Tôhoku Math. Journ.
31 (1979), 57-62.

ON THE ABSOLUTE CONTINUITY OF MEASURES RELATIVE TO A POISSON MEASURE

HIROAKI MORIMOTO

(Received November 7, 1977, revised March 1, 1978)

1. Introduction. Let (Ω, F, P) be a complete probability space with a non-decreasing right continuous family $(F_t)_{0 \leqq t < \infty}$ of sub σ-fields of F such that F_0 contains all null sets and $F = \bigvee_{t \geqq 0} F_t$. Let $N = (N_t)$ be a stochastic point process on (Ω, F, P), that is, an F_t-adapted process with right continuous paths taking values in $Z_+ = \{0, 1, 2, \cdots\}$ such that $N_0 = 0$ and $\Delta N_t = N_t - N_{t-} = 0$ or 1 for all t. We assume here that F_t is the σ-field generated by $(N_s, s \leqq t)$. If there exists a positive predictable process $\lambda = (\lambda_t)$ such that the process $\hat{N}$ defined by

$$\hat{N}_t = N_t - \int_0^t \lambda_s ds$$

is a local martingale, then (N, P) is said to be a stochastic point process with the intensity λ. Now, let P_0 be a Poisson measure; that is, (N_t) is a Poisson process with respect to P_0. J. H. Van Schuppen and E. Wong proved in [6] that if P is equivalent to P_0, then (N, P) is a stochastic point process with some intensity. Our interest lies in giving a necessary and sufficient condition for the equivalence $P \sim P_0$ under some assumptions.

Our aim is to prove the following theorems.

THEOREM 1. *Let (N, P) be a stochastic point process with the intensity λ and assume that*

$$P\left(\int_0^\infty (|\lambda_s - 1| + |\lambda_s \log \lambda_s|) ds < \infty\right) = 1 .$$

Then, P is equivalent to P_0 if and only if

$$E\left[\left(\prod_{\substack{s \\ N_s \neq N_{s-}}} \lambda_s^{-1}\right) \exp\left(\int_0^\infty (\lambda_s - 1) ds\right)\right] = 1 . \tag{1}$$

The sample paths of the process N have only a finite number of discontinuities in $[0, 1]$, and so we get:

THEOREM 2. *Let (N, P) be a stochastic point process with the intensity λ. Then, P is equivalent to P_0 on F_1 if and only if*

$$(2) \qquad E\Big[\Big(\prod_{\substack{s\leq 1\\ N_s\neq N_{s-}}} \lambda_s^{-1}\Big)\exp\Big(\int_0^1(\lambda_s-1)ds\Big)\Big]=1\,.$$

2. Preliminary lemmas. The reader is assumed to be familiar with the martingale theory given in [3].

Let M be a local martingale and L be the unique solution of the stochastic integral equation:

$$L_t = 1 + \int_0^t L_{s-}dM_s\,.$$

Obviously, L is a local martingale.

Our proofs are based on the following lemma obtained by J. H. Van Schuppen and E. Wong [6].

LEMMA 1. *Let X be a local martingale such that the process $\langle X, M\rangle$ exists. If L is a uniformly integrable positive martingale, then $dP' = L_\infty dP$ defines a probability measure and the process $(X_t - \langle X, M\rangle_t)$ is a P'-local martingale.*

For the proof, see [6]. We note that if $(L_{t\wedge 1})$ is a positive martingale, then $dP'' = L_1 dP$ defines also a probability measure and $X_{t\wedge 1} - \langle X, M\rangle_{t\wedge 1}$ is a P''-local martingale. As is proved in [3], if μ is a predictable process with $P\Big(\int_0^t|\mu_s|\lambda_s ds < \infty\Big) = 1$ for every t, then the Stieltjes integral $\int_0^t \mu_s d\hat{N}_s$ is a local martingale. The following lemma gives a sufficient condition for its integral to be a uniformly integrable martingale.

LEMMA 2. *If $\mu=(\mu_t)$ is a predictable process such that $E\Big[\int_0^\infty|\mu_s|\lambda_s ds\Big] < \infty$, then the process $\Big(\int_0^t\mu_s d\hat{N}_s\Big)$ is a uniformly integrable martingale.*

PROOF. It is sufficient to show that $\Big(\int_0^t\mu_s d\hat{N}_s\Big)$ is uniformly integrable. Let $T_n \uparrow \infty$ be stopping times such that for each n, $\Big(\int_0^{t\wedge T_n}|\mu_s|d\hat{N}_s\Big)$ is a martingale. Then for every n

$$E\Big[\int_0^{t\wedge T_n}|\mu_s|dN_s\Big] = E\Big[\int_0^{t\wedge T_n}|\mu_s|\lambda_s ds\Big]\,,$$

and so, letting $n\to\infty$, we find that $E\Big[\int_0^t|\mu_s|dN_s\Big] = E\Big[\int_0^t|\mu_s|\lambda_s ds\Big]$. Therefore,

$$E\Big[\sup_t\Big|\int_0^t\mu_s d\hat{N}_s\Big|\Big] \leqq E\Big[\int_0^\infty|\mu_s|\,|d\hat{N}_s|\Big] \leqq 2E\Big[\int_0^\infty|\mu_s|\lambda_s ds\Big]\,.$$

From the assumption it follows that $\Big(\int_0^t\mu_s d\hat{N}_s\Big)$ is uniformly integrable.

LEMMA 3. *Let $\mu = (\mu_t)$ be a predictable process with $P\left(\int_0^\infty |\mu_s|\lambda_s ds < \infty\right) = 1$. Then for any $a, b > 0$,*

$$P\left(\sup_t \left|\int_0^t \mu_s d\hat{N}_s\right| > a\right) \leqq (2b/a) + P\left(\int_0^\infty |\mu_s|\lambda_s ds > b\right).$$

PROOF. Let $T = \inf\left\{t; \int_0^t |\mu_s|\lambda_s ds > b\right\}$ and $\mu'_t = \mu_t I_{\{t \leqq T\}}$. Then the process μ' is a predictable process such that $E\left[\int_0^\infty |\mu'_s|\lambda_s ds\right] \leqq b$, and so by Lemma 2 $\left(\int_0^t \mu'_s d\hat{N}_s\right)$ is a uniformly integrable martingale. Therefore, from the definition of T and Doob's maximal inequality it follows that

$$\begin{aligned} P\left(\sup_t \left|\int_0^t \mu_s d\hat{N}_s\right| > a\right) &\leqq P\left(\sup_t \left|\int_0^t \mu_s d\hat{N}_s\right| > a,\ T = \infty\right) + P(T < \infty) \\ &\leqq P\left(\sup_t \left|\int_0^t \mu'_s d\hat{N}_s\right| > a\right) + P\left(\int_0^\infty |\mu_s|\lambda_s ds > b\right) \\ &\leqq \left(E\left[\int_0^\infty |\mu'_s|\,|d\hat{N}_s|\right]\Big/ a\right) + P\left(\int_0^\infty |\mu_s|\lambda_s ds > b\right). \end{aligned}$$

By the definition of μ', the first term on the right hand side is smaller than $2b/a$. Thus the lemma is proved.

3. Proof of Theorem 1. Sufficiency: Let $M_t = \int_0^t (\lambda_s^{-1} - 1) d\hat{N}_s$, which is a local martingale. Then the solution L of $L_t = 1 + \int_0^t L_{s-} dM_s$ is given by

$$L_t = \left(\prod_{\substack{s \leqq t \\ N_s \neq N_{s-}}} \lambda_s^{-1}\right) \exp\left(\int_0^t (\lambda_s - 1) ds\right),$$

which can be rewritten as $\exp\left(-\int_0^t \log \lambda_s d\hat{N}_s - \int_0^t (1 - \lambda_s + \lambda_s \log \lambda_s) ds\right)$. Let $L_\infty = (\prod_{\substack{s \\ N_s \neq N_{s-}}} \lambda_s^{-1}) \exp\left(\int_0^\infty (\lambda_s - 1) ds\right)$. By Lemma 3,

$$P\left(\sup_t \left|\int_0^t \log \lambda_s d\hat{N}_s\right| > a\right) \leqq (2b/a) + P\left(\int_0^\infty |\log \lambda_s|\lambda_s ds > b\right)$$

and letting $a \to \infty$ and then $b \to \infty$ we find that $P\left(\left|\int_0^\infty \log \lambda_s d\hat{N}_s\right| < \infty\right) = 1$. Thus L_t converges to $L_\infty > 0$ a.s. By the assumption (1), $E[L_\infty] = 1$, and so L is a uniformly integrable positive martingale. Let $dP' = L_\infty dP$. As $L_\infty > 0$, P' is equivalent to P. From Lemma 1, it follows that $\hat{N}_t - \langle \hat{N}, M \rangle_t = N_t - t$ is a P'-local martingale, that is, $P' = P_0$ on F.

Necessity: The process L defined by $L_t = E[dP_0/dP \mid F_t]$ is a uniformly integrable positive martingale with $L_0 = 1$. Let us consider the local martingale M defined by $M_t = \int_0^t L_{s-}^{-1} dL_s$. Then, L is a solution of $L_t = 1 + \int_0^t L_{s-} dM_s$, and the process M has a representation $M_t = \int_0^t \nu_s d\hat{N}_s$ for a predictable and integrable process ν ([1], p. 1018). Therefore, setting $\mu = \nu + 1$, we have

$$L_t = \Big(\prod_{\substack{s \leqq t \\ N_s \neq N_{s-}}} \mu_s\Big) \exp\Big(-\int_0^t (\mu_s - 1)\lambda_s ds\Big) .$$

By Lemma 1, $\hat{N}_t - \langle \hat{N}, M \rangle_t = N_t - \int_0^t \mu_s \lambda_s ds$ is a P_0-local martingale, and so we have $\mu = \lambda^{-1}$ because $N_t - t$ is a P_0-local martingale. Then, L_t is rewritten as follows:

$$L_t = \Big(\prod_{\substack{s \leqq t \\ N_s \neq N_{s-}}} \lambda_s^{-1}\Big) \exp\Big(\int_0^t (\lambda_s - 1) ds\Big) .$$

Consequently, letting $t \to \infty$, the theorem is established.

4. **Proof of Theorem 2.** Sufficiency: Let $M_t = \int_0^{t \wedge 1} (\lambda_s^{-1} - 1) d\hat{N}_s$, which is a local martingale. The solution L of $L_t = 1 + \int_0^t L_{s-} dM_s$ is given by

$$L_t = \Big(\prod_{\substack{s \leqq t \wedge 1 \\ N_s \neq N_{s-}}} \lambda_s^{-1}\Big) \exp\Big(\int_0^{t \wedge 1} (\lambda_s - 1) ds\Big) .$$

Since the sample paths of N have only a finite number of discontinuities in $[0, 1]$, L is a positive supermartingale. Furthermore, by the assumption (2), $E[L_1] = 1$, and so L is a martingale. Let $dP'' = L_1 dP$. Then P'' is equivalent to P on F_1, and by Lemma 1, $\hat{N}_{t\wedge 1} - \langle \hat{N}, M \rangle_{t \wedge 1} = N_{t \wedge 1} - t \wedge 1$ is a P''-local martingale. This implies that $P'' = P_0$ on F_1.

Necessity: Let D be the Radon-Nikodym derivative of P_0 with respect to P on F_1. Then the process L defined by $L_t = E[D | F_t]$ is a uniformly integrable positive martingale such that $L_0 = 1$ and $L_t = L_{t \wedge 1}$ for all t. The same argument as in the proof of the necessity of Theorem 1 gives

$$L_t = \Big(\prod_{\substack{s \leqq t \wedge 1 \\ N_s \neq N_{s-}}} \lambda_s^{-1}\Big) \exp\Big(\int_0^{t \wedge 1} (\lambda_s - 1) ds\Big) .$$

Thus, the theorem is established.

5. **Sufficient conditions.** We first give a sufficient condition for P and P_0 to be equivalent.

PROPOSITION 1. *Suppose that for some constant K,*

$$P\left(\int_0^\infty |\lambda_s \log \lambda_s| ds < \infty, \int_0^\infty |\lambda_s - 1| ds \leqq K\right) = 1 .$$

Then λ *satisfies* (1).

PROOF. Recall that

$$L_t = \Big(\prod_{\substack{s \leq t \\ N_s \neq N_{s-}}} \lambda_s^{-1}\Big) \exp\left(\int_0^t (\lambda_s - 1) ds\right)$$

is a positive local martingale. Thus to prove (1), it suffices to show that L is uniformly integrable. For that purpose, define the local martingale M' by

$$M'_t = \int_0^t (\lambda_s^{-1} - 1) I_{\{\lambda_s < 1\}} d\hat{N}_s .$$

Then the solution L' of the equation $L'_t = 1 + \int_0^t L'_{s-} dM'_s$ is given by

$$\begin{aligned} L'_t &= \Big(\prod_{\substack{s \leq t \\ N_s \neq N_{s-}}} (1 + (\lambda_s^{-1} - 1) I_{\{\lambda_s < 1\}})\Big) \exp\left(\int_0^t (\lambda_s - 1) I_{\{\lambda_s < 1\}} ds\right) \\ &= \exp\left(- \int_0^t (\log \lambda_s) I_{\{\lambda_s < 1\}} d\hat{N}_s - \int_0^t (1 - \lambda_s + \lambda_s \log \lambda_s) I_{\{\lambda_s < 1\}} ds\right) . \end{aligned}$$

By Lemma 3 it is easy to see that $P\left(\left|\int_0^\infty (\log \lambda_s) I_{\{\lambda_s < 1\}} d\hat{N}_s\right| < \infty\right) = 1$. Thus L'_t converges to L'_∞ a.s. Furthermore, we have $E[L'_\infty] \leqq 1$, because L' is a positive supermartingale. By a simple calculation,

$$\begin{aligned} \sup_t L_t &= \sup_t \exp\left(- \int_0^t \log \lambda_s dN_s + \int_0^t (\lambda_s - 1) ds\right) \\ &\leqq \exp\left(- \int_0^\infty (\log \lambda_s) I_{\{\lambda_s < 1\}} dN_s + \int_0^\infty |\lambda_s - 1| ds\right) \\ &\leqq L'_\infty \exp\left(2 \int_0^\infty |\lambda_s - 1| ds\right) , \end{aligned}$$

so that we have $E[\sup_t L_t] < \infty$. Consequently, L is uniformly integrable.

Similarly, we can give a sufficient condition for P and P_0 to be equivalent on F_1.

PROPOSITION 2. *Suppose that for some constant* K,

$$P\left(\int_0^1 \lambda_s ds \leqq K\right) = 1 .$$

Then λ *satisfies* (2).

The proof is the same as that of Proposition 1 with t replaced by $t \wedge 1$.

REFERENCES

[1] R. BOEL, P. VARAIYA, AND E. WONG, Martingales on jump processes I, II, SIAM J. Control, 13 (1975), 999-1021, 1022-1061.

[2] P. BREMAUD, The martingale theory of point processes over the real half-line admitting an intensity, Lecture Notes in Economics and Mathematical Systems, 107, Springer-Verlag, (1974), 519-542.

[3] C. DOLÉANS-DADE AND P. A. MEYER, Intégrales stochastiques par rapport aux martingales locales, Séminaire de Probabilités IV, Université de Strasbourg, Lecture Notes in Math. 124, Springer-Verlag, (1970), 77-107.

[4] T. KAILATH AND M. ZAKAI, Absolute continuity and Radon-Nikodym derivatives for certain measures relative to Wiener measure, Ann. of Math. Statist., 42 (1971), 130-140.

[5] R. S. LIPCER AND A. N. SIRYAEV, On the absolute continuity of measures corresponding to processes of diffusion type relative to a Wiener measure, Math. USSR Izvestija, 6 (1972), 839-882.

[6] J. H. VAN SCHUPPEN AND E. WONG, Transformation of local martingales under a change of law, Ann. of Prob., 2 (1974), 879-888.

TÔHOKU INSTITUTE OF TECHNOLOGY
SENDAI, 982 JAPAN