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ON THE MULTIPLICITIES OF THE SPECTRUM FOR QUASI-CLASSICAL MECHANICS ON SPHERES

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0. Introduction. A. Weinstein [8] presented a quasi-classical calculation of the energy spectrum for a free particle moving on a sphere of constant curvature in any dimension. He showed that the quasi-classical spectrum resembles quite closely, in terms of both eigenvalues and multiplicities, the spectrum of the quantum hamiltonian $\Delta/2$. For the case of d-sphere S^d of constant sectional curvature one, his result is as follows: The quasi-classical eigenvalues are

$$\lambda_n = \frac{1}{2}\left(n + \frac{d-1}{2}\right)^2 \qquad \left(n > \frac{d-1}{2}\right),$$

and the multiplicity of λ_n is

$$m(\lambda_n) = \begin{cases} 2\begin{pmatrix} n + \dfrac{d-3}{2} \\ d-1 \end{pmatrix} & d \text{ odd}, \\ 2\begin{pmatrix} n + \dfrac{d-2}{2} \\ d-1 \end{pmatrix} & d \text{ even}. \end{cases}$$

Note that the counting starts with $n = (d+1)/2 (d \text{ odd})$ or $n = d/2 (d \text{ even})$. It is well-known that the eigenvalues of the quantum hamiltonian $\Delta/2$ on S^d are

$$\mu_n = \frac{1}{2}n(n+d-1) = \frac{1}{2}\left(n + \frac{d-1}{2}\right)^2 - \frac{(d-1)^2}{8},$$

and the multiplicity of μ_n is

$$m(\mu_n) = \frac{2n+d-1}{n}\begin{pmatrix} n+d-2 \\ n-1 \end{pmatrix},$$

where the counting starts with $n = 0$. See Berger-Gauduchon-Mazet [2].

In this note, we will present a slightly modified calculation of the quasi-classical energy spectrum for a free particle moving on S^d and show that the multiplicities of the resulting quasi-classical eigenvalues are equal to the multiplicities of the corresponding quantum eigenvalues. Our result is as follows: The quasi-classical eigenvalues are

$$\nu_n = \frac{1}{2}\left(n + \frac{d-1}{2}\right)^2,$$

and the multiplicity of ν_n is

$$m(\nu_n) = \frac{2n+d-1}{n}\binom{n+d-2}{n-1},$$

where the counting starts with $n = 0$.

1. **Preliminaries.** Let $(\ |\ )$ denote the hermitian inner product and $\sqrt{-1}$ denote the complex structure in the complex d-space $\boldsymbol{C}^d = \boldsymbol{R}^d \oplus \sqrt{-1}\boldsymbol{R}^d$. The symplectic structure $[\ ,\ ]$ in $\boldsymbol{C}^d$ is defined by $[u, v] = \operatorname{Im}(u|v)$ for $u, v \in \boldsymbol{C}^d$. A linear subspace L in $\boldsymbol{C}^d$ is called Lagrangian if $\dim L = d$ and $[u, v] = 0$ for all $u, v \in L$. The real subspace $\boldsymbol{R}^d$ and the imaginary subspace $\sqrt{-1}\boldsymbol{R}^d$ are Lagrangian. Let $\Lambda(d)$ denote the set of all Lagrangian subspaces in $\boldsymbol{C}^d$. $\Lambda(d)$ is a manifold, $\Lambda(d) \cong U(d)/O(d)$ (Arnol'd [1]). For any $L \in \Lambda(d)$, there exists a $U \in U(d)$ such that $L = U(\sqrt{-1}\boldsymbol{R}^d)$. Although U is not determined uniquely, $W(L) = U\bar{U}^{-1}$ is determined uniquely for each L. $W(L)$ is a symmetric unitary matrix (Leray[4]). Let us define $\mathrm{Det}^2\colon \Lambda(d) \to S^1$ by $\mathrm{Det}^2(L) = \det W(L)$, where S^1 is the circle $\{e^{\sqrt{-1}\theta}\}$, oriented counterclockwise. The one-dimensional homology and cohomology groups of $\Lambda(d)$ are free cyclic: $H_1(\Lambda(d), \boldsymbol{Z}) \cong H^1(\Lambda(d), \boldsymbol{Z}) \cong \boldsymbol{Z}$. For the generator of the cohomology group $H^1(\Lambda(d), \boldsymbol{Z})$, we may take the cocycle α whose value on an oriented closed curve $\tilde{\gamma}\colon S^1 \to \Lambda(d)$ is equal to the degree of the composition $\mathrm{Det}^2 \circ \tilde{\gamma}\colon S^1 \to S^1$ (Arnol'd [1]).

EXAMPLE (Arnol'd [1]). Consider a one-parameter group of automorphisms

$$T_k(\theta) = \begin{pmatrix} e^{\sqrt{-1}\theta} & & & & & \\ & \ddots & & & & \\ & & e^{\sqrt{-1}\theta} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{matrix} \}k \\ \\ \}d-k\,. \end{matrix}$$

For any $L = U(\sqrt{-1}\boldsymbol{R}^d) \in \Lambda(d)$, let us define an oriented closed curve $\tilde{\gamma}_{k,L}\colon [0, \pi] \to \Lambda(d)$ by $\tilde{\gamma}_{k,L}(\theta) = T_k(\theta)(L)$. Then

$$W(T_k(\theta)(L)) = (T_k(\theta)U)(\overline{T_k(\theta)U})^{-1} = T_k(\theta)U\bar{U}^{-1}\overline{T_k(\theta)}^{-1} = T_k(\theta)W(L)T_k(\theta) .$$

Therefore

$$\mathrm{Det}^2(T_k(\theta)(L)) = \det(T_k(\theta)W(L)T_k(\theta)) = e^{2\sqrt{-1}k\theta}\,\mathrm{Det}^2(L) .$$

Thus the value of the class α on the curve $\tilde{\gamma}_{k,L}$ is equal to k.

Now, following Fujiwara [3], let us define the Maslov index. Let M be a d-dimensional complete Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$ and Levi-Civita connection ∇. We may identify the tangent bundle TM with the cotangent bundle T^*M by means of the metric, and we then have the connection map K: $TT^*M \to TM$. Let π: $T^*M \to M$ be the projection. For any $\xi \in T^*M$, the d-dimensional subspaces $h_\xi =$ kernel $(K|_{T_\xi T^*M})$ and $v_\xi = \mathrm{kernel}(\pi_*|_{T_\xi T^*M})$ are called the horizontal subspace and the vertical subspace in $T_\xi T^*M$, respectively. Note that $T_\xi T^*M = h_\xi \oplus v_\xi$. The Sasaki metric $\langle\langle\ ,\ \rangle\rangle$ in T^*M is defined by $\langle\langle X, Y\rangle\rangle = \langle\pi_*(X), \pi_*(Y)\rangle + \langle K(X), K(Y)\rangle$ for $X, Y \in T_\xi T^*M$. The restrictions $K|_{v_\xi}$: $v_\xi \to T_{\pi(\xi)}M$ and $\pi_*|_{h_\xi}$: $h_\xi \to T_{\pi(\xi)}M$ are isometries. Let J: $TT^*M \to TT^*M$ denote an almost complex structure defined by $\pi_* \circ J = -K$ and $K \circ J = \pi_*$. In local coordinates $(y_1, \cdots, y_d; \eta_1, \cdots, \eta_d)$ in T^*M associated with a normal coordinate system $(y_1, \cdots, y_d)$ about $\pi(\xi)$ in M, $J_\xi = J|_{T_\xi T^*M}$: $T_\xi T^*M \to T_\xi T^*M$ is given by $J_\xi(\partial/\partial y_j) = \partial/\partial\eta_j$ and $J_\xi(\partial/\partial\eta_j) = -\partial/\partial y_j$ for $j = 1, 2, \cdots, d$. Note that $J(v_\xi) = h_\xi$ and $J(h_\xi) = v_\xi$.

The canonical one-form on T^*M will be denoted by ω. Let us recall $\omega|_\xi = \pi^*(\xi)$ for $\xi \in T^*M$. The symplectic form Ω is defined by $\Omega = d\omega$. In local coordinates $(y_1, \cdots, y_d; \eta_1, \cdots, \eta_d)$ in T^*M associated with local coordinates $(y_1, \cdots, y_d)$ in M, we have $\omega|_\xi = \sum_{j=1}^d \xi_j dy_j$ and $\Omega|_\xi = \sum_{j=1}^d d\eta_j \wedge dy_j$, where $\xi_j = \xi(\partial/\partial y_j)$. $\Omega|_\xi$ introduces a symplectic structure in $T_\xi T^*M$. Thus we have a notion of Lagrangian subspaces in $T_\xi T^*M$. The following three lemmas are due to Fujiwara [3].

LEMMA 1. *h_ξ and v_ξ are Lagrangian subspaces in $T_\xi T^*M$.*

For a linear isometry φ_ξ: $h_\xi \to \boldsymbol{R}^d$, we define a linear map Φ_ξ: $T_\xi T^*M \to \boldsymbol{C}^d = \boldsymbol{R}^d \oplus \sqrt{-1}\boldsymbol{R}^d$ as follows: $\Phi_\xi|_{h_\xi} = \varphi_\xi$ and $\Phi_\xi(X) = -\sqrt{-1} \circ \varphi_\xi \circ J_\xi(X)$ for $X \in v_\xi$. Then Φ_ξ is bijective and we have

LEMMA 2. (i) *Φ_ξ: $T_\xi T^*M \to \boldsymbol{C}^d$ is a symplectic linear map.*

(ii) *If Ψ_ξ is another symplectic map such as Φ_ξ, then $\Psi_\xi \circ \Phi_\xi^{-1} \in O(d) \subset U(d)$.*

Let $\Lambda(T_\xi T^*M)$ denote the set of all Lagrangian subspaces in the symplectic space $T_\xi T^*M$. Since the image of a Lagrangian subspace under a symplectic linear map is a Lagrangian subspace, Φ_ξ induces a map $\tilde{\Phi}_\xi: \Lambda(T_\xi T^*M) \to \Lambda(d)$.

LEMMA 3. *$\tilde{\Phi}_\xi = \tilde{\Psi}_\xi$, where Ψ_ξ is as in Lemma 2.*

For the proof, it is sufficient to note that $\Lambda(d) \cong U(d)/O(d)$. A submanifold L' in T^*M is called isotropic if the pull-back of Ω to L' is identically zero. An isotropic submanifold L is called Lagrangian if $\dim L = \dim M$. Let $\gamma: S^1 \to L$ be an oriented closed curve on a Lagrangian submanifold L. Then the tangent space $T_{\gamma(t)}L$ to L at $\gamma(t)$ is a Lagrangian subspace in $T_{\gamma(t)}T^*M$. Thus we have an oriented closed curve $\tilde{\gamma}: S^1 \to \Lambda(d)$ defined by $\tilde{\gamma}(t) = \tilde{\Phi}_{\gamma(t)}(T_{\gamma(t)}L)$.

DEFINITION (Fujiwara [3]). The Maslov index $\mathrm{Ind}_L \gamma$ of γ on L is defined by $\mathrm{Ind}_L \gamma = \alpha(\tilde{\gamma})$.

The free Hamiltonian $h: T^*M \to \boldsymbol{R}$ is defined by $h(v) = 1/2\langle v, v\rangle$. The Hamiltonian vector field H on T^*M for which $H \lrcorner\, \Omega = dh$ generates the geodesic flow $\chi = \{\chi_t\}$ on T^*M. In local coordinates $(y_1, \cdots, y_d; \eta_1, \cdots, \eta_d)$ in T^*M associated with a normal coordinate system $(y_1, \cdots, y_d)$ about $\pi(\xi)$ in M, $H|_\xi$ is written as $H|_\xi = \sum_{j=1}^d \xi_j(\partial/\partial\eta_j)$. A closed Lagrangian submanifold L in T^*M is called a quasi-classical state if it satisfies the Maslov's quantization condition (Maslov [5]):

For any oriented closed curve γ on L,

$$\frac{1}{2\pi}\int_\gamma \omega - \frac{1}{4}\mathrm{Ind}_L(\gamma) \quad \text{is an integer}\,.$$

A quasi-classical state L in T^*M is called a quasi-classical eigenstate if h is constant on L. The value of h on a quasi-classical eigenstate L is called the eigenvalue of L. A foliation $\mathscr{L}$ on T^*M is called a Jacobi foliation if its leaves are Lagrangian submanifolds in T^*M. A Jacobi foliation corresponds classically to an orthogonal decomposition of the quantum state space. See Slawionowski [6], [7]. In the next section, we will construct a Jacobi foliation $\mathscr{L}_d$ on the cotangent bundle T^*S^d of the d-sphere S^d, and select from its leaves all those which satisfy the quantization condition. See Weinstein [8].

2. Calculations for spheres.

We will consider the spheres

$$S^k = \{(x_0, \cdots, x_d) \in \boldsymbol{R}^{d+1} \mid x_0^2 + \cdots + x_d^2 = 1,\ x_{k+1} = \cdots = x_d = 0\}$$

for $k = 1, 2, \cdots, d$, with the Riemannian metric induced from the Euclidean metric $dx_0^2 + \cdots + dx_d^2$ on $\boldsymbol{R}^{d+1}$. Let $1 \leqq k < m \leqq d$. S^k is a totally geodesic submanifold in S^m. The (co)tangent bundle of S^k with the Sasaki metric is naturally imbedded as a totally geodesic submanifold in that of S^m. The canonical one-form and the symplectic form on T^*S^m pull back to the corresponding forms on T^*S^k. The restriction of the free Hamiltonian on T^*S^m to T^*S^k is that of T^*S^k. The geodesic flow on T^*S^m leaves T^*S^k invariant, and its restriction to T^*S^k is the geodesic flow on T^*S^k. Thus there may be no confusion if we write the canonical one-form ω, the symplectic form Ω, the free Hamiltonian h, the Hamiltonian vector field H, the geodesic flow $\chi = \{\chi_t \mid t \in \boldsymbol{R}\}$ on T^*S^k, for any $k = 1, 2, \cdots, d$. As before, we will identify the tangent bundle and the cotangent bundle of S^k. Restricting the vector field dx_{k+1} on $\boldsymbol{R}^{d+1}$ to S^k, we have a normal cross-section $X_{k+1} = dx_{k+1}|_{S^k}: S^k \to T^*S^{k+1}$. X_1 denotes the tangent vector $dx_1|_p$ to S^1 at $p = (1, 0, \cdots, 0) \in \boldsymbol{R}^{d+1}$. We will first construct a Jacobi foliation $\mathscr{L}_d$ on T^*S^d with a small exceptional set. For any $a_1 \in \boldsymbol{R}$, let $L(a_1) = \{\chi_t(a_1X_1) \in T^*S^1 \mid t \in \boldsymbol{R}\}$. Then $\mathscr{L}_1 = \{L(a_1) \mid a_1 \in \boldsymbol{R}\}$ is a Jacobi foliation on T^*S^1. Let $\pi: T^*S^k \to S^k$ denote the projection. Regarding $L(a_1)$ as a subset of T^*S^2, we have a submanifold $L(a_1) \# [a_2] = \{Y_1 + a_2X_2|_{\pi(Y_1)} \mid Y_1 \in L(a_1)\}$ in T^*S^2, for any $a_2 > 0$. $L(a_1, a_2) = \{\chi_t(Y_2) \mid Y_2 \in L(a_1) \# [a_2]\}$ is a two-dimensional submanifold in T^*S^2. Thus we have a foliation $\mathscr{L}_2 = \{L(a_1, a_2) \mid (a_1, a_2) \in \boldsymbol{R}^2, a_2 > 0\}$ on T^*S^2 with a small exceptional set. Iterating this procedure, we have a foliation $\mathscr{L}_k = \{L(a_1, \cdots, a_k) \mid (a_1, \cdots, a_k) \in \boldsymbol{R}^k, a_j > 0 \text{ for } j \geqq 2\}$ on T^*S^k with a small exceptional set. From the construction, it is easy to see that the leaves $L(a_1, \cdots, a_k)$ are k-dimensional tori which are invariant under the geodesic flow. The free Hamiltonian h is constant on $L(a_1, \cdots, a_k)$ and is equal to $(a_1^2 + \cdots + a_k^2)/2$.

LEMMA 4. *$L(a_1, \cdots, a_k)$ is a Lagrangian submanifold in T^*S^k.*

PROOF. We will prove this by induction. $L(a_1)$ is a Lagrangian submanifold in T^*S^1. If we assume that $L(a_1, \cdots, a_j)$ is a Lagrangian submanifold in T^*S^j, then an easy computation shows that $L(a_1, \cdots, a_j) \# [a_{j+1}]$ is an isotropic submanifold in T^*S^{j+1}. For any tangent vector Z to $L(a_1, \cdots, a_j) \# [a_{j+1}]$, $\Omega(Z, H) = dh(Z) = Z(h) = 0$, since h is constant on $L(a_1, \cdots, a_j) \# [a_{j+1}]$. The tangent space of $L(a_1, \cdots, a_{j+1})$ at a point $Y_{j+1} \in L(a_1, \cdots, a_j) \# [a_{j+1}]$ is spanned by the tangent space of $L(a_1, \cdots, a_j) \# [a_{j+1}]$ at Y_{j+1} and the Hamiltonian vector $H|_{Y_{j+1}}$ at Y_{j+1}. Since the geodesic flow leaves Ω invariant, it follows that $L(a_1, \cdots, a_{j+1})$ is a Lagrangian submanifold in T^*S^{j+1}. q.e.d.

Thus we have a Jacobi foliation $\mathscr{L}_d$ on T^*S^d with a small exceptional set. Now we will calculate the Maslov index for oriented closed curves on $L(a_1, \cdots, a_d)$. Let $S^1(r) \approx [0, 2\pi r]/\{0\} \cup \{2\pi r\}$ denote the oriented circle of radius r. For $k = 1, 2, \cdots, d$, let us define a curve $\gamma_k': S^1(r_k) \to$

$L(a_1, \cdots, a_k)$ by $\gamma'_k(t) = \chi_t(\sum_{j=1}^k a_j X_j|_p)$, where $r_k = (\sum_{j=1}^k a_j^2)^{-1/2}$ and $p = (1, 0, \cdots, 0)$. γ'_k is an oriented closed curve. $c_k = \pi\circ\gamma'_k$ is a closed geodesic, its tangent vector $\dot{c}_k(t)$ is equal to $\gamma'_k(t)$. Define a curve $\gamma_k: S^1(r_k) \to L(a_1, \cdots, a_d)$ by $\gamma_k(t) = \gamma'_k(t) + \sum_{j=k+1}^d a_j X_j|_{c_k(t)}$. γ_k is an oriented closed curve on $L(a_1, \cdots, a_d)$, $\pi\circ\gamma_k = c_k$. $\gamma_1, \cdots, \gamma_d$ are generators of the one-dimensional homology group $H_1(L(a_1, \cdots, a_d), \boldsymbol{Z}) \cong \bigoplus^d \boldsymbol{Z}$ of $L(a_1, \cdots, a_d)$.

LEMMA 5. $\mathrm{Ind}_{L(a_1,\ldots,a_d)}\gamma_k = 2(k-1)$.

PROOF. Let $Y_j^{(k)}: S^1(r_k) \to TS^d$, $j = 1, 2, \cdots, d$, be parallel vector fields along c_k such that $\{Y_j^{(k)}(t)\}_{j=1,\ldots,d}$ forms an orthonormal basis for the tangent space to S^d at $c_k(t)$, $Y_1^{(k)}(t) = r_k\dot{c}_k(t)$, and $Y_j^{(k)}(t)$, $j = 1, \cdots, k$, are tangent to S^k. For a small $\varepsilon > 0$, a map $f(y_1, \cdots, y_d) = \exp_{c_k(y_1)}(\sum_{j=2}^d y_j Y_j^{(k)}(y_1))$ is a diffeomorphism from $S^1(r_k) \times (-\varepsilon, \varepsilon)^{d-1}$ into S^d, where exp is the exponential map. Thus we have a Fermi coordinate system $(y_1, \cdots, y_d)$ along c_k. Let $(y_1, \cdots, y_d; \eta_1, \cdots, \eta_d)$ denote the associated coordinate system in T^*S^d. We represent a tangent vector to T^*S^d by its component with respect to this coordinate system. We regard $L(a_1, \cdots, a_k)$ as a submanifold in T^*S^d. If the tangent space to $L(a_1, \cdots, a_k)$ at $\gamma'_k(t)$ is spanned by vectors $(a_{i1}, \cdots, a_{ik}, 0, \cdots, 0; b_{i1}, \cdots, b_{ik}, 0, \cdots, 0)$, $i = 1, \cdots, k$, then the tangent space to $L(a_1, \cdots, a_k) \# [a_{k+1}]$ at $\gamma'_k(t) + a_{k+1}X_{k+1}|_{c_k(t)}$ is spanned by $(a_{i1}, \cdots, a_{ik}, 0, \cdots, 0; b_{i1}, \cdots, b_{ik}, 0, \cdots, 0)$, $i = 1, \cdots, k$. Since the Hamiltonian vector at $\gamma'_k(t) + a_{k+1}X_{k+1}|c_{k(t)}$ is given by $(r_k^{-1}, 0, \cdots, 0, a_{k+1}, 0, \cdots, 0; 0, \cdots, 0)$, the tangent space to $L(a_1, \cdots, a_{k+1})$ at $\gamma'_k(t) + a_{k+1}X_{k+1}|_{c_k(t)}$ is spanned by the vectors $(a_{i1}, \cdots, a_{ik}, 0, \cdots, 0; b_{i1}, \cdots, b_{ik}, 0, \cdots, 0)$, $i = 1, \cdots, k$, and $(r_k^{-1}, 0, \cdots, 0, a_{k+1}, 0, \cdots, 0; 0, \cdots, 0)$. Similarly, the tangent space to $L(a_1, \cdots, a_{k+j})$ at $\gamma'_k(t) + \sum_{i=1}^j a_{k+i}X_{k+i}|_{c_k(t)}$ is spanned by the vectors $(a_{i1}, \cdots, a_{ik}, 0, \cdots, 0; b_{i1}, \cdots, b_{ik}, 0, \cdots, 0)$, $i = 1, \cdots, k$, and $(r_k^{-1}, 0, \cdots, 0, a_{k+1}, 0, \cdots, 0; 0, \cdots, 0), \cdots, (r_k^{-1}, 0, \cdots, 0, a_{k+1}, a_{k+2}, \cdots, a_{k+j}, 0, \cdots, 0; 0, \cdots, 0)$. The vector $(1, 0, \cdots, 0; 0, \cdots, 0)$ is tangent to $L(a_1, \cdots, a_d)$ at $\gamma_k(t)$. It follows that the tangent space to $L(a_1, \cdots, a_d)$ at $\gamma_k(t)$ is spanned by the vectors $(a_{i1}, \cdots, a_{ik}, 0, \cdots, 0; b_{i1}, \cdots, b_{ik}, 0, \cdots, 0)$, $i = 1, \cdots, k$, and $\partial/\partial y_{k+1}, \cdots, \partial/\partial y_d$. Since $(y_1, \cdots, y_d)$ is a normal coordinate system about $c_k(t)$, $\{\partial/\partial y_1, \cdots, \partial/\partial y_d, \partial/\partial\eta_1, \cdots, \partial/\partial\eta_d\}$ is an orthonormal basis for the tangent space to T^*S^d at $\gamma_k(t)$. $\partial/\partial y_1, \cdots, \partial/\partial y_d$ are horizontal vectors at $\gamma_k(t)$. $\partial/\partial\eta_1, \cdots, \partial/\partial\eta_d$ are vertical vectors at $\gamma_k(t)$. $J(\partial/\partial y_j) = \partial/\partial\eta_j$ for $j=1, \cdots, d$. On the other hand, the tangent space to $L(a_1, \cdots, a_k)$ in T^*S^k at $\gamma'_k(t)$ is spanned by the vectors $(a_{i1}, \cdots, a_{ik}, 0, \cdots, 0; b_{i1}, \cdots, b_{ik}, 0, \cdots, 0)$, $i = 1, \cdots, k$. Thus, from the definition of the Maslov index, it follows easily that the Maslov index of the curve γ'_k on the Lagrangian submanifold $L(a_1, \cdots, a_k)$ in T^*S^k is equal to the Maslov index of the curve γ_k on the Lagrangian submanifold $L(a_1, \cdots, a_d)$ in T^*S^d. Since γ'_k is an orbit of the geodesic flow on T^*S^k, we have

$$\begin{aligned}\mathrm{Ind}_{L(a_1,\ldots,a_k)}\gamma'_k &= (\text{Morse index of } c_k \text{ in } S^k)\\ &= 2(k-1)\,.\end{aligned}$$

See Maslov [5], Weinstein [8]. From this our lemma follows. q.e.d.

Now we will calculate the action integral along γ_k.

$$\begin{aligned}\int_{\gamma_k}\omega &= \int_{\gamma_k}\pi^*(\gamma_k) = \int_{c_k}\gamma_k = \int_0^{2\pi r_k}\langle\gamma_k, \dot{c}_k\rangle dt = \int_0^{2\pi r_k}\langle\gamma'_k, \gamma'_k\rangle dt\\ &= 2\pi\Big(\sum_{j=1}^{k} a_j^2\Big)^{1/2}\,.\end{aligned}$$

Therefore, the quantization condition for the Lagrangian submanifold $L(a_1, \cdots\cdot, a_d)$ is written as

$$\Big(\sum_{j=1}^{k} a_j^2\Big)^{1/2} - \frac{k-1}{2} = n_k \in \boldsymbol{Z}\,,$$

for $k = 1, \cdots, d$. Thus we have

$$\begin{aligned}a_1 &= n_1\,,\\ a_2 &= \Big(\Big(n_2 + \frac{1}{2}\Big)^2 - n_1^2\Big)^{1/2} \quad \text{for} \quad n_2 \geqq |n_1|\,,\\ a_k &= \Big(\Big(n_k + \frac{k-1}{2}\Big)^2 - \Big(n_{k-1} + \frac{k-2}{2}\Big)^2\Big)^{1/2} \quad \text{for} \quad n_k \geqq n_{k-1}\,,\\ &\qquad\qquad k = 3, \cdots, d\,.\end{aligned}$$

It follows that the quasi-classical spectrum $\{\nu_n\}_{n=0}^{\infty}$ for the sphere S^d is given by

$$\nu_n = \frac{1}{2}\sum_{j=1}^{d} a_j^2 = \frac{1}{2}\Big(n + \frac{d-1}{2}\Big)^2\,.$$

See Weinstein [8]. The multiplicity $m(\nu_n)$ of ν_n is the number of the Lagrangian submanifolds $L(a_1, \cdots, a_d)$ which satisfy the quantization condition and $\sum_{j=1}^{d} a_j^2/2 = \nu_n$. Therefore $m(\nu_n)$ is equal to the number of d-tuples $(n_1, \cdots, n_d)$ of integers satisfying $0 \leqq |n_1| \leqq n_2 \leqq \cdots \leqq n_{d-1} \leqq n_d = n$. Thus we have

$$m(\nu_n) = \frac{2n+d-1}{n}\binom{n+d-2}{n-1}\,,$$

where the counting starts with $n = 0$. Despite the incompleteness of the Jacobi foliation $\mathscr{L}_d$, the quasi-classical multiplicity $m(\nu_n)$ of ν_n agrees with the quantum multiplicity $m(\mu_n)$ of μ_n.

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