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ON ABSTRACT MEAN ERGODIC THEOREMS

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1. **Introduction.** In [8] Sine showed an interesting mean ergodic theorem. His theorem states that the ergodic averages $(1/n)\sum_{i=0}^{n-1} T^i$ converge in the strong operator topology if and only if the fixed points of T separate the fixed points of the adjoint operator T^*, T being any linear contraction on a Banach space. Later, this theorem was generalized and extended by Nagel [5] to a bounded right amenable operator semigroup in a Banach space. Another generalization was also done by Lloyd [4]. In the present paper we intend to apply the notion of "ergodicity", given for an operator semigroup in a locally convex topological vector space and originally introduced by Eberlein [3], and obtain abstract mean ergodic theorems which generalize Sine's, Nagel's and Lloyd's ergodic theorems.

2. **Definitions and examples.** Throughout this paper, E will denote a *complete* locally convex topological vector space (t. v. s.) and $\mathfrak{S}$ a semigroup of continuous linear operators on E. For $x \in E$ we denote by $A(x)$ the affine subspace of E determined by the set $\{Tx: T \in \mathfrak{S}\}$, i.e.,

$$A(x) = \left\{y: y = \sum_{i=1}^{n} a_i T_i x,\ \sum_{i=1}^{n} a_i = 1,\ T_i \in \mathfrak{S},\ 1 \leqq n < \infty\right\},$$

and by $\bar{A}(x)$ the closure of $A(x)$ in E. Let $(T_\alpha, \alpha \in \Lambda)$ be a net of linear operators on E. $(T_\alpha, \alpha \in \Lambda)$ is said to be a (*weakly*) *right* [resp. (*weakly*) *left*] $\mathfrak{S}$-*ergodic net* if it satisfies:

(I) For every $x \in E$ and all $\alpha \in \Lambda$, $T_\alpha x \in \bar{A}(x)$.

(II) The transformations T_α are equicontinuous.

(III) For every $x \in E$ and all $T \in \mathfrak{S}$,

$$(\text{weak-})\lim_\alpha T_\alpha Tx - T_\alpha x = 0\ [\text{resp. } (\text{weak-})\lim_\alpha TT_\alpha x - T_\alpha x = 0]\,.$$

$\mathfrak{S}$ is said to be a (*weakly*) *right* [resp. (*weakly*) *left*] *ergodic semigroup* (in the sense of Eberlein [3]) if it possesses at least one (weakly) right [resp. (weakly) left] $\mathfrak{S}$-ergodic net $(T_\alpha, \alpha \in \Lambda)$. Whenever $(T_\alpha, \alpha \in \Lambda)$ is a (weakly) right and left both $\mathfrak{S}$-ergodic net, we call it simply a (*weakly*) $\mathfrak{S}$-*ergodic net*. And if $\mathfrak{S}$ possesses at least one (weakly) $\mathfrak{S}$-ergodic net

$(T_\alpha, \alpha \in \Lambda)$, $\mathfrak{S}$ is said to be a (*weakly*) *ergodic semigroup*. (See also Day [1].) Here we note that our definition of ergodicity is somewhat different from that of Eberlein [3]. Instead of our condition (I), he used the following stronger condition:

(S-I) For every $x \in E$ and all $\alpha \in \Lambda$, $T_\alpha x \in \overline{\mathrm{co}}\, \mathfrak{S}x$, where $\overline{\mathrm{co}}\, \mathfrak{S}x$ denotes the closed convex hull of the set $\{Tx: T \in \mathfrak{S}\}$.

EXAMPLES. (1) Suppose either (i) $0 \in \mathfrak{S}$ or (ii) $\lambda_i I \in \mathfrak{S}$ $(i = 1, 2)$ with $\lambda_1 \neq \lambda_2$, I being the identity operator. It follows, in either case, that $0 \in \bar{A}(x)$ for every $x \in E$. Thus the sequence $(T_n, n \geqq 1)$, defined by $T_n = 0$ for all $n \geqq 1$, is an $\mathfrak{S}$-ergodic net.

(2) Suppose T is a bounded linear operator of spectral radius $r(T) \leqq 1$ on a Banach space. If $\sup_{0<r<1} \|(1-r)\sum_{n=0}^\infty r^n T^n\| < \infty$, then $\mathfrak{S} = \{T^n: n \geqq 0\}$ is an ergodic semigroup. In fact, putting $T_r = (1-r)\sum_{n=0}^\infty r^n T^n$ $(0 < r < 1)$, we have an $\mathfrak{S}$-ergodic net $(T_r, 0 < r < 1)$.

(3) Let $C(\mathfrak{S})$ denote the space of all bounded continuous functions on $\mathfrak{S}$, $\mathfrak{S}$ being equipped with the weak operator topology. It is then easily seen that, for each f in $C(\mathfrak{S})$ and each S in $\mathfrak{S}$, ${}_Sf$ and f_S are again in $C(\mathfrak{S})$, where ${}_Sf$ and f_S are defined by

$$ {}_Sf(T) = f(ST) \text{ and } f_S(T) = f(TS) \qquad (T \in \mathfrak{S}) . $$

A linear functional μ on $C(\mathfrak{S})$ is said to be a *right* [resp. *left*] *invariant mean* if μ satisfies $\|\mu\| = 1 = \langle 1, \mu\rangle$ and $\langle f_S, \mu\rangle = \langle f, \mu\rangle$ [resp. $\langle {}_Sf, \mu\rangle = \langle f, \mu\rangle$] for every $S \in \mathfrak{S}$ and all $f \in C(\mathfrak{S})$.

PROPOSITION 1. *Suppose $\mathfrak{S}$ is an equicontinuous semigroup of linear operators on E. If there exists a right* [*resp. left*] *invariant mean on $C(\mathfrak{S})$, then $\mathfrak{S}$ is a weakly right* [*resp. weakly left*] *ergodic semigroup.*

PROOF. Let $l_1(\mathfrak{S})$ denote the space of all functions ξ defined on $\mathfrak{S}$ for which the norm is given by

$$ \|\xi\|_1 = \sum_{S \in \mathfrak{S}} |\xi(S)| < \infty . $$

From Section 10 of Day [2] it follows that if $C(\mathfrak{S})$ has a right [resp. left] invariant mean, then there exists a net $(\xi_\alpha, \alpha \in \Lambda)$ of elements in $l_1(\mathfrak{S})$ such that

(i) for every $\alpha \in \Lambda$, $\xi_\alpha \geqq 0$ on $\mathfrak{S}$, $\sum_{S \in \mathfrak{S}} \xi_\alpha(S) = 1$ and $\{S \in \mathfrak{S}: \xi_\alpha(S) > 0\}$ is a finite set,

(ii) for every $f \in C(\mathfrak{S})$ and all $T \in \mathfrak{S}$,

$$ \lim_\alpha \sum_{S \in \mathfrak{S}} f(S)(\xi_\alpha * \delta_T(S) - \xi_\alpha(S)) = 0 \text{ [resp. } \lim_\alpha \sum_{S \in \mathfrak{S}} f(S)(\delta_T * \xi_\alpha(S) - \xi_\alpha(S)) = 0] , $$

where $\xi_\alpha * \delta_T$ and $\delta_T * \xi_\alpha (\in l_1(\mathfrak{S}))$ are defined by

$$\xi_\alpha * \delta_T(S) = \sum_{RT=S} \xi_\alpha(R) \text{ and } \delta_T * \xi_\alpha(S) = \sum_{TR=S} \xi_\alpha(R)$$

for all $S \in \mathfrak{S}$.

Let us put $T_\alpha = \sum_{S \in \mathfrak{S}} \xi_\alpha(S) S$. We shall prove that the net $(T_\alpha, \alpha \in \Lambda)$ is a weakly right [resp. weakly left] $\mathfrak{S}$-ergodic net.

To do this, fix an $x \in E$ and an $x^* \in E^*$ arbitrarily, E^* being the dual space of E, and define a function f on $\mathfrak{S}$ by the relation: $f(S) = \langle Sx, x^* \rangle$ for all $S \in \mathfrak{S}$. Since $\mathfrak{S}$ is equicontinuous, f is a bounded function on $\mathfrak{S}$, and hence f is in $C(\mathfrak{S})$. It follows that for all $T \in \mathfrak{S}$,

$$\begin{aligned}\lim_\alpha \langle T_\alpha T x - T_\alpha x, x^* \rangle &= \lim_\alpha \sum_{S \in \mathfrak{S}} f(S)(\xi_\alpha * \delta_T(S) - \xi_\alpha(S)) \\ &= 0 \text{ [resp. } \lim_\alpha \langle T T_\alpha x - T_\alpha x, x^* \rangle = 0] ,\end{aligned}$$

and therefore we have weak-$\lim_\alpha T_\alpha T x - T_\alpha x = 0$ [resp. weak-$\lim_\alpha T T_\alpha x - T_\alpha x = 0$]. Clearly, $(T_\alpha, \alpha \in \Lambda)$ satisfies conditions (S-I) and (II). The proof is complete.

3. Abstract mean ergodic theorems.

THEOREM 1. *Let E be a complete locally convex t.v.s. and $\mathfrak{S}$ a semigroup of continuous linear operators on E. Suppose $(T_\alpha, \alpha \in \Lambda)$ is a weakly right $\mathfrak{S}$-ergodic net, and define*

$$D = \{x \in E: \text{weak-}\lim_\alpha T_\alpha x \text{ exists}\}$$

and

$$T_\infty x = \text{weak-}\lim_\alpha T_\alpha x \qquad (x \in D) .$$

Then we have:

(a) *D is a closed linear subspace of E such that $TD \subset D$ for all $T \in \mathfrak{S}$.*

(b) *$T_\infty D \subset D$ and T_∞ is linear and continuous on D.*

(c) *$T_\infty T = T_\infty$ on D for all $T \in \mathfrak{S}$.*

PROOF. It is easily seen that D is a linear subspace of E. Since $(T_\alpha, \alpha \in \Lambda)$ is a weakly right $\mathfrak{S}$-ergodic net, it follows that $TD \subset D$ and $T_\infty T = T_\infty$ for all $T \in \mathfrak{S}$. Thus (c) is proved. To see that D is closed, we show that D is complete. Let (x_β) be a Cauchy net in D. Since the transformations T_α are equicontinuous, $(T_\infty x_\beta)$ is also a Cauchy net in E. Thus if we let

$$x = \lim_{\beta} x_\beta \quad \text{and} \quad y = \lim_{\beta} T_\infty x_\beta ,$$

then, given a weak convex neighborhood U of $0 \in E$, there exists a β_0 such that

$$T_\infty x_{\beta_0} - y \in (1/3)U \quad \text{and} \quad T_\alpha(x - x_{\beta_0}) \in (1/3)U$$

for all $\alpha \in \Lambda$. Since $T_\infty x_{\beta_0} = \text{weak-lim}_\alpha T_\alpha x_{\beta_0}$, there exists an α_0 such that

$$T_\alpha x_{\beta_0} - T_\infty x_{\beta_0} \in (1/3)U$$

for all $\alpha \in \Lambda$ with $\alpha > \alpha_0$. Then, for all $\alpha \in \Lambda$ with $\alpha > \alpha_0$, we have

$$\begin{aligned} T_\alpha x - y &= T_\alpha(x - x_{\beta_0}) + T_\alpha x_{\beta_0} - T_\infty x_{\beta_0} + T_\infty x_{\beta_0} - y \\ &\in (1/3)U + (1/3)U + (1/3)U = U , \end{aligned}$$

so that $y = \text{weak-lim}_\alpha T_\alpha x$ and $x \in D$. Thus (a) is proved. By (a), $T_\alpha x \in \bar{A}(x) \subset D$ for all $x \in D$ and all $\alpha \in \Lambda$, and this implies that $T_\infty D \subset D$. It is easily seen that T_∞ is linear and continuous on D. The proof is complete.

From now on we shall always assume that $(T_\alpha, \alpha \in \Lambda)$ is a weakly right $\mathfrak{S}$-ergodic net, unless the contrary is explicitly specified. Let $\mathfrak{S}^* = \{T^*: T \in \mathfrak{S}\}$ denote the adjoint semigroup of $\mathfrak{S}$. Define

$$\begin{aligned} F &= \{x \in E: Tx = x \text{ for all } T \in \mathfrak{S}\} , \\ D(F) &= \{x \in D: T_\infty x \in F\} , \\ D(0) &= \{x \in D: T_\infty x = 0\} , \end{aligned}$$

and

$$F^* = \{x^* \in E^*: T^* x^* = x^* \text{ for all } T^* \in \mathfrak{S}^*\} .$$

Then we have

THEOREM 2.

(a) *F and $D(F)$ are closed linear subspaces of E such that $F \subset D(F) \subset D$.*

(b) *$T_\infty D(F) \subset D(F)$ and $TD(F) \subset D(F)$ for all $T \in \mathfrak{S}$.*

(c) *$TT_\infty = T_\infty T = T_\infty$ on $D(F)$ for all $T \in \mathfrak{S}$.*

(d) *$D(0)$ is the closed linear subspace of E determined by the set $\{x - Tx: x \in E$ and $T \in \mathfrak{S}\}$.*

(e) *$y \in \bar{A}(x) \cap F$ if and only if $x \in D(F)$ and $y = T_\infty x$.*

PROOF. Since (a), (b), and (c) are direct from Theorem 1, we omit the details.

To prove (d), we first notice that $D(0)$ is a closed linear subspace of E. If we denote by N the closed linear subspace of E determined by the set $\{x - Tx: x \in E \text{ and } T \in \mathfrak{S}\}$, then $N \subset D(0)$, since weak-$\lim_\alpha T_\alpha Tx - T_\alpha x = 0$ for every $x \in E$ and all $T \in \mathfrak{S}$. Now suppose $x_0 \in D(0)$. If $x^* \in E^*$ satisfies $\langle y, x^* \rangle = 0$ for all $y \in N$, then $x^* \in F^*$, because $\langle x - Tx, x^* \rangle = 0$ for every $x \in E$ and all $T \in S$. Thus

$$\langle x_0, x^* \rangle = \langle T_\infty x_0, x^* \rangle = \langle 0, x^* \rangle = 0 ,$$

because $T_\infty x_0 \in \bar{A}(x_0)$. Hence, by the separation theorem (see, for example, Theorem 3.5 of [6]), $x_0 \in N$. This proves (d).

To prove (e), let $y \in \bar{A}(x) \cap F$. Given a weak convex and balanced neighborhood U of $0 \in E$, there exists a neighborhood V of $0 \in E$ (in the topology originally given in E) such that

$$T_\alpha V \subset (1/2) U$$

for all $\alpha \in \Lambda$, since the transformations T_α are equicontinuous. Choose $\sum_{i=1}^n a_i T_i x \in A(x)$ so that

$$y - \sum_{i=1}^n a_i T_i x \in V .$$

Since $(T_\alpha, \alpha \in \Lambda)$ is a weakly right $\mathfrak{S}$-ergodic net, then there exists an α_0 such that

$$T_\alpha T_i x - T_\alpha x \in \left(1/2 \sum_{i=1}^n |a_i|\right) U$$

for all $i = 1, \cdots, n$ and all $\alpha \in \Lambda$ with $\alpha > \alpha_0$. Then, for all $\alpha \in \Lambda$ with $\alpha > \alpha_0$,

$$\begin{aligned} y - T_\alpha x = T_\alpha y - T_\alpha x \in T_\alpha\left(\sum_{i=1}^n a_i (T_i x - x) \right) \\ + T_\alpha V \in \sum_{i=1}^n a_i \left(1/2 \sum_{i=1}^n |a_i|\right) U + (1/2) U = U , \end{aligned}$$

thus $y = \text{weak-}\lim_\alpha T_\alpha x$ and $x \in D(F)$. The converse implication is obvious. The proof is complete.

THEOREM 3. *C is a closed linear subspace of $T_\infty D$ and separates F^* if and only if $T_\infty D = C$ and $D = E$.*

PROOF. First suppose that C is a closed linear subspace of $T_\infty D$ and separates F^*. Write

$$D(C) = \{x \in D: T_\infty x \in C\} .$$

$D(C)$ is a closed linear subspace of D, and thus it is also a closed linear subspace of E. Let $x^* \in E^*$ be such that $\langle x, x^* \rangle = 0$ for all $x \in D(C)$.

Since, by Theorem 2, $x - Tx \in D(0) \subset D(C)$ for every $x \in E$ and all $T \in S$, it follows that $x^* \in F^*$ and hence

$$\langle T_\infty x, x^* \rangle = \langle x, x^* \rangle = 0 \qquad (x \in D(C)) .$$

Thus $x^* = 0$, because $C = T_\infty D(C)$ separates F^*. This and the separation theorem imply that $D(C) = E$.

Conversely, suppose $T_\infty D = C$ and $D = E$. For an $x^* \in F^*$ with $x^* \neq 0$, choose an $x \in E$ so that $\langle x, x^* \rangle \neq 0$. Then we have

$$\langle T_\infty x, x^* \rangle = \langle x, x^* \rangle \neq 0 ,$$

which proves that $C = T_\infty D$ separates F^*. The proof is complete.

COROLLARY 1. *Let E be a complete locally convex t.v.s. and $\mathfrak{S}$ a weakly right ergodic semigroup. Then the following conditions are equivalent*:

(a) *There exists a (unique) continuous linear operator P on E such that, for every $x \in E$ and all $T \in \mathfrak{S}$,*

$$Px \in \bar{A}(x) \quad \textit{and} \quad PT = TP = P^2 = P .$$

(b) *E is the direct sum of F and N, where N is the closed linear subspace of E determined by the set $\{x - Tx: x \in E \text{ and } T \in \mathfrak{S}\}$.*

(c) *F separates F^*.*

(d) *The set $\{x \in E: \bar{A}(x) \cap F \neq \emptyset\}$ is weakly dense in E.*

PROOF. (a) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (c): For an $x^* \in E^*$ with $x^* \neq 0$, take an $x \in E$ such that $\langle x, x^* \rangle \neq 0$. If $y \in \bar{A}(x) \cap F$ then we have

$$\langle y, x^* \rangle = \langle x, x^* \rangle \neq 0 .$$

Hence the implication (d) $\Rightarrow$ (c) follows.

(c) $\Rightarrow$ (b): Let $(T_\alpha, \alpha \in \Lambda)$ be a weakly right $\mathfrak{S}$-ergodic net, and define D and T_∞ as in Theorem 1. It is then clear that $F = T_\infty F \subset T_\infty D$, therefore if F separates F^* then Theorem 3 implies that $T_\infty D = F$ and $D = E$. Now Theorem 2 implies that $TT_\infty = T_\infty T = T_\infty$ (on E) for all $T \in \mathfrak{S}$. Therefore any $x \in E$ can be written as $x = T_\infty x + (x - T_\infty x)$, where $T_\infty x \in F$ and $x - T_\infty x \in D(0) = N$. Clearly $F \cap N = \{0\}$.

(b) $\Rightarrow$ (a): Suppose E is the direct sum of F and N. Then, by Theorem 2, we have $D = E$. Hence, letting $P = T_\infty$ (on E), (a) follows. This completes the proof.

COROLLARY 2. *Let E and $\mathfrak{S}$ be as in Corollary 1. Then the following conditions are equivalent*:

(a) *For every $x \in E$, $0 \in \bar{A}(x)$.*

(b) $E = N$.

(c) $F^* = \{0\}$.

(d) *The set* $\{x \in E: 0 \in \bar{A}(x)\}$ *is weakly dense in* E.

We omit the proof of Corollary 2.

PROPOSITION 2. *Let E be a complete locally convex t.v.s. and $\mathfrak{S}$ a semigroup of continuous linear operators on E. Suppose that $\mathfrak{S}$ posseses a weakly left $\mathfrak{S}$-ergodic net $(T_\alpha, \alpha \in \Lambda)$ satisfying condition* (S-I). *If the set $\{Tx: T \in \mathfrak{S}\}$ is relatively weakly compact in E, then $\bar{A}(x) \cap F \neq \emptyset$.*

PROOF. Since E is a complete locally convex t.v.s., Krein's theorem (cf. Theorem IV. 11.4 of [7]) implies that $\overline{\mathrm{co}}\, \mathfrak{S}x$ is again weakly compact. On the other hand, we have $T(\overline{\mathrm{oc}}\, \mathfrak{S}x) \subset \overline{\mathrm{oc}}\, \mathfrak{S}x$ and weak-$\lim_\alpha TT_\alpha x - T_\alpha x = 0$ for every $T \in \mathfrak{S}$. Thus, it follows from an easy compactness argument that there exists an x in $\overline{\mathrm{co}}\, \mathfrak{S}x$ which is a fixed point of $\mathfrak{S}$. This completes the proof.

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