:

$$S \in I(\rho_2, \rho_1) . \tag{2.4}$$

We have:

$$SS^* = T_0 T_0^* + D_{T_0^*}(1 - P) Y^* D_{T_0^*} \leqq T_0 T_0^* + D_{T_0^*}^2 = 1 , \quad \text{so}$$

$$\|S\| = \|SS^*\|^{1/2} \leqq 1 . \tag{2.5}$$

The relations (2.2), (2.4) and (2.5) prove that S is a CIL for T_0; the relation (2.3) contradicts the uniqueness of a CIL for T_0.

(II) $\Rightarrow$ (I). Let T be a CIL for T_0; with respect to the decompositions:

$$H_1 = H_0 \oplus (H_1 \ominus H_0) \quad \text{and} \quad H_2 = T_0(H_0)^- \oplus \ker T_0^*,$$

T is the matrix

$$\begin{pmatrix} T_0 & T_1 \\ 0 & T_2 \end{pmatrix} ,$$

where $T_1 \in I((\rho_2)_{T_0(H_0)^-}, (\rho_1)_{1-P})$ and $T_2 \in I((\rho_2)_{\ker T_0^*}, (\rho_1)_{1-P})$. Using the hypothesis and the fact that $\ker T_0^* \subset \mathscr{D}_{T_0^*}$ we have that $T_2 = 0$. Now, let us denote:

$$H_{10} = T_0(H_0)^- \ominus T_0(\mathscr{D}_{T_0})^-; \; H_{11} = T_0(\mathscr{D}_{T_0})^- .$$

From this, $T_1 = T_{10} + T_{11}$, where $T_{10} \in I((\rho_2)_{H_{10}}, (\rho_1)_{1-P})$ and $T_{11} \in I((\rho_2)_{H_{11}}, (\rho_1)_{1-P})$.

We have (see [9], ch. I, section 3):

$$\mathscr{D}_{T_0^*} = T_0(\mathscr{D}_{T_0})^- \oplus \ker T_0^* = T_0(\mathscr{D}_{T_0})^- \oplus (H_2 \ominus T_0(H_0)^-) . \tag{2.6}$$

Using (2.6) we infer that:

$$H_{11} \subset \mathscr{D}_{T_0^*} \quad \text{and} \quad H_{10} \subset H_2 \ominus \mathscr{D}_{T_0^*} = \ker D_{T_0^*} = \{h_2 \in H_2 : \| T_0^* h_2 \| = \| h_2 \| \} .$$

So, by hypothesis, $T_{11} = 0$. Let $h_2 \in \ker D_{T_0^*}$; we have

$$\| T^* h_2 \|^2 = \| (T_0^* + T_{10}^*) h_2 \|^2 = \| T_0^* h_2 \|^2 + \| T_{10}^* h_2 \|^2 = \| h_2 \|^2 + \| T_{10}^* h_2 \|^2 .$$

But $\| T \| \leqq 1$, so $T_{10} = 0$. This proves that $T = T_0 P$, thus T_0 has a unique CIL.

(II) $\Rightarrow$ (III). The condition (a) follows from (2.6) and the hypothesis. We infer also that

$$(2.7) \qquad (\rho_1)_{1-E} \between (\rho_2)_{1-F} .$$

Let us denote by $\hat{T}_0$ the operator T_0 from $(1 - E)P(H_1)$ into $R(H_2) = T_0(1 - E)P(H_1)^-$, where $R \in [\rho_2(\mathfrak{a})]'$ is a projection. We have

$$(2.8) \qquad \hat{T}_0 \in I((\rho_2)_R , \quad (\rho_1)_{P-E}) .$$

But $(1 - E)P = P - E \leqq 1 - E$ and (2.7) implies that $R \leqq F$, which means that $R(H_2) \subset \ker D_{T_0^*}$. Because $\hat{T}_0^* = T_0^* |_{R(H_2)}$, this proves that $\hat{T}_0$ is a co-isometry.

Moreover, R is central in $[\rho_2(\mathfrak{a})]'$. Indeed, let R_1 be the central support of R in $[\rho_2(\mathfrak{a})]'$. Because $R \leqq F$ and F is central in $[\rho_2(\mathfrak{a})]'$, $R_1 \leqq F$. On the other hand, $\hat{T}_0$ is with dense range, so $(\rho_2)_R$ is equivalent to a subrepresentation of $(\rho_1)_{1-E}$ (see (2.3)). But $(\rho_1)_E \between (\rho_1)_{1-E}$, so $(\rho_1)_E \between (\rho_2)_R$ and then $(\rho_1)_E \between (\rho_2)_{R_1}$. This implies that

$$T_0 E(H_1) \subset (1 - R_1)(H_2) ,$$

so

$$(2.9) \qquad T_0^* R_1(H_2) \subset (1 - E)P(H_1) .$$

But $R_1 \leqq F$ implies that:

$$(2.10) \qquad T_0 T_0^* h_2 = h_2, \quad \text{for every} \quad h_2 \in R_1(H_2) .$$

Using (2.10) in (2.9), we obtain that

$$R_1(H_2) = T_0 T_0^* R_1(H_2) \subset T_0(1 - E)P(H_1)^- = R(H_2)$$

which means $R_1 = R$.

(III) $\Rightarrow$ (IV). We must prove (IV)b. We have:

$$T_0 D_{T_0} |_{(1-E)P(H_1)} = D_{T_0^*} T_0 |_{(1-E)P(H_1)} = 0 ,$$

using that T_0 is a partial isometry on $(1 - E)P(H_1)$.
Thus:

$$T_0D_{T_0}(PH_1) \subset T_0D_{T_0}(E(H_1)) = T_0ED_{T_0}(EH_1) \subset T_0E(H_1) .$$

(IV) $\Rightarrow$ (II). From $T_0(\mathscr{D}_{T_0}) \subset T_0E(H_1)$ it follows that

$$(\rho_1)_{1-P} \mathrel{\delta} (\rho_2)_{T_0(\mathscr{D}_{T_0})^-} . \tag{2.11}$$

The conclusion results from (2.6), (2.11) and (IV)a. The theorem is completely proved.

REMARK 2.1. The conditions (IIIb) and (IVb) of Theorem 2.1 can be replaced respectively by:

(IIIb′) *There exists a central projection* $E_1 \in [\rho_1(\mathfrak{a})]'$ *such that* $E_1 \leqq P$ *and* T_0 *is a partial isometry on* $(1 - E_1)P(H_1)$.

(IVb′) *There exists a central projection* $E_1 \in [\rho_1(\mathfrak{a})]'$ *such that* $E_1 \leqq P$ *and* $T_0(\mathscr{D}_{T_0}) \subset T_0E_1(H_1)$.

COROLLARY 2.1. *With the notations of Theorem 2.1, if* ρ_1 *and* ρ_2 *are non-disjoint factorial representations of* $\mathfrak{a}$, *then* T_0 *has a unique* CIL *iff* $H_0 = H_1$ *or* T_0 *is a co-isometry.*

PROOF. Two non-disjoint factorial representations of $\mathfrak{a}$ are equivalent, so two subrepresentations of them are disjoint iff one of the subrepresentations is trivial. The corollary follows now from the condition (II) of Theorem 2.1.

COROLLARY 2.2. *Let* $T \in \mathscr{L}(H_1, H_2)$ *be a contraction,* $H_0 \subset H_1$ *a subspace of* H_1 *and* $T_0 = T|_{H_0}$. *The following conditions are equivalent*:

(1) *If* $S \in \mathscr{L}(H_1, H_2)$ *is a contraction such that* $S|_{H_0} = T_0$, *then* $S = T$.

(2) $H_0 = H_1$ *or* T_0 *is a co-isometry.*

PROOF. Let $\mathfrak{a} = \boldsymbol{C}$ and $\rho_j\colon \boldsymbol{C} \ni \lambda \mapsto \lambda I_{H_j} \in \mathscr{L}(H_j)$ and apply Corollary 2.1.

REMARK 2.2. Corollary 2.2 appeared (with a direct proof) in one of the preliminary versions of [3], (namely T. Ando's one).

COROLLARY 2.3. *Let* $U_j \in \mathscr{L}(H_j)$ *be unitary on* H_j*—separable Hilbert space—*$(j = 1, 2)$, H_0 *a reducing subspace for* U_1, $P = P_{H_0}$, $V_1 = U_1|_{H_0}$ *and* $T_0 \in I(U_2, V_1)$. *The following conditions are equivalent*:

(i) T_0P *is the only contraction* $S \in I(U_2, U_1)$ *such that* $S|_{H_0} = T_0$.

(ii) *If* $W_1 = U_1|_{H_1 \ominus H_0}$ *and* $W_2 = U_2|_{\mathscr{D}_{T_0^*}}$, *then* $I(W_2, W_1) = \{0\}$.

(iii) If $W_2' = U_2|_{\ker T_0^*}$, then:

a) $I(W_2', W_1) = \{0\}$.

b) *There exists* ω_j*-Borel set in the spectrum of* U_j*-*$(j = 1, 2)$, *such* *that* $P_{\omega_1} \geqq 1 - P$ *and* T_0 *is a co-isometry from* $P_{\omega_1}(H_0)$ *onto* $P_{\omega_2}(H_2)$, *where* P_{ω_j} *is the spectral projection of* U_j *corresponding to* ω_j $(j = 1, 2)$.

(iv) a) $I(W_2', W_1) = \{0\}$

b) *There exists* ω_1'*-Borel set in the spectrum of* U_1*—such that* $P_{\omega_1'} \leqq P$ *and* $T_0(\mathscr{D}_{T_0}) \subset T_0 P_{\omega_1'}(H_1)$.

PROOF. Let $\mathfrak{a}$ be the C^*-algebra of continuous (complex valued) functions on $\boldsymbol{T} = \{z \in \boldsymbol{C}: |z| = 1\}$, ρ_j the representation of $\mathfrak{a}$ given by U_j $(j = 1, 2)$. Since H_j is separable, by a theorem of J. von Neumann, every central projections of $[\rho_j(\mathfrak{a})]'$ corresponds (by the Borel functional calculus) to a Borel subset of the spectrum of U_j $(j=1, 2)$. Using Putnam-Fuglede theorem, the corollary follows from Theorem 2.1 (see also Remark 2.1).

3. Consider again the situation of the first section: let $\mathfrak{a}$ be a C^*-algebra, ρ_j a representation of $\mathfrak{a}$ in $\mathscr{L}(H_j)$, $T_j \in [\rho_j(\mathfrak{a})]'$ a contraction, $U_j \in \mathscr{L}(K_j)$ the minimal isometric dilation of T_j and $P_j = P_{H_j} \in \mathscr{L}(K_j)$ $(j = 1, 2)$. Let $A \in I(T_1, T_2) \cap I(\rho_1, \rho_2)$ be a contraction. We will prove that in the Definition 1.2, T_2 can be replaced by U_2. To this end, consider $\tilde{A} = AP_2 \in \mathscr{L}(K_2, H_1)$. It is clear that $\tilde{A} \in I(T_1, U_2) \cap I(\rho_1, \rho_2^{(\infty)})$.

Since U_2 is an isometry, $\mathscr{R}(\tilde{A} \cdot U_2)$ is identified in the sequel with $P_{\mathscr{D}_A}\mathscr{R}(\tilde{A} \cdot U_2)$. With this identification we write $(\rho_2^{(\infty)})_{\mathscr{R}(\tilde{A} \cdot U_2)}$ instead of $((\rho_2^{(\infty)})^{(1)})_{\mathscr{R}(\tilde{A} \cdot U_2)}$.

LEMMA 3.1. a) *An operator* $B \in \mathscr{L}(K_2, K_1)$ *is a* $(\rho_1, \rho_2; T_1, T_2)$-CID *for* A *iff* B *is a* $(\rho_1, \rho_2^{(\infty)}; T_1, U_2)$-CID *for* $\tilde{A}$.

b) A *is* $(\rho_1, \rho_2; T_1, T_2)$*-regular iff* $\tilde{A}$ *is* $(\rho_1, \rho_2^{(\infty)}; T_1, U_2)$*-regular*.

PROOF. a) is an easy computation.

b) because U_2 is an isometry, we can write that:

$$\mathscr{R}(\tilde{A} \cdot U_2) = \mathscr{D}_{\tilde{A}} \ominus D_{\tilde{A}} U_2(K_2)^- . \tag{3.1}$$

Let $i_2: H_2 \mapsto K_2$ *be defined by*

$$i_2(h_2) = h_2 \oplus 0 \oplus 0 \oplus \cdots \qquad (h_2 \in H_2) .$$

Since $\tilde{A}^* = i_2 A^*$, we infer that

$$D_{\tilde{A}}^2 = I_{K_2} - \tilde{A}^*\tilde{A} = I_{K_2} - i_2 A^* A P_2 = D_A^2 \oplus I_{K_2 \ominus H_2}$$

thus

$$D_{\tilde{A}} = D_A \oplus I_{K_2 \ominus H_2} = \begin{pmatrix} D_A & 0 & 0 \cdots \\ 0 & I & 0 \cdots \\ 0 & 0 & I \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Using this, we obtain that:

$$\mathscr{D}_{\tilde{A}} = D_{\tilde{A}}(K_2)^- = \mathscr{D}_A \oplus \mathscr{D}_{T_2} \oplus \mathscr{D}_{T_2} \oplus \cdots . \tag{3.2}$$

Since

$$D_{\tilde{A}}U_2 = \begin{pmatrix} D_A & 0 & 0\cdots \\ 0 & I & 0\cdots \\ 0 & 0 & I\cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} T_2 & 0 & 0\cdots \\ D_{T_2} & 0 & 0\cdots \\ 0 & I & 0\cdots \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} D_A T_2 & 0 & 0\cdots \\ D_{T_2} & 0 & 0\cdots \\ 0 & I & 0\cdots \\ \vdots & & \ddots \end{pmatrix}$$

we infer that:

$$D_{\tilde{A}}U_2(K_2)^- = \{D_A T_2 k_2 \oplus D_{T_2}k_2 : k_2 \in K_2\}^- \oplus \mathscr{D}_{T_2} \oplus \mathscr{D}_{T_2} \oplus \cdots . \tag{3.3}$$

From (3.1), (3.2) and (3.3) it follows that:

$$\mathscr{R}(\tilde{A}\cdot U_2) = \mathscr{R}(A\cdot T_2) \oplus \{0\} \oplus \{0\} \oplus \cdots ,$$

which means that

$$(\rho_2^{(1)})_{\mathscr{R}(A\cdot T_2)} \text{ is equivalent to } (\rho_2^{(\infty)})_{\mathscr{R}(\tilde{A}\cdot U_2)} . \tag{3.4}$$

On the other hand

$$\begin{aligned}\mathscr{R}(T_1\cdot\tilde{A}) &= \mathscr{D}_{T_1} \oplus \mathscr{D}_{\tilde{A}} \ominus \{D_{T_1}\tilde{A}k_2 \oplus D_{\tilde{A}}k_2 : k_2 \in K_2\}^- \\ &= (\mathscr{D}_{T_1} \oplus \mathscr{D}_A \oplus \mathscr{D}_{T_2} \oplus \mathscr{D}_{T_2} \oplus \cdots) \\ &\quad \ominus \{D_{T_1}Ah_2 \oplus D_A h_2 \oplus h_2' : h_2 \in H_2,\ h_2' \in K_2 \ominus H_2\}^- \\ &= \mathscr{R}(T_1\cdot A) \oplus \{0\} \oplus \{0\} \oplus \cdots ,\end{aligned}$$

which implies that

$$(\rho_1 \oplus \rho_2)_{\mathscr{R}(T_1\cdot A)} \text{ is equivalent to } (\rho_1 \oplus \rho_2{}^{(\infty)})_{\mathscr{R}(T_1\cdot\tilde{A})} . \tag{3.5}$$

The relations (3.4) and (3.5) prove the lemma.

We will give now another characterization of the notion of $(\rho_1, \rho_2; T_1, T_2)$-regularity.

Let $S = T_1 A \in \mathscr{L}(H_2, H_1)$ and $Z: \mathscr{D}_S \mapsto \mathscr{D}_{T_1} \oplus \mathscr{D}_A$ be defined by:

$$Z(D_S h_2) = D_{T_1}Ah_2 \oplus D_A h_2 , \qquad (h_2 \in H_2) . \tag{3.6}$$

The operator Z is an isometry (see [9], ch. VII, section 3) and

$$\mathscr{R}(T_1\cdot A) = \mathscr{D}_{Z^*} . \tag{3.7}$$

Put

$$Z_1 = P_{\mathscr{D}_{T_1}} Z : \mathscr{D}_S \mapsto \mathscr{D}_{T_1} .$$

We have that $S \in I(\rho_1, \rho_2)$, $Z \in I((\rho_1 \oplus \rho_2)_{\mathscr{D}_{T_1}\oplus\mathscr{D}_A}, (\rho_2)_{\mathscr{D}_S})$, $Z_1 \in I((\rho_1)_{\mathscr{D}_{T_1}}, (\rho_2)_{\mathscr{D}_S})$

and $D_{Z_1^*} \in [(\rho_1)_{\mathscr{D}_{T_1}}]'$.

LEMMA 3.2. *The representations* $(\rho_1 \oplus \rho_2)_{\mathscr{R}(T_1 \cdot A)}$ *and* $(\rho_1)_{\mathscr{D}_{Z_1^*}}$ *are equivalent.*

PROOF. Let $V: \mathscr{D}_{Z_1^*} \mapsto \mathscr{D}_{Z^*}$ be defined by:

$$V(D_{Z_1^*}h_1) = D_{Z^*}(h_1 \oplus 0) , \qquad (h_1 \in \mathscr{D}_{T_1}) .$$

From the equalities

$$\begin{aligned}||D_{Z_1^*}h_1||^2 &= ||h_1||^2 - ||Z_1^* h_1||^2 = ||h_1||^2 - ||Z^*(h_1 \oplus 0)||^2 \\ &= ||D_{Z^*}(h_1 \oplus 0)||^2 , \qquad (h_1 \in H_1) ,\end{aligned}$$

we obtain that V is isometric.

Note now that V is unitary, that is $D_{Z^*}(\mathscr{D}_{T_1} \oplus \{0\})^- = \mathscr{D}_{Z^*}$. Indeed, consider $h_1 \oplus h_2 \in \mathscr{D}_{Z^*} \ominus D_{Z^*}(\mathscr{D}_{T_1} \oplus \{0\})^-$. Then, from (3.7) we obtain

$$\langle h_1 \oplus h_2, D_{T_1}Ah_2' \oplus D_A h_2' \rangle = 0 , \quad \text{for every} \quad h_2' \in H_2 ,$$

which means

$$\langle A^* D_{T_1} h_1 + D_A h_2, h_2' \rangle = 0 , \quad \text{for every} \quad h_2' \in H_2 ,$$

therefore

$$A^* D_{T_1} h_1 + D_A h_2 = 0 . \tag{3.8}$$

But $h_1 \oplus h_2$ is orthogonal on $D_{Z^*}(\mathscr{D}_{T_1} \oplus \{0\})^-$, therefore

$$\langle h_1 \oplus h_2, D_{Z^*}(h_1' \oplus 0) \rangle = 0 , \quad \text{for every} \quad h_1' \in \mathscr{D}_{T_1} . \tag{3.9}$$

Because Z is an isometry, $D_{Z^*} = P_{\mathscr{D}_{Z^*}}$ and from (3.9) we obtain that $\langle h_1 \oplus h_2, h_1' \oplus 0 > = 0$, for every $h_1' \in \mathscr{D}_{T_1}$, which means that $h_1 = 0$.

Using (3.8), we deduce that $h_2 = 0$ and therefore V is unitary. For $x \in \mathfrak{a}$, we have

$$\begin{aligned}(\rho_1 \oplus \rho_2)_{\mathscr{D}_{T_1} \oplus \mathscr{D}_A}(x) V(D_{Z_1^*}h_1) &= (\rho_1 \oplus \rho_2)_{\mathscr{D}_{T_1} \oplus \mathscr{D}_A}(x) D_{Z^*}(h_1 \oplus 0) \\ &= D_{Z^*}(\rho_1(x)h_1 \oplus 0) = V D_{Z_1^*}\rho_1(x)(h_1) = V(\rho_1)_{\mathscr{D}_{T_1}}(x) D_{Z_1^*}(h_1) ,\end{aligned}$$

for every $h_1 \in \mathscr{D}_{T_1}$, which implies that

$$V \in I((\rho_1 \oplus \rho_2)_{\mathscr{R}(T_1 \cdot A)} , \quad (\rho_1)_{\mathscr{D}_{Z_1^*}}) .$$

The lemma is now completely proved.

COROLLARY 3.1. *A is* $(\rho_1, \rho_2; T_1, T_2)$*-regular iff*

$$(\rho_1)_{\mathscr{D}_{Z_1^*}} \,\natural\, (\rho_2^{(\infty)})_{\mathscr{R}(\tilde{A} \cdot U_2)} . \tag{3.10}$$

PROOF. Since the proof of Lemma 3.1 shows that $(\rho_1 \oplus \rho_2)_{\mathscr{R}(T_1 \cdot A)}$ is equivalent to $(\rho_1 \oplus \rho_2^{(\infty)})_{\mathscr{R}(T_1 \cdot \tilde{A})}$, the lemma follows from Lemma 3.1 and Lemma 3.2.

4. We will analyze now the iterative construction (see [9], ch II, section 2, or [3], section 3) of a (T_1, U_2)-CID, in order to prove that the relation (3.10) can be also iterated, and that the presence of the representations is not difficult to handle. Let us start with $H_1^{(0)} = H_1$, $T_1^{(0)} = T_1$ and $B_0 = \tilde{A}$.

The first step consists in the following construction:

Let $H_1^{(1)} = H_1^{(0)} \oplus \mathscr{D}_{T_1}$ and $B_1: K_2 \mapsto H_1^{(1)}$ defined by:

$$B_1 = \begin{pmatrix} B_0 \\ X_1 \end{pmatrix}, \quad \text{where} \quad X_1: K_2 \mapsto \mathscr{D}_{T_1} . \tag{4.1}$$

The problem is to find X_1 such that:

$$\begin{aligned} &\text{a)} \quad \|B_1\| \leqq 1 \\ &\text{b)} \quad B_1 \in I(T_1^{(1)}, U_2) \\ &\text{c)} \quad B_1 \in I(\rho_1^{(1)}, \rho_2^{(\infty)}) , \end{aligned} \tag{4.2}$$

where $T_1^{(1)} = \begin{pmatrix} T_1 & 0 \\ D_{T_1} & 0 \end{pmatrix}$.

As in [9] or [3], we take $X_1 = \hat{C}_1 D_{B_0}$, where $\hat{C}_1$ is a "suitable" extension of the contraction $C_1: D_{B_0} U_2(K_2)^- \mapsto \mathscr{D}_{T_1}$, defined by:

$$C_1 D_{B_0} U_2 = D_{T_1^{(0)}} B_0 . \tag{4.3}$$

Note that from (4.3) it is clear that $C_1 \in I((\rho_1)_{\mathscr{D}_{T_1}}, (\rho_2^{(\infty)})_{D_{B_0} U_2(K_2)^-})$. Using that, we deduce that there exists an extension of C_1 such that B_1 fulfills (4.2) (take $\hat{C}_1 = C_1 P_{D_{B_0} U_2(K_2)^-}$).

LEMMA 4.1. *A is $(\rho_1, \rho_2; T_1, T_2)$-regular iff*

$$(\rho_1)_{\mathscr{D}_{C_1^*}} \between (\rho_2^{(\infty)})_{\mathscr{R}(B_0 \cdot U_2)} . \tag{4.4}$$

PROOF. The construction made in relations (3.6) and (3.7) can be made for every factorization: let $\tilde{Z}$ (resp. W) be the operators constructed like Z in (3.6) for factorization $T_1 \cdot \tilde{A}$ (resp. $B_0 \cdot U_2$). Because U_2 is an isometry, W can be identified with the unitary from $\mathscr{D}_{B_0 U_2}$ onto $D_{B_0} U_2(K_2)^-$, defined by:

$$W(D_{B_0 U_2} k_2) = D_{B_0} U_2 k_2 , \qquad (k_2 \in K_2) . \tag{4.5}$$

From (4.3) and (4.5) we infer that:

$$C_1 = P_{\mathscr{D}_{T_1}} \tilde{Z} W^* . \tag{4.6}$$

Let $i_{\mathscr{D}_{T_1}}: \mathscr{D}_{T_1} \mapsto H_1$ be the operator $i_{\mathscr{D}_{T_1}}(h_1) = h_1$, $(h_1 \in \mathscr{D}_{T_1})$. Then:

$$D_{C_1^*}^2 = I_{\mathscr{D}_{T_1}} - P_{\mathscr{D}_{T_1}} \tilde{Z} W^* W \tilde{Z}^* i_{\mathscr{D}_{T_1}} = I_{\mathscr{D}_{T_1}} - P_{\mathscr{D}_{T_1}} \tilde{Z} \tilde{Z}^* i_{\mathscr{D}_{T_1}} = D_{\tilde{Z}_1^*}^2 = D_{Z_1^*}^2 .$$

Now lemma follows from Corollary 3.1.

Next steps consist in repeating the construction with the new objects; more precisely:

$$(4.7)\qquad H_1^{(n)} = H_1 \oplus \overbrace{\mathscr{D}_{T_1} \oplus \cdots \oplus \mathscr{D}_{T_1}}^{n\text{-times}},\ B_n\colon K_2 \mapsto H_1^{(n)} \quad \text{by}$$

$$B_n = \begin{pmatrix} B_{n-1} \\ X_n \end{pmatrix}, \quad \text{where} \quad X_n\colon K_2 \to \mathscr{D}_{T_1}, \qquad (n \geqq 2),$$

such that

$$(4.8)\qquad \begin{cases} \text{a)} & \|B_n\| \leqq 1 \\ \text{b)} & B_n \in I(T_1^{(n)}, U_2) \\ \text{c)} & B_n \in I(\rho_1^{(n)}, \rho_2^{(\infty)}), \quad \text{where} \end{cases}$$

$$T_1^{(n)} = \begin{pmatrix} T_1 & 0 & 0 \cdots 0 \\ D_{T_1} & 0 & 0 \cdots 0 \\ 0 & I & 0 \cdots 0 \\ \vdots & \vdots & \vdots \quad \vdots \\ 0 & 0 & 0 \cdot\cdot I \; 0 \end{pmatrix}, \quad ((n+1) \times (n+1)\text{-matrix}).$$

We take $X_n = \hat{C}_n D_{B_{n-1}}$, $(n \geqq 2)$, where $\hat{C}_n$ is a "suitable" extension of the contraction $C_n\colon D_{B_{n-1}} U_2(K_2)^- \mapsto \mathscr{D}_{T_1}$, defined by

$$(4.9)\qquad C_n D_{B_{n-1}} U_2 = D_{T_1^{(n-1)}} B_{n-1}, \qquad (n \geqq 2).$$

The same argument as in the first step shows that such a "suitable" extension always exists. Note also that $D_{T_1^{(n)}} = \begin{pmatrix} 0 & & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \cdot\, 0 & 0 \\ 0 & 0 & I \end{pmatrix}$ $((n+1) \times (n+1)$-matrix), for every $n \geqq 1$, therefore (4.9) implies that:

$$(4.10)\qquad C_n D_{B_{n-1}} U_2 = X_{n-1} = \hat{C}_{n-1} D_{B_{n-2}}, \qquad (n \geqq 2).$$

LEMMA 4.2. (1) $\mathscr{D}_{C_n^*} = \mathscr{D}_{\hat{C}_{n-1}^*}$, for every $n \geqq 2$.

(2) If $\hat{C}_n = C_n P_{D_{B_{n-1}} U_2(K_2)^-}$, then the representations $(\rho_2^{(\infty)})_{\mathscr{R}(B_{n-1} \cdot U_2)}$ and $(\rho_2^{(\infty)})_{\mathscr{R}(B_n \cdot U_2)}$ are equivalent, $(n \geqq 1)$.

PROOF. (1) Define $M_n\colon D_{B_n} U_2(K_2)^- \mapsto \mathscr{D}_{B_{n-1}}$ by

$$M_n(D_{B_n} U_2 k_2) = D_{B_{n-1}} k_2, \qquad (k_2 \in K_2,\ n \geqq 1).$$

Then:

$$\begin{aligned} \|D_{B_n} U_2 k_2\|^2 &= \|U_2 k_2\|^2 - \|B_n U_2 k_2\|^2 = \|k_2\|^2 - \|B_{n-1} U_2 k_2\|^2 - \|X_n U_2 k_2\|^2 \\ &= \|k_2\|^2 - \|B_{n-1} U_2 k_2\|^2 - \|\hat{C}_n D_{B_{n-1}} U_2 k_2\|^2 \\ &= \|k_2\|^2 - \|B_{n-1} U_2 k_2\|^2 - \|C_n D_{B_{n-1}} U_2 k_2\|^2 \\ &= \|k_2\|^2 - \|B_{n-1} U_2 k_2\|^2 - \|D_{T_1^{(n-1)}} B_{n-1} k_2\|^2 \end{aligned}$$

$$= \|k_2\|^2 - \|B_{n-1}U_2k_2\|^2 - \|B_{n-1}k_2\|^2 + \|T_1^{(n-1)}B_{n-1}k_2\|^2$$
$$= \|D_{B_{n-1}}k_2\|^2 - \|B_{n-1}U_2k_2\|^2 + \|B_{n-1}U_2k_2\|^2 = \|D_{B_{n-1}}k_2\|^2 , \quad (k_2 \in K_2,\ n \geqq 1) .$$

Therefore M_n is an isometry with dense range, that is, a unitary ($n \geqq 1$). Using (4.10), we infer that

$$(4.11) \qquad C_n = \hat{C}_{n-1}M_{n-1} , \qquad (n \geqq 2) ,$$

therefore

$$D^2_{C^*_n} = I_{\mathscr{D}_{T_1}} - C_nC^*_n = I_{\mathscr{D}_{T_1}} - \hat{C}_{n-1}M_{n-1}M^*_{n-1}\hat{C}^*_{n-1} = D^2_{\hat{C}^*_{n-1}} \qquad (n \geqq 2) ,$$

which implies that $\mathscr{D}_{C^*_n} = \mathscr{D}_{\hat{C}^*_{n-1}}$, ($n \geqq 2$).

(2) Define Q_n: $D_{\hat{C}_n}D_{B_{n-1}}(K_2)^- \mapsto \mathscr{D}_{B_n}$ by

$$(4.12) \qquad Q_n(D_{\hat{C}_n}D_{B_{n-1}}k_2) = D_{B_n}k_2 , \qquad (k_2 \in K_2,\ n \geqq 1) .$$

Then:

$$\|D_{B_n}k_2\|^2 = \|k_2\|^2 - \|B_nk_2\|^2 = \|k_2\|^2 - \|B_{n-1}k_2\|^2$$
$$- \|\hat{C}_nD_{B_{n-1}}k_2\|^2 = \|D_{B_{n-1}}k_2\|^2 - \|\hat{C}_nD_{B_{n-1}}k_2\|^2$$
$$= \|D_{\hat{C}_n}D_{B_{n-1}}k_2\|^2 , \qquad (k_2 \in K_2,\ n \geqq 1) .$$

Therefore Q_n is unitary.

Because $\hat{C}_n = C_nP_{D_{B_{n-1}}U_2(K_2)^-}$, we have that:

$$\mathscr{D}_{\hat{C}_n} = \mathscr{R}(B_{n-1}\cdot U_2) \oplus D_{C_n}D_{B_{n-1}U_2(K_2)^-} .$$

Using this, we infer that:

$$\mathscr{R}(B_n\cdot U_2) = \mathscr{D}_{B_n} \ominus D_{B_n}U_2(K_2)^- = Q_n(D_{\hat{C}_n}(D_{B_{n-1}}(K_2)^-)$$
$$\ominus D_{B_n}U_2(K_2)^- = Q_n(\mathscr{R}(B_{n-1}\cdot U_2)) \oplus Q_n(D_{C_n}D_{B_{n-1}}U_2(K_2)^-)$$
$$\ominus D_{B_n}U_2(K_2)^- = Q_n(\mathscr{R}(B_{n-1}\cdot U_2)) , \qquad (n \geqq 1) .$$

It is easy to deduce from (4.12) that

$$Q_n \in I((\rho_2^{(\infty)})_{\mathscr{R}(B_n\cdot U_2)}, (\rho_2^{(\infty)})_{\mathscr{R}(B_{n-1}\cdot U_2)}) ,$$

which proves the lemma.

5. Proof of Theorem 1.1. (1) Since B_n satisfies (4.8), ($n \geqq 1$), taking B the strong limit of the sequence $\{B_n\}$ $n \geqq 1$, it is easy to prove that B is a ($\rho_1, \rho_2^{(\infty)}$; T_1, U_2)-CID for $\tilde{A}$, so (using Lemma 3.1(a)) B is a (ρ_1, ρ_2; T_1, T_2)-CID for A; moreover, every (ρ_1, ρ_2; T_1, T_2)-CID for A arises in this way.

(2) Let A be (ρ_1, ρ_2; T_1, T_2)-regular; using Lemma 4.1 we obtain that $(\rho_1)_{\mathscr{D}_{C_1^*}} \between (\rho_2^{(\infty)})_{\mathscr{R}(\tilde{A}\cdot U_2)}$, which means by (3.1) that $(\rho_2^{(\infty)})_{\mathscr{D}_{\tilde{A}}\ominus D_{\tilde{A}}U_2(K_2)^-} \between (\rho_1)_{\mathscr{D}_{C_1^*}}$.

The application of Theorem 2.1 with $H_1 = \mathscr{D}_{\tilde{A}}$, $H_0 = D_{\tilde{A}} U_2(K_2)^-$, $T_0 = C$, $\rho_1 = (\rho_2^{(\infty)})_{\mathscr{D}_{\tilde{A}}}$ and $\rho_2 = \rho_1$, shows that the only $\hat{C}_1$ such that B_1 satisfies (4.2) is $\hat{C}_1 = C_1 P_{D_{\tilde{A}} U_2(K_2)^-}$. Therefore $\mathscr{D}_{\hat{C}_1^*} = \mathscr{D}_{C_1^*}$ and Lemma 4.2 implies that

$$(\rho_2^{(\infty)})_{\mathscr{D}_{B_1} \ominus D_{B_1} U_2(K_2)^-} \between (\rho_1)_{\mathscr{D}_{C_2^*}} .$$

Theorem 2.1 shows again that $\hat{C}_2$ is unique such that B_2 satisfies (4.8) for $n = 2$. By induction, $\hat{C}_n$ is unique such that B_n satisfies (4.8) and therefore A has a unique $(\rho_1, \rho_2; T_1, T_2)$-CID.

Conversely, if A has a unique $(\rho_1, \rho_2; T_1, T_2)$-CID, then $\hat{C}_1 = C_1 P_{D_{\tilde{A}} U_2(K_2)^-}$, therefore (by Theorem 2.1)

$$(\rho_2^{(\infty)})_{\mathscr{D}_{\tilde{A}} \ominus D_{\tilde{A}} U_2(K_2)^-} \between (\rho_1)_{\mathscr{D}_{C_1^*}} .$$

This condition implies (by Lemma 4.1) that A is $(\rho_1, \rho_2; T_1, T_2)$-regular. The theorem is completely proved.

COROLLARY 5.1. *If ρ_1 and ρ_2 are two non-disjoint factorial representations of $\mathfrak{a}$, then A has a unique $(\rho_1, \rho_2; T_1, T_2)$-CID iff one of the factorizations $T_1 \cdot A$ or $A \cdot T_2$ is regular.*

PROOF. Use a similar argument as in the proof of Corollary 2.1.

COROLLARY 5.2. ([3]) *A has a unique (T_1, T_2)-CID iff one of the factorizations $T_1 \cdot A$ or $A \cdot T_2$ is regular.*

PROOF. Take $\mathfrak{a} = \boldsymbol{C}$ and $\rho_j : \boldsymbol{C} \ni \lambda \mapsto \lambda I_{H_j} \in \mathscr{L}(H_j)$ $(j = 1, 2)$ and apply Corollary 5.1.

6. We give now some applications to the case of a pair of commuting contractions. Fix the following notations. Let $T_1, T_2 \in \mathscr{L}(H)$ be a pair of commuting contractions, $\mathfrak{a}$ a C^*-algebra, $\rho : \mathfrak{a} \mapsto \mathscr{L}(H)$ a representation of $\mathfrak{a}$ such that $T_j \in [\rho(\mathfrak{a})]'$, $(j = 1, 2)$. From Ando's theorem [1], the pair $\{T_1, T_2\}$ always has a minimal isometric dilation $\{U_1, U_2\}$, $U_j \in \mathscr{L}(K)$, $(j = 1, 2)$.

DEFINITION 6.1. A minimal isometric dilation of $\{T_1, T_2\}$ on K namely $\{U_1, U_2\}$, is called *ρ-adequate* if there exists a representation $\tilde{\rho} : \mathfrak{a} \mapsto \mathscr{L}(K)$ such that H is an invariant subspace for $\tilde{\rho}$, $(\tilde{\rho})_H = \rho$ and $U_j \in [\tilde{\rho}(\mathfrak{a})]'$, $(j = 1, 2)$.

Note that, given $\{U_1, U_2\}$, if such a $\tilde{\rho}$ exists, then it is unique.

THEOREM 6.1. 1) *The pair $\{T_1, T_2\}$ always has a ρ-adequate minimal isometric dilation.*

2) *The pair $\{T_1, T_2\}$ has a unique (up to an isomorphism) ρ-adequate*

minimal isometric dilation iff $(\rho \oplus \rho)_{\mathscr{R}(T_1 \cdot T_2)} \,\delta\, (\rho \oplus \rho)_{\mathscr{R}(T_2 \cdot T_1)}$.

3) *If* ρ *is a factor representation, then the pair* $\{T_1, T_2\}$ *has a unique* ρ*-adequate minimal isometric dilation iff one of the factorizations* $T_1 \cdot T_2$ *or* $T_2 \cdot T_1$ *is regular.*

PROOF. (1) Because $T_2 \in I(T_1, T_1) \cap I(\rho, \rho)$ we can apply Theorem 1.1 (1) to find a $(\rho, \rho; T_1, T_1)$-CID for T_2. This means that if $U_1 \in \mathscr{L}(K_1)$ is the minimal isometric dilation of T_1, then there exists a contraction $\tilde{T}_2 \in \mathscr{L}(K_1)$ such that $\tilde{T}_2 U_1 = U_1 \tilde{T}_2$, $P_H \tilde{T}_2 = T_2 P_H$ and $\tilde{T}_2 \in [\rho^{(\infty)}(\mathfrak{a})]'$, where $\rho^{(\infty)}$ is defined by (1.1). Now $U_1 \in I(\tilde{T}_2, \tilde{T}_2) \cap I(\rho^{(\infty)}, \rho^{(\infty)})$ and we apply again Theorem 1.1 (1) to find a $(\rho^{(\infty)}, \rho^{(\infty)}; \tilde{T}_2, \tilde{T}_2)$-CID for U_1. This means that if $\tilde{U}_2 \in \mathscr{L}(\tilde{K}_2)$ is the minimal isometric dilation of $\tilde{T}_2$, then there exists an unique *isometry* $\tilde{U}_1 \in \mathscr{L}(\tilde{K}_2)$ such that $P_{K_1} \tilde{U}_1 = U_1 P_{K_1}$, $\tilde{U}_1 \tilde{U}_2 = \tilde{U}_2 \tilde{U}_1$ and $\tilde{U}_1 \in [(\rho^{(\infty)})^{(\infty)}(\mathfrak{a})]'$, (see [8], Proposition 10.8). The pair $\{\tilde{U}_1, \tilde{U}_2\}$, which is an isometric dilation for $\{T_1, T_2\}$, contains a minimal isometric dilation $\{\hat{U}_1, \hat{U}_2\}$ on the space $K = \bigvee_{n=0, m=0}^{\infty} \tilde{U}_1^n \tilde{U}_2^m (H)$. It is clear that $P_K \in [(\rho^{(\infty)})^{(\infty)}(\mathfrak{a})]'$, therefore, taking $\tilde{\rho} = ((\rho^{(\infty)})^{(\infty)})_K$, we see that $\{\hat{U}_1, \hat{U}_2\}$ is ρ-adequate.

(2) Let $\{U_1, U_2\}$ be a ρ-adequate minimal isometric dilation (on K) for $\{T_1, T_2\}$ and let $K_1 = \bigvee_{n=0}^{\infty} U_1^n(H)$. The minimality of $\{U_1, U_2\}$ implies that H is invariant for U_1^* and therefore K_1 is reducing for U_1. Denote $V_1 = U_1|_{K_1}$ and $V_2 = P_{K_1} U_2|_{K_1}$; then V_1 is a minimal isometric dilation for T_1 and, up to an isomorphism of dilations (see [9], ch. I, section 4.1 for definition), we can consider that $V_1 \in \mathscr{L}(K_1)$ is the "standard" minimal isometric dilation described in section 1. Let $\tilde{\rho}$ be the representation which appears in the definition of the fact that $\{U_1, U_2\}$ is ρ-adequate. Then K_1 is invariant for $\tilde{\rho}$ and because $V_1 \in [(\tilde{\rho})_{K_1}(\mathfrak{a})]'$ we have (up to an isomorphism) that $(\tilde{\rho})_{K_1} = \rho^{(\infty)}$ (see 1.1). This implies that V_2 is a $(\rho, \rho; T_1, T_1)$-CID for T_2, and, by Theorem 1.1 (1), that $\{U_1, U_2, \tilde{\rho}\}$ is unitary equivalent to the $\rho^{(\infty)}$-adequate minimal isometric dilation obtained from $\{V_1, V_2, \rho^{(\infty)}\}$ (see the second part of (1)). Because the factorization $V_1 \cdot V_2$ is always regular (see [9], ch. VII, Proposition 3.2 (b)), the uniqueness problem for a ρ-adequate minimal isometric dilation of $\{T_1, T_2\}$ is solved by the uniqueness of V_2. So we can apply Theorem 1.1 (2) in order to get the conclusion.

(3) is a consequence of (2).
The theorem is completely proved.

COROLLARY 6.1. *A pair* $\{T_1, T_2\}$ *of commuting contractions has a unique minimal unitary dilation iff one of the factorizations* $T_1 \cdot T_2$ *or* $T_2 \cdot T_1$ *is regular.*

PROOF. Apply theorem 6.1 (3) for $\mathfrak{a} = \boldsymbol{C}$ and $\rho \colon \boldsymbol{C} \ni \lambda \mapsto \lambda I_H \in \mathscr{L}(H)$.

REMARK 6.1. This corollary was communicated to us by Professor C. Foiaş in connection to [3].

T. Ando proved in [2] that if T_1, T_2, T_3 are contractions on H such that T_3 doubly commutes with T_1 and T_2 and T_1 commutes with T_2, then the system $\{T_1, T_2, T_3\}$ has a unitary dilation. Using the techniques of [2], one can prove the following more general result, which we present here as a consequence of technique involved in Theorem 6.1.

COROLLARY 6.2. *Let* $\{T_1, T_2, \{S_\omega\}_{\omega\in\Omega}\}$ *be contractions on* H *such that* S_ω *doubly commutes with* T_1 *and* T_2, *for every* $\omega\in\Omega$, T_1 *commutes with* T_2 *and the system* $\{S_\omega\}_{\omega\in\Omega}$ *has regular unitary dilation* (*see* [9], *ch.* I, §9). *Then the system* $\{T_1, T_2, \{S_\omega\}_{\omega\in\Omega}\}$ *has a unitary dilation.*

PROOF. Let α be the C^*-algebra generated by $\{S_\omega\}_{\omega\in\Omega}$ and ρ the identical representation of α on H. Making the same construction as in the proof of Theorem 6.1 (1), we obtain the system $\{U_1, \widetilde{T}_2, \{\widetilde{S}_\omega\}_{\omega\in\Omega}\}$ on K_1, where $U_1\in\mathscr{L}(K_1)$ is the minimal isometric dilation of T_1, $\widetilde{T}_2$ is a dilation of T_2 which commutes with U_1 and doubly commutes with

$$\widetilde{S}_\omega = S_\omega \oplus S_\omega|_{\mathscr{D}_{T_1}} \oplus S_\omega|_{\mathscr{D}_{T_1}} \oplus \cdots\,, \qquad (\omega\in\Omega)\,.$$

It is easy to see (using for example the condition (9.12) from [9], ch. I) that the system $\{\widetilde{S}_\omega\}_{\omega\in\Omega}$ has a regular unitary dilation. The Proposition 9.2, ch. I of [9] finishes the proof.

Added in proof. Recently, Z. Ceauşescu and C. Foiaş proved in "On intertwining dilations. V" (Acta Sci. Math. 40 (1978), 9-32) that there exists an explicit bijection between the intertwining dilations of a contraction A and the sequences of "choice operators" for A (that means the sequences of contractions $\{\Gamma_n\}_n$ with Γ_1: $\mathscr{R}(T_1\cdot A)\mapsto\mathscr{R}(A\cdot T_2)$ and Γ_n: $\mathscr{D}_{\Gamma_{n-1}}\to\mathscr{D}_{\Gamma^*_{n-1}}$ for $n\geqq 2$).

Using the methods of this paper and the preceding result, one can prove the following (in the conditions of Theorem 1.1 above) "There exists an explicit bijection between the $(\rho_1, \rho_2; T_1, T_2)$-CID for A and the choice sequences for A which fulfil the conditions:

(1) $\Gamma_1\in I((\rho_1\oplus\rho_2)_{\mathscr{R}(T_1\cdot A)}, (\rho_2^{(1)})_{\mathscr{R}(A\cdot T_2)})$

(2) $\Gamma_n\in I((\rho_1\oplus\rho_2)_{\mathscr{D}_{\Gamma_{n-1}}}, (\rho_2^{(1)})_{\mathscr{D}_{\Gamma^*_{n-1}}})$, $(n\geqq 2)$".

REFERENCES

[1] T. ANDO, On a pair of commutative contractions, Acta Sci. Math. 24 (1963), 88-90.

[2] T. ANDO, Unitary dilation for a triple of commuting contractions, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. 24 (1976), 851-853.

[3] T. ANDO, Z. CEAUŞESCU AND C. FOIAŞ, On intertwining dilations II, Acta Sci. Math. 39 (1977), 3-14.

[4] J. G. W. CARSWELL AND C. F. SCHUBERT, Lifting of operators that commute with shifts, Michigan Math. J. 22 (1975), 65–69.
[5] Z. CEAUŞESCU, Lifting of a contraction intertwining two isometries, preprint, 1976.
[6] Z. CEAUŞESCU AND C. FOIAŞ, On intertwining dilations III, Rev. Roum. Math. Pures et Appl. 22 (1977), 1387–1396.
[7] J. DIXMIER, Les C^*-algèbres et leurs représentations, Gauthier-Villars, 1969.
[8] I. SUCIU, Function algebras, Bucharest, 1973.
[9] B. SZ.-NAGY AND C. FOIAŞ, Harmonic Analysis of operators on Hilbert space, Budapest-Amsterdam-London, 1970.

NATIONAL INSTITUTE FOR SCIENTIFIC AND TECHNICAL CREATION
DEPARTMENT OF MATHEMATICS
BD. PǍCII 220, BUCHAREST 77538
ROMANIA