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SCHWARZIAN DERIVATIVES OF SOME CONFORMAL MAPPINGS

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1. Introduction. As is well known, Nehari [5] proved the following theorem.

If f is a univalent meromorphic function defined in the unit disc, then

$$\sup_{|z|<1} |[f](z)|(1-|z|^2)^2 \leqq 6 , \tag{1}$$

where $[f]$ is the Schwarzian derivative of f.

In this note, we are concerned with the case where the equality in (1) holds.

It is also well known that the Schwarzian derivatives of conformal mappings of the unit disc onto circular polygons are certain rational functions (see, for example, Goluzin [2]). First, by using such conformal mappings, we show that there exists a univalent meromorphic function for which the equality in (1) holds and whose Schwarzian derivative lies on the boundary of the Teichmüller space for a cyclic Fuchsian group. We also give a necessary condition in order that the equality in (1) holds and give an application of it.

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2. Notations and definitions. Let D be a simply connected domain in the extended complex plane $\hat{C}$ with more than one boundary point and let ρ_D be the Poincaré density of D, for example, $\rho_D(z) = (1-|z|^2)^{-1}$ if D is the unit disc. For a function ϕ holomorphic in D we introduce the norm

$$\|\phi\|_D = \sup_{z \in D} |\phi(z)| \rho_D(z)^{-2} .$$

We denote by $B_2(D, 1)$ the Banach space consisting of all the holomorphic functions ϕ in D which satisfy $\|\phi\|_D < \infty$.

For a locally univalent meromorphic function f in D, let $[f]$ be the Schwarzian derivative of f, that is,

$$[f](z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 .$$

It is well known that $\|[f]\|_D \leqq 12$ for functions f univalent meromorphic in D and, in particular, $\|[f]\|_\Delta \leqq 6$ for functions f univalent meromorphic in the unit disc Δ (Lehto [4] and Nehari [5]).

Let A be an open disc in $\hat{C}$. We define the subset $T(A, 1)$ of $B_2(A, 1)$ as the set of functions $\phi \in B_2(A, 1)$, each of which is the Schwarzian derivative of some univalent meromorphic function f in A with the image domain $f(A)$ bordered by a quasi-circle. It is well known that $T(A, 1)$ is a bounded domain in $B_2(A, 1)$.

Let Γ be a Fuchsian group keeping a disc A invariant. We denote by $B_2(A, \Gamma)$ the closed subspace of $B_2(A, 1)$ consisting of those $\phi \in B_2(A, 1)$ which satisfy $\phi(\gamma(z))(\gamma'(z))^2 = \phi(z)$ for every $\gamma \in \Gamma$ and every $z \in A$. We define the subset $T(A, \Gamma)$ of $B_2(A, \Gamma)$ as the connected component of $T(A, 1) \cap B_2(A, \Gamma)$ containing the origin of $B_2(A, \Gamma)$. For a Fuchsian group Γ with $\dim T(A, \Gamma) > 0$, we set

$$o(A, \Gamma) = \sup_{\phi \in T(A, \Gamma)} \|\phi\|_A$$

and call it the outradius of $T(A, \Gamma)$.

Let g be a Möbius transformation. The mapping χ which takes $\phi \in B_2(g(A), g\Gamma g^{-1})$ into $(\phi \circ g)(g')^2 \in B_2(A, \Gamma)$ is a norm-preserving linear isomorphism and the image $\chi(T(g(A), g\Gamma g^{-1}))$ of $T(g(A), g\Gamma g^{-1})$ under χ coincides with $T(A, \Gamma)$. In particular, we have $o(g(A), g\Gamma g^{-1}) = o(A, \Gamma)$.

In the special case where A is the unit disc Δ, we write briefly ρ, $\|\ \|$, $B_2(1)$, $T(1)$, $B_2(\Gamma)$, $T(\Gamma)$ and $o(\Gamma)$ without indicating the disc Δ and we call $T(1)$, $T(\Gamma)$ and $o(\Gamma)$ the universal Teichmüller space, the Teichmüller space for Γ and the outradius of $T(\Gamma)$, respectively.

3. Domains bounded by circular polygons.

3.1. Let P be a simply connected polygonal domain in $\hat{C}$ with its boundary consisting of n circular arcs or straight line segments. Straight line segments are regarded as arcs on circles with infinite radius. We denote by $A_1, \cdots, A_n$ the endpoints of these n arcs which are the vertices of the polygonal domain P, and we denote by $\pi\alpha_j$ the interior angle (with respect to P) at the vertex A_j ($j = 1, \cdots, n$).

There exists a function f which maps the unit disc Δ onto P conformally and maps $\bar{\Delta}$ onto $\bar{P}$ homeomorphically. We denote by a_j the point on the unit circle $\partial\Delta$ corresponding to the vertex A_j ($j = 1, \cdots, n$).

As is seen in Goluzin [2], it is known that

$$[f](z) = \sum_{j=1}^{n} \left(\frac{1-\alpha_j^2}{2(z-a_j)^2} + \frac{C_j}{z-a_j} \right), \tag{2}$$

where C_j $(j = 1, \cdots, n)$ are constants satisfying

$$\begin{cases} \sum_{j=1}^{n} C_j = 0 , \\ \sum_{j=1}^{n} (1-\alpha_j^2) + 2\sum_{j=1}^{n} C_j a_j = 0 , \\ \sum_{j=1}^{n} (1-\alpha_j^2) a_j + \sum_{j=1}^{n} C_j a_j^2 = 0 . \end{cases} \tag{3}$$

It should be noted that three of the points a_j $(j = 1, \cdots, n)$ can be chosen arbitrarily on $\partial\Delta$. In the case of a circular triangle, all the constants C_j $(j = 1, 2, 3)$ are determined by (3) for arbitrarily given points a_j $(j = 1, 2, 3)$. Furthermore, if $P_{n,q}$ is the interior of an n-sided circular polygon with vertices at $A_j = e^{i2\pi j/n}$ $(j = 1, \cdots, n)$ and with the interior angles $\alpha_j \pi = q\pi$ $(j = 1, \cdots, n;\ 0 \leqq q \leqq 2)$, then the mapping function $f_{n,q}$ of Δ onto $P_{n,q}$ with $f_{n,q}(0) = 0$ and $f'_{n,q}(0) > 0$ satisfies

$$[f_{n,q}](z) = n^2(1-q^2)z^{n-2}/2(z^n-1)^2 \tag{4}$$

(see Goluzin [2], p. 83).

Using (2), (3) and (4), we obtain the following proposition easily.

PROPOSITION. *Under the above notations, the following hold*:

(i) $\|[f]\| \geqq 2 \max_{1 \leqq j \leqq n} |1-\alpha_j^2|$.

(ii) *If* $\alpha_j = 2$ *for some* j, *then* $\|[f]\| = 6$.

(iii) *If* P *is a circular triangle with* $\alpha_j = 0$ *or* 2 *for some* j $(j = 1, 2, 3)$, *then* $[f]$ *lies on* $\partial T(1)$.

(iv) $\|[f_{n,q}]\| = 2|1-q^2|$.

(v) $[f_{n,q}]$ *lies on* $\partial T(1)$ *if* $q = 0$ *or* 2.

From (iv) we have $\|[f_{n,q}]\| \leqq 2$ for $0 \leqq q \leqq \sqrt{2}$. We also see that $P_{n,q}$ is not convex for $0 \leqq q < 1 - 2/n$ or $q > 1$. This shows that the *converse to the following theorem of Lehto* [4] *is not true*: If f is a conformal mapping of Δ onto a convex domain, then $\|[f]\| \leqq 2$.

3.2. Well known results of Nehari, Earle and Hille yield that $2 < o(\Gamma) \leqq 6$ for an arbitrary Fuchsian group Γ and $o(1) = 6$, where 1 denotes the group consisting of only the identity transformation. We also see $o(\Gamma) < 6$ for a finitely generated Fuchsian group Γ of the first kind (see [6]). Here we prove the following theorem.

THEOREM 1. *If* Γ *is a cyclic Fuchsian group acting on the unit disc* Δ, *then* $o(\Gamma) = 6$.

PROOF. Assume that Γ is a hyperbolic cyclic group. Let U be the upper half plane. Since $o(\Gamma)=o(\Delta,\Gamma)=o(U, g\Gamma g^{-1})$ for a Möbius transformation g which maps Δ onto U, we have only to show that $o(U, \Gamma')=6$, where Γ' is the hyperbolic cyclic group generated by $\gamma(z)=\lambda z$ $(\lambda>0, \lambda\neq 1)$. We set $g_m(z)=z^m$ $(0<m<2)$. The function g_m is univalent holomorphic in U and the boundary of $g_m(U)$ is a quasi-circle. Hence $[g_m]$ is in $T(U, 1)$. On the other hand, $[g_m]$ is in $B_2(U, \Gamma')$ and $\|[g_{m_2}]-[g_{m_1}]\|_U=2|m_2^2-m_1^2|$ for $m_1, m_2 \in (0, 2)$. Hence we see that the mapping $m \mapsto [g_m]$ is continuous so that $[g_m]$ is in $T(U, \Gamma')$, the connected component of $T(U, 1)\cap B_2(U, \Gamma')$ containing the origin of $B_2(U, \Gamma')$. Therefore we obtain, by letting $m_1=1$ and $m_2 \to 2$, $o(U, \Gamma')=6$.

Next assume that Γ is an elliptic cyclic group of order n. We have only to show $o(\Gamma')=6$ for the elliptic cyclic group Γ' generated by $\gamma(z)=ze^{i2\pi/n}$. Let $f_{n,q}$ $(0<q<2)$ be as same as §3.1. Since the boundary of $f_{n,q}(\Delta)$ is a quasi-circle, we see $[f_{n,q}]\in T(1)$. On the other hand, $[f_{n,q}]$ is in $B_2(\Gamma')$ and $\|[f_{n,q_2}]-[f_{n,q_1}]\|=2|q_2^2-q_1^2|$ for $q_1, q_2\in(0, 2)$. Hence we see $[f_{n,q}]\in T(\Gamma')$. Therefore we obtain, by letting $q_1=1$ and $q_2\to 2$, $o(\Gamma')=6$.

Finally assume that Γ is a parabolic cyclic group. We have only to show $o(U, \Gamma')=6$ for the parabolic cyclic group Γ' generated by $\gamma_0(z)=z+2$. Let P be the set $\{z\in\hat{C};\ 0<\operatorname{Re} z<1 \text{ and } \operatorname{Im} z>0\}$ and let Q_α be the circular triangle with vertices at $0, 1$ and ∞ and with the interior angles $\alpha\pi$ $(0<\alpha<1)$ at 0 and 1 and the interior angle 0 at ∞. We set

$$f(z)=\left(\frac{1+e^{i\pi z}}{1-e^{i\pi z}}\right)^2 \quad\text{and}\quad g(z)=i\frac{z-i}{z+i}\,.$$

The function f maps P conformally onto U and the function g maps U onto Δ. Let h_α $(0<\alpha<1)$ be the conformal mapping of Δ onto Q_α with $h_\alpha(0)=\infty$, $h_\alpha(i)=0$ and $h_\alpha(-i)=1$. Then $\psi_\alpha=h_\alpha\circ g\circ f$ maps P conformally onto Q_α keeping $0, 1$ and ∞ fixed. According to the symmetry principle, the function ψ_α can be extended to the conformal mapping $\tilde{\psi}_\alpha$ defined in U such that

$$\tilde{\psi}_\alpha\circ\gamma=\gamma\circ\tilde{\psi}_\alpha \quad \text{for every} \quad \gamma\in\Gamma'\,. \tag{5}$$

By the construction of $\tilde{\psi}_\alpha$ and the geometric characterization of quasi-circles (Ahlfors [1]), it follows that the boundary of $\tilde{\psi}_\alpha(U)$ is a quasi-circle. Hence we see $[\tilde{\psi}_\alpha]\in T(U, 1)$. By (5) we also have $[\tilde{\psi}_\alpha]\in B_2(U, \Gamma')$. On the other hand, (2) and (3) give

$$\begin{aligned}[h_\alpha](z)&=1/2(z-1)^2+C_1/(z-1)+(1-\alpha^2)/2(z-i)^2\\&\quad+C_2/(z-i)+(1-\alpha^2)/2(z+i)^2+C_3/(z+i)\,,\end{aligned} \tag{6}$$

where

$$(7) \quad C_1 = -1/2, C_2 = 1/4 + i(1-\alpha^2)/2 \quad \text{and} \quad C_3 = 1/4 - i(1-\alpha^2)/2 .$$

By (6) and (7) we have

$$[\psi_{\alpha_2}] - [\psi_{\alpha_1}] = (\alpha_1^2 - \alpha_2^2)(r\circ g\circ f)((g\circ f)')^2 ,$$

where

$$r(z) = (1/2)\{(z-i)^{-2} + i(z-i)^{-1} + (z+i)^{-2} - i(z+i)^{-1}\} .$$

Since $\|[\tilde{\psi}_{\alpha_2}] - [\tilde{\psi}_{\alpha_1}]\|_U \leqq 12$ for all α_1 and $\alpha_2 \in (0, 1)$, we have

$$\sup_{z\in P} \rho_U(z)^{-2}|[\psi_{\alpha_2}](z) - [\psi_{\alpha_1}](z)| \leqq M|\alpha_2^2 - \alpha_1^2|$$

for all α_1 and $\alpha_2 \in (0, 1)$, where M is a constant (determined by f and g). Hence it follows from the construction of $\tilde{\psi}_\alpha$ that $\|[\tilde{\psi}_{\alpha_2}] - [\tilde{\psi}_{\alpha_1}]\|_U \leqq M|\alpha_2^2 - \alpha_1^2|$. Therefore we have $[\tilde{\psi}_\alpha] \in T(U, \Gamma')$. On the other hand, since

$$f'(z) = 4i\pi e^{i\pi z}(1 + e^{i\pi z})^{-1}(1 - e^{i\pi z})^{-1}f(z)$$

and

$$\lim_{P\ni z\to 0,\,(z/y)\to 1} (1 - e^{i\pi z})/y = -i\pi \qquad (z = x + iy) ,$$

we have

$$\lim_{P\ni z\to 0,\,(z/y)\to 1} \rho_U(z)^{-2}(r\circ g\circ f)(z)((g\circ f)')^2(z) = -8 .$$

Hence it holds

$$\begin{aligned} 8|\alpha_2^2 - \alpha_1^2| &\leqq \sup_{z\in P} \rho_U(z)^{-2}|[\psi_{\alpha_2}](z) - [\psi_{\alpha_1}](z)| \\ &\leqq \|[\tilde{\psi}_{\alpha_2}] - [\tilde{\psi}_{\alpha_1}]\|_U . \end{aligned}$$

Therefore we obtain, by letting $\alpha_1 = 1/2$ and $\alpha_2 \to 1$, $o(U, \Gamma') = 6$. Thus the theorem is proved.

REMARK. The proof of Theorem 2 shows the fact that there does exist a point ϕ on the boundary of $T(\Gamma)$ with $\|\phi\| = 6$.

4. A necessary condition for $\|[f]\| = 6$.

4.1. First we prove the following.

THEOREM 2. *Let A be an open disc in $\hat{C}$ and let f be a univalent meromorphic function defined in A. Let $d(z, \partial f(A))$ be the distance between the point $z \in C$ and the boundary $\partial f(A)$ of $f(A)$. Assume that $\|[f]\|_A = 6$. Then there exists a sequence of points $\{\beta_n\}$ in $f(A)$ converging to a point of $\partial f(A)$ and such that*

$$\lim_{n\to\infty} d(\beta_n, \partial f(A))^2 \iint_{C-f(A)} \frac{dxdy}{|z-\beta_n|^4} = 0 . \tag{8}$$

To prove this theorem, we use two lemmas. The following Lemma 1 is well known. (For example, see Kra [3].)

LEMMA 1. *Let D be a simply connected domain in $\hat{C}$.*

(i) *If h is univalent meromorphic in D, then $\rho_{h(D)}(h(z))|h'(z)| = \rho_D(z)$ at every $z \in D$ with $z \neq \infty$ and $h(z) \neq \infty$.*

(ii) *$\rho_D(z)d(z, \partial D) \leqq 1$ for every $z \in D$ with $z \neq \infty$.*

LEMMA 2. *Let f be a univalent meromorphic function defined in $\Delta^* = \{z \in C; 1 < |z| \leqq \infty\}$ and let $\alpha \in \Delta^* - \{\infty\}$ with $f(\alpha) \neq \infty$. Set*

$$V_\alpha(z) = \frac{\alpha z + 1}{z + \bar{\alpha}}, \qquad \eta_\alpha(z) = -\frac{(|\alpha|^2 - 1)f'(\alpha)}{z - f(\alpha)}$$

and $F_\alpha = \eta_\alpha \circ f \circ V_\alpha$. Then F_α has the expansion

$$F_\alpha(z) = z + b_0(\alpha) + b_1(\alpha)z^{-1} + b_2(\alpha)z^{-2} + \cdots \tag{9}$$

in Δ^* and

$$\rho_{\Delta^*}(\alpha)^{-2}|[f](\alpha)| = 6|b_1(\alpha)| . \tag{10}$$

PROOF. The function F_α is univalent meromorphic in Δ^* and keeps ∞ fixed. We also see $\lim_{z\to\infty} F'_\alpha(z) = 1$. Hence we have the expansion (9) in Δ^*. Noting $[f] = [\eta_\alpha \circ f] = [F_\alpha \circ V_\alpha^{-1}]$ and $\lim_{z\to\infty} z^4[F_\alpha](z) = -6b_1(\alpha)$, we have (10).

Now we give a proof of Theorem 2. We may assume $A = \Delta^*$. Assume $\alpha \in \Delta^* - \{\infty\}$ and $f(\alpha) \neq \infty$ and set $E = \hat{C} - f(\Delta^*)$. Consider V_α, η_α and F_α in Lemma 2 and set $w = \eta_\alpha(z)$ and $w = u + iv$. Then the Bieberbach area theorem shows

$$\pi\Big(1 - \sum_{n=1}^{\infty} n|b_n(\alpha)|^2\Big) = \iint_{\eta_\alpha(E)} dudv . \tag{11}$$

On the other hand, using Lemmas 1 and 2, we have

$$\begin{aligned}\iint_{\eta_\alpha(E)} dudv &= \iint_E \frac{\rho_{\Delta^*}(\alpha)^{-2}|f'(\alpha)|^2}{|z - f(\alpha)|^4} dxdy \\ &= \rho_{f(\Delta^*)}(f(\alpha))^{-2} \iint_E \frac{dxdy}{|z - f(\alpha)|^4} \\ &\geqq d(f(\alpha), \partial f(\Delta^*))^2 \iint_E \frac{dxdy}{|z - f(\alpha)|^4} .\end{aligned} \tag{12}$$

Hence it follows from (11) and (12) that

$$\pi(1-|b_1(\alpha)|^2) \geqq \pi\Big(1-\sum_{n=1}^{\infty} n|b_n(\alpha)|^2\Big) \geqq d(f(\alpha),\ \partial f(\Delta^*))^2 \iint_E \frac{dxdy}{|z-f(\alpha)|^4}\ . \tag{13}$$

If $\rho_{\Delta^*}(\alpha_0)^{-2}|[f](\alpha_0)| = 6$ for some $\alpha_0 \in \Delta^*$ (where we may assume $\alpha_0 \in \Delta^* - \{\infty\}$ and $f(\alpha_0) \neq \infty$), then (10) and the Bieberbach area theorem imply that $[f] = [k\circ g]$, where $k(z) = z/(1-z)^2$ is the Koebe extremal function and g is a Möbius transformation keeping Δ^* invariant. Hence the 2-dimensional Lebesgue measure of E equals 0. Therefore, (8) holds for an arbitrary sequence of points $\{\beta_n\}$ in $f(\Delta^*)$ converging to a point on $\partial f(\Delta^*)$.

If $\rho_{\Delta^*}(\alpha')^{-2}|[f](\alpha')| < 6$ for every $\alpha' \in \Delta^*$, then there exists a sequence of points $\{\alpha_n\}$ in Δ^* converging to a boundary point of Δ^* and satisfying

$$\lim_{n\to\infty} \rho_{\Delta^*}(\alpha_n)^{-2}|[f](\alpha_n)| = \lim_{n\to\infty} 6|b_1(\alpha_n)| = 6 \tag{14}$$

(see (10)). On the other hand, the sequence of points $\{f(\alpha_n)\}$ contains a subsequence $\{\beta_n\}$ which converges to a boundary point of $f(\Delta^*)$. Therefore, by (13) and (14) we have (8).

4.2. LEMMA 3. *Let $\Omega \subset \boldsymbol{C}$ be a convex domain and $\Omega' \subset \boldsymbol{C}$ a domain. Let φ be a C^1-diffeomorphism of Ω onto Ω'. Then, for any convex subdomain Ω_0' whose closure is contained in Ω' and is compact, there exists a constant $M = M(\Omega_0') > 0$ such that*

$$M^{-1}|z_2 - z_1| \leqq |\varphi(z_2) - \varphi(z_1)| \leqq M|z_2 - z_1| \tag{15}$$

for all $z_1, z_2 \in \varphi^{-1}(\Omega_0')$.

PROOF. Set $z = x + iy$ and $\varphi(z) = u(z) + iv(z)$. We write as $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $x_0 = x_2 - x_1$ and $y_0 = y_2 - y_1$. Then there exist two points ζ_1 and ζ_2 lying on the open line segment joining z_1 and z_2 so that

$$\begin{aligned} &|(\varphi(z_2) - \varphi(z_1))/(z_2 - z_1)| \\ &\quad = (\alpha + 2\beta(y_0/x_0) + \gamma(y_0/x_0)^2)/(1 + (y_0/x_0)^2)\ , \qquad \text{if} \quad x_0 \neq 0\ , \\ &\quad = (\alpha(x_0/y_0)^2 + 2\beta(x_0/y_0) + \gamma)/((x_0/y_0)^2 + 1)\ , \qquad \text{if} \quad y_0 \neq 0\ , \end{aligned} \tag{16}$$

where $\alpha = (u_x(\zeta_1))^2 + (v_x(\zeta_2))^2$, $\beta = u_x(\zeta_1)u_y(\zeta_1) + v_x(\zeta_2)v_y(\zeta_2)$ and $\gamma = (u_y(\zeta_1))^2 + (v_y(\zeta_2))^2$. Here α and γ do not vanish simultaneously, for φ is a diffeomorphism. Hence $\max(\alpha, \gamma) > 0$. On the other hand, we have an inequality

$$(a + 2bt + ct^2)/(1 + t^2) \leqq c + ((a-c)^2 + 4b^2)^{1/2}\ , \tag{17}$$

where a, b, c and t are real numbers with $c > 0$. Let K be a compact subset of Ω and $\hat{K}$ $(\subset \Omega)$ the convex hull of K. Then it follows from (16) and (17) that

$$|\varphi(z_2) - \varphi(z_1)| \leqq M|z_2 - z_1|$$

for all $z_1, z_2 \in K$, where M is a positive constant satisfying

$$M^2 \geqq \max_{\zeta_1, \zeta_2 \in \hat{K}} (\max(\alpha, \gamma) + ((\alpha - \gamma)^2 + 4\beta^2)^{1/2}) .$$

Since the closure of $\varphi^{-1}(\Omega_0')$ is contained in Ω and is compact, we obtain (15) by considering the inverse mapping φ^{-1}.

Now we prove one more lemma.

LEMMA 4. *Let* $\omega \in (0, 2\pi)$ *and* $r_0 \in (0, 1)$. *Set* $A_{\omega,r_0} = \{z \in \boldsymbol{C}; |z| \leqq r_0$ *and* $0 \leqq \arg z \leqq \omega\}$ *and* $A_{2\pi,r_0} = \{z \in \boldsymbol{C}; |z| \leqq r_0\} \cap \{z \in \boldsymbol{C}; |z + i| \leqq 1$ *or* $|z - i| \leqq 1$ *or* $\operatorname{Re} z \geqq 0\}$ *and* $A'_{\pi,r_0} = A_{2\pi,r_0} \cap \{z \in \boldsymbol{C}; \operatorname{Im} z \geqq 0\}$. *Then*

$$\liminf_{A \ni \alpha \to 0} d(\alpha, \partial A)^2 \iint_A \frac{dxdy}{|z - \alpha|^4} > 0 \tag{18}$$

where $A = A_{\omega,r_0}$ $(\omega \in (0, 2\pi])$ *or* $A = A'_{\pi,r_0}$.

PROOF. Set $A_\omega = A_{\omega,r_0}$, $A'_\pi = A'_{\pi,r_0}$, $d_\omega(\alpha) = d(\alpha, \partial A_\omega)$ and $d(\alpha) = d_{\pi/2}(\alpha)$. First we prove (18) for $A = A_{\pi/2}$. Let $\delta \in (0, 1)$ be sufficiently small and set $V_\delta = \{z \in \boldsymbol{C}; |z| < \delta$ and $\pi/2 < \arg z < 2\pi\}$. For $\alpha \in V_\delta$ and $r \in (2d(\alpha), r_0 - \delta)$, we define $\theta_1(\alpha, r)$ and $\theta_2(\alpha, r)$ $(-\pi/2 < \theta_j(\alpha, r) < \pi, j = 1, 2)$ by the condition

$$A_{\pi/2} \cap \{z \in \boldsymbol{C}; |z - \alpha| = r\} = \{\alpha + re^{i\theta}; \theta_1(\alpha, r) \leqq \theta \leqq \theta_2(\alpha, r)\} .$$

We also write $\theta(\alpha, r) = \theta_2(\alpha, r) - \theta_1(\alpha, r)$. It is not difficult to verify the existence of a positive constant θ_0 such that $\theta(\alpha, r) > \theta_0$ for all α and r. Hence

$$\begin{aligned} \iint_{A_{\pi/2}} \frac{dxdy}{|z - \alpha|^4} &\geqq \int_{2d(\alpha)}^{r_0 - \delta} \int_{\theta_1(r)}^{\theta_2(r)} \frac{drd\theta}{r^3} \\ &\geqq \theta_0(1/8d(\alpha)^2 - 1/2(r_0 - \delta)^2) \end{aligned}$$

for $\alpha \in V_\delta$. Therefore we have (18).

Next we prove (18) for $A = A_\omega$ $(\omega \in (0, \pi))$. Consider an affine transformation F_a (from the z-plane to the w-plane) given by

$$F_a: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix} = a\begin{pmatrix} 1 & \cos\omega \\ 0 & \sin\omega \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad (a > 0) ,$$

where $z = x + iy$ and $w = u + iv$. Then $F_a(A_{\pi/2}) \subset A_\omega$ for a sufficiently small a. By Lemma 3 there exist an open set V in C with $V \supset A_{\pi/2}$ and a constant $M = M(V) > 0$ such that

$$|F_a(z_2) - F_a(z_1)| \leqq M|z_2 - z_1|$$

for $z_1, z_2 \in V$ and

$$d_\omega(F_a(\alpha)) \geqq M^{-1}d(\alpha)$$

for $\alpha \in V$. Hence we have

$$d_\omega(F_a(\alpha))^2 \iint_{A_\omega} \frac{dudv}{|w - F_a(\alpha)|^4} \geqq M^{-6}a^2 (\sin \omega) d(\alpha)^2 \iint_{A_{\pi/2}} \frac{dxdy}{|z - \alpha|^4}$$

for $\alpha \in V - \bar{A}_{\pi/2}$. By the conclusion of the case $\omega = \pi/2$ we obtain (18).

Next we prove (18) for $A = A_\omega$ ($\omega \in [\pi, 2\pi)$). Because of the symmetry of A with respect to the line $y \cos(\omega/2) = x \sin(\omega/2)$ (with $z = x + iy$), we obtain (18) by the conclusion for $A = A_\omega$ ($\omega \in (0, \pi)$).

Next we prove (18) for $A = A'_\pi$. Since there exists a Möbius transformation g which maps $\{z \in C;\ |z - i| < 1\}$ onto the upper half plane U with $g(0) = 0$ and $g(\infty) = \infty$, we obtain (18), by using the conclusion for $A = A_\pi$ (and the conclusion for $A = A_{\pi/2}$), in a manner similar to that for $A = A_\omega$ ($\omega \in (0, \pi)$).

Finally we prove (18) for $A = A_{2\pi}$. Since $A_{2\pi}$ is symmetric with respect to the real axis, we obtain (18) by the conclusion for $A = A'_\pi$. The proof of Lemma 4 is hereby complete.

Now we prove the following as a corollary of Theorem 2.

THEOREM 3. *Let P be a polygonal domain defined in §3.1. Assume $0 \leqq \alpha_j < 2$ ($j = 1, \cdots, n$) for the interior angle $\pi\alpha_j$ at the vertex A_j of P. Let V be a neighborhood of ∂P and let φ be a C^1-diffeomorphism of V into $\hat{C}$. If Ω is the connected component of $\hat{C} - \varphi(\partial P)$ with $\varphi(P \cap V) \subset \Omega$, then conformal mappings f of Δ onto Ω satisfy $\|[f]\| < 6$.*

PROOF. We may assume that P and Ω are bounded domains in $\hat{C}$. We set $V(p, \varepsilon) = \{z \in C;\ |z - p| < \varepsilon\}$ for $p \in C$ and $\varepsilon > 0$ and also set $C(E) = C - E$ for a subset E of C. Let $p_0 \in \partial P$, $q_0 = \varphi(p_0)$ and $\gamma = \varphi(\partial P)$. Let $r \in (0, 1)$ and ω ($\in (0, 2\pi]$) be the exterior angle of P at p_0 with respect to P. Then there exists a Möbius transformation g with $g(0) = p_0$, $g(\bar{V}(0, r)) \subset V$, $g(A_r) \subset C(P)$ and $g(\partial A_r) \subset \partial P$, where $A_r = A_{\omega,r}$ if $\omega \neq \pi$ and $A_r = A_{\pi,r}$ or $A'_{\pi,r}$ if $\omega = \pi$ (see Lemma 4). We take $V_0 = V(q_0, \varepsilon)$ so that $V_1 = V(q_0, 2\varepsilon) \subset \psi(V(0, r))$ $(\subset \varphi(V))$, where $\psi = \varphi \circ g$. Then $g(A_s) \subset \varphi^{-1}(C(\Omega) \cap V_0)$ for some $s \in (0, r)$.

For any point $q \in V_0$, there exists a point $q' \in \gamma \cap V_1$ with $d(q, \gamma) =$

$|q - q'|$. Hence, by Lemma 3, there exists a constant $M = M(V_1) > 0$ such that

$$|\psi(z) - \psi(\alpha)| \leqq M|z - \alpha|$$

for $z, \alpha \in \psi^{-1}(V_1)$ and such that

$$d(q, \gamma) \geqq M^{-1} d(\alpha, \partial A_r)$$

for $q \in V_0$ and $\alpha = \psi^{-1}(q)$.

Let $q \in V_0$, $q = \psi(\alpha)$, $w = \psi(z)$, $z = x + iy$ and $w = u + iv$. Then we have

$$\begin{aligned} d(q, \partial\Omega)^2 \iint_{C(\Omega)} \frac{dudv}{|w - q|^4} &\geqq d(q, \gamma)^2 \iint_{C(\Omega)\cap V_0} \frac{dudv}{|w - q|^4} \\ &\geqq M^{-2} d(\alpha, \partial A_r)^2 \iint_{A_s} \frac{|J_\psi(z)|}{(M|z - \alpha|)^4} dxdy \\ &\geqq M' d(\alpha, \partial A_r)^2 \iint_{A_s} \frac{dxdy}{|z - \alpha|^4}\,, \end{aligned}$$

where J_ψ is the Jacobian of ψ and M' is a positive constant. Hence by Lemma 4 we have

$$\liminf_{q \to q_0, q \in \Omega} d(q, \partial\Omega)^2 \iint_{C(\Omega)} \frac{dxdy}{|z - q|^4} > 0\,.$$

Therefore, Theorem 2 implies $\|[f]\| < 6$.

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