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FUNDAMENTAL MATRICES OF LINEAR AUTONOMOUS RETARDED EQUATIONS WITH INFINITE DELAY

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. **Introduction.** If $x: (-\infty, A) \to \boldsymbol{C}^n$, then for any t in $(-\infty, A)$ we let $x_t: (-\infty, 0] \to \boldsymbol{C}^n$ be defined by $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leqq 0$. The linear autonomous retarded equation with infinite delay is an equation

$$dx/dt = L(x_t)\,, \tag{1.1}$$

where $L: \mathscr{B} \to \boldsymbol{C}^n$ is linear and continuous, and $\mathscr{B}$ is a linear space of some functions $\phi: (-\infty, 0] \to \boldsymbol{C}^n$. Hypotheses $(\mathrm{H}_0), \cdots, (\mathrm{H}_4)$ imposed on the space $\mathscr{B}$ are stated in Section 2. In [6], under these hypotheses the fundamental matrix $X(t)$ of this equation is defined for $t > 0$ in terms of the inverse Laplace transform. It has also been proved that X gives the variation-of-constants formula of solutions of the nonhomogeneous equation corresponding to Equation (1.1). The objective of this paper is to establish that, if we set $X(0) = I$ and $X(t) = 0$ for $t < 0$, then X satisfies Equation (1.2) below which is naturally induced from Equation (1.1) (Theorem 5.2).

To obtain this result, in Section 3 we first consider the representation of the operator L. From Hypotheses (H_1) and (H_2) the operator L induces a linear operator L_0 on the space $\mathscr{C}$ of continuous functions mapping $(-\infty, 0]$ into $\boldsymbol{C}^n$ with compact support. Furthermore, L_0 becomes a "Radon" measure on $(-\infty, 0]$. A well known result of measure theory implies that L_0 has a unique "Borel" prolongation $\tilde{L}$ over the space Γ of bounded and Borel measurable functions mapping $(-\infty, 0]$ into $\boldsymbol{C}^n$ with compact support. Introducing this operator, we define an $n \times n$ matrix function $\eta(\theta)$, $-\infty < \theta \leqq 0$, which becomes a kernel function of the linear operator L when this is represented by a Stieltjes integral. More precisely, the representation of $L(\phi)$ is proved only for the functions ϕ which are either an element of the space $\mathscr{C}$ or an exponential function $\exp(\lambda\theta)b$ with a lower bound α_0 for $\mathrm{Re}\,\lambda$, where b is in $\boldsymbol{C}^n$. If we set $\zeta(t) = -\eta(-t)$ for $t \geqq 0$, then the representation of $G(\lambda) \equiv L(\exp(\lambda\cdot)I)$ with respect to η is interpreted as a Laplace-Stieltjes transform of ζ. In Section 4, a classical theorem on the characterization of generating functions is applied for $G(\lambda)$. Consequently, under an additional Hypothesis (H_5) for $\mathscr{B}$ the lower bound α_0 for $\operatorname{Re}\lambda$ is replaced by the best possible one. Thus the representation of $L(\phi)$ is obtained for all of the concrete functions ϕ which are known to be the elements of every space $\mathscr{B}$ satisfying Hypotheses $(H_0), \cdots, (H_5)$.

Observe that, for every $t \geqq 0$, $L(X_t)$ may not have a meaning but $\tilde{L}(X_t)$ is well defined since X_t obviously lies in Γ. Hence Equation (1.1) with L replaced by $\tilde{L}$ is naturally introduced. As final results, we prove that

$$dX/dt = \tilde{L}(X_t) \quad \text{a.e. in } t \geqq 0 , \tag{1.2}$$

and that, if X_r lies in $\mathscr{B}$ for some $r \geqq 0$, then $X(t)$ satisfies Equation (1.1) for every $t \geqq r$. From the results established in Section 3, these assertions are obtained by the method of Laplace and Laplace-Stieltjes transform. We emphasize that $\tilde{L}$ is continuous in Lebesgue; roughly speaking, the bounded convergence theorem holds for $\tilde{L}$ on every compact interval of $(-\infty, 0]$. This property makes the proofs of the above results easy to follow.

In case the delay is finite and the phase space is $C([-r, 0], \boldsymbol{C}^n)$, the general theory of the fundamental matrix is well known (cf. [3]). Kappel [5] introduced the method of Laplace-Stieltjes transform into the study of neutral functional differential equations. Under several conditions on phase spaces and linear operators, Corduneanu [1] treated the fundamental matrix in case the delay is infinite. The Laplace transform was also used. See Hale and Kato [4] for examples of the space $\mathscr{B}$ satisfying Hypotheses $(H_0), \cdots, (H_5)$. Corduneanu and Lakshmikantham [2] contains complete references for the papers concerning equations with infinite delay.

2. The space $\mathscr{B}$ and basic results. Let $\mathscr{B}$ be a linear space of functions mapping $(-\infty, 0]$ into $\boldsymbol{C}^n$ with elements $\phi, \psi, \cdots$ having seminorm $|\phi|_{\mathscr{B}}, |\psi|_{\mathscr{B}}, \cdots$. We say that ϕ and ψ in $\mathscr{B}$ are equivalent if $|\phi - \psi|_{\mathscr{B}} = 0$, and denote by $\hat{\phi}$ the equivalence class of ϕ. The collection of equivalence classes, designated by $\hat{\mathscr{B}}$, becomes a normed linear space if we define $|\hat{\phi}|_{\hat{\mathscr{B}}} = |\phi|_{\mathscr{B}}$. On the spaces $\mathscr{B}$ and $\hat{\mathscr{B}}$, we impose the following hypotheses. The presentation is apparently different from the one in [6] but both hypotheses are equivalent to each other.

(H_0) $\hat{\mathscr{B}}$ is a Banach space.

(H_1) If x is a function mapping $(-\infty, \sigma + A)$ into $\boldsymbol{C}^n$ with $A > 0$ such that x is continuous on $[\sigma, \sigma + A)$ and x_σ lies in $\mathscr{B}$, then x_t also lies in $\mathscr{B}$ and x_t is a continuous function of t for t in $[\sigma, \sigma + A)$.

(H_2) There exist functions $K(t)$ and $M(t)$ of $t \geqq 0$ with the following properties:

(i) $K(t)$ is continuous for t in $[0, \infty)$.

(ii) $M(t)$ is locally bounded on $[0, \infty)$ and submultiplicative, that is, $M(t+s) \leqq M(t)M(s)$ for $t, s \geqq 0$.

(iii) For every function x which arises in (H_1), it holds that, for $\sigma \leqq t < \sigma + A$,

$$|x_t|_{\mathscr{B}} \leqq K(t-\sigma)\sup\{|x(s)|: \sigma \leqq s \leqq t\} + M(t-\sigma)|x_\sigma|_{\mathscr{B}} .$$

(H_3) $|\phi(0)| \leqq K|\phi|_{\mathscr{B}}$ for all ϕ in $\mathscr{B}$ and some constant K.

(H_4) If $\{\hat{\phi}^k\}$ is a Cauchy sequence of $\hat{\mathscr{B}}$ and $\{\phi^k(\theta)\}$ converges to $\phi(\theta)$ uniformly for θ in each compact set of $(-\infty, 0]$, then ϕ also lies in $\mathscr{B}$ and $\hat{\phi}^k \to \hat{\phi}$ as $k \to \infty$.

Now, from the papers [4] and [6] let us introduce some results which will be needed in the following sections. Suppose $L: \mathscr{B} \to C^n$ is linear and continuous. Hypotheses (H_1), (H_2) and (H_3) guarantee the unique existence of the solution $x(\phi)(t)$ on $[0, \infty)$ of Equation (1.1) with the initial condition $x_0 = \phi$ in $\mathscr{B}$. For ϕ in $\mathscr{B}$, we set

$$T_L(t)\phi = x_t(\phi) \quad \text{for} \quad t \geqq 0 .$$

Then $T_L(t)$ is a continuous linear operator on $\mathscr{B}$ into $\mathscr{B}$. If we set $\hat{T}_L(t)\hat{\phi} = (T_L(t)\phi)^\wedge$ for $\hat{\phi}$ in $\hat{\mathscr{B}}$, then $\hat{T}_L(t): \hat{\mathscr{B}} \to \hat{\mathscr{B}}$ is also linear and continuous. Furthermore, Hypothesis (H_1) means that $\hat{T}_L(t)$ is a strongly continuous semigroup on the space $\hat{\mathscr{B}}$. This is called the solution semigroup of Equation (1.1).

It is well known that the type number α_L of the semigroup $\hat{T}_L(t)$ is defined as

$$\alpha_L = \lim_{t\to\infty} [\log|\hat{T}_L(t)|]/t = \inf_{t>0} [\log|\hat{T}_L(t)|]/t ,$$

which may be $-\infty$ but not $+\infty$. For bounded sets B of a Banach space X, let $\alpha(B)$ denote the Kuratowski measure of noncompactness of B. It induces the semi-norm $\alpha(T)$ for bounded linear operators $T: X \to X$ defined by $\alpha(T) = \inf\{k: \alpha(TB) \leqq k\alpha(B)$ for all bounded sets B in $X\}$. Using this semi-norm, we define the "essential" type number β_L of $\hat{T}_L(t)$ as

$$\beta_L = \lim_{t\to\infty} [\log \alpha(\hat{T}_L(t))]/t = \inf_{t>0} [\log \alpha(\hat{T}_L(t))]/t .$$

In addition to a direct result that $\beta_L \leqq \alpha_L$, we can prove that β_L is independent of L [6, p. 79]. Therefore, if we denote by β this common value of β_L, then $\beta \leqq \alpha_L$ for all L. Furthermore, following the proof of [6, Theorem 4.5, p. 81], we know that

(A_1) $\beta \leqq \alpha_0$, and $\beta < \alpha_0$ if and only if $\beta < 0$.

It need hardly be said that α_0 is the type number of the solution semigroup $\hat{T}_0(t)$ of the trivial equation $dx/dt = 0$. Because of its importance, $T_0(t)$ is designated by a special symbol $S(t)$. Clearly, it is given by

$$(2.1) \qquad (S(t)\phi)(\theta) = \begin{cases} \phi(0) & \text{for} \quad t + \theta \geqq 0 \\ \phi(t + \theta) & \text{for} \quad t + \theta < 0 \, . \end{cases}$$

The number β has also the following relation with the structure of the space $\mathscr{B}$ [6, Theorem 4.4, p. 79]. For λ in $\boldsymbol{C}$ and b in $\boldsymbol{C}^n$, let $\omega(\lambda)b$ denote the function of θ in $(-\infty, 0]$ defined as

$$[\omega(\lambda)b](\theta) = e^{\lambda\theta}b \quad \text{for} \quad \theta \leqq 0 \, .$$

Then $\omega(\lambda)b$ lies in $\mathscr{B}$ for λ in $\boldsymbol{C}_\beta = \{\lambda \in \boldsymbol{C} \colon \operatorname{Re}\lambda > \beta\}$, and

(A_2) $(\omega(\lambda)b)\hat{}$ is an analytic function of λ in $\boldsymbol{C}_\beta$ into $\hat{\mathscr{B}}$.

For simplicity, let the symbol $\hat{\omega}(\lambda)b$ mean $(\omega(\lambda)b)\hat{}$.

3. Representation theory for continuous linear functionals on $\mathscr{B}$. It is well known that every linear and continuous operator $L \colon C([-r, 0], \boldsymbol{C}^n) \to \boldsymbol{C}^n$ is represented by a Stieltjes integral with respect to a matrix function of bounded variation in $[-r, 0]$. In this section, an analogues result will be proved for linear and continuous operators $L \colon \mathscr{B} \to \boldsymbol{C}^n$. However, the representation of $L(\phi)$ is restricted to the following functions; that is, ϕ is in $\mathscr{C}$ introduced in Section 1 or $\phi = \omega(\lambda)b$ for $\operatorname{Re}\lambda > \alpha_0$ and b in $\boldsymbol{C}^n$, where α_0 is the type number of $S(t)$.

By Hypothesis (H_1), the space $\mathscr{C}$ is a linear subspace of $\mathscr{B}$. For each ϕ in $\mathscr{C}$, $\operatorname{supp}\phi$ denotes the support of ϕ, and $|\phi|_{\mathscr{C}} = \sup\{|\phi(\theta)| \colon -\infty < \theta \leqq 0\}$. If L is a linear and continuous operator on $\mathscr{B}$ into $\boldsymbol{C}^n$, then the restriction of L on $\mathscr{C}$ is clearly a linear operator on $\mathscr{C}$ into $\boldsymbol{C}^n$ which we denote by L_0. Hypothesis (H_2) implies that the operator L_0 is continuous on $\mathscr{C}$ in the sense that, if $\operatorname{supp}\phi$ lies in $[-t, 0]$, then

$$(3.1) \qquad |L_0(\phi)| \leqq |L| K(t) |\phi|_{\mathscr{C}} \, .$$

Now, we introduce some results from measure theory (cf. [7, pp. 521, 1-521, 12]). Suppose X is a locally compact metric space. Denote by $\mathscr{C}(X)$ the linear space of continuous functions mapping X into $\boldsymbol{C}$ with compact support. A linear operator μ mapping $\mathscr{C}(X)$ into a Banach space E is called a Radon measure on X into E if μ is continuous in the sense that, for each compact set K of X, there exists a constant c_K such that $|\mu(\phi)| \leqq c_K \sup\{|\phi(x)| \colon x \in X\}$ provided $\operatorname{supp}\phi$ lies in K. Let $\Gamma(X)$ be the linear space of bounded and Borel measurable functions $\phi \colon X \to \boldsymbol{C}$ with compact support. Obviously, $\mathscr{C}(X)$ is a linear subspace of $\Gamma(X)$. A sequence $\{\phi^k\}$ of $\Gamma(X)$ is said to converge in Lebesgue (or L-converge) to a function ϕ in $\Gamma(X)$ if $\{\phi^k(x)\}$ are uniformly bounded, their supports are all contained in a compact set and $\phi^k(x) \to \phi(x)$ as $k \to \infty$ for each x in X. A linear operator ν on $\Gamma(X)$ into E is said to be continuous in Lebesgue (or L-continuous) if the sequence $\{\nu(\phi^k)\}$ converges to $\nu(\phi)$ for any sequence $\{\phi^k\}$ of $\Gamma(X)$ which converges in Lebesgue to ϕ. A Borel prolongation of a Radon measure μ is a linear operator $\nu: \Gamma(X) \to E$ such that $\nu(\phi) = \mu(\phi)$ for ϕ in $\mathscr{C}(X)$ and ν is continuous in Lebesgue. It is known that, if E is of finite dimension, then every Radon measure on X into E has a unique Borel prolongation.

The space Γ introduced in Section 1 is the product space of n-copies of $\Gamma((-\infty, 0])$. Clearly, $\mathscr{C}$ is the subspace of Γ. Is Γ contained in $\mathscr{B}$ or not? At present, we have no answer to this question under Hypotheses $(\mathrm{H}_0), \cdots, (\mathrm{H}_4)$. For $\mathscr{C}$ and Γ, give similar definitions of "Radon" measure, "Borel" prolongation, etc. Then, Inequality (3.1) implies that L_0 is a "Radon" measure on $(-\infty, 0]$. Applying the above result to L_0, one can state the following theorem.

THEOREM 3.1. *Suppose L is a linear and continuous operator on $\mathscr{B}$ into C^n. Then there exists one and only one linear operator $\tilde{L}$ on Γ into C^n which has the following properties:*

(i) *$\tilde{L}(\phi) = L(\phi)$ for all $\phi \in \mathscr{C}$.*

(ii) *$\tilde{L}$ is continuous in Lebesgue.*

Now, define a function $\chi: \boldsymbol{R} \to \boldsymbol{R}$ by

$$\chi(t) = 1 \quad \text{for} \quad t \geqq 0\,, \quad \text{and} \quad \chi(t) = 0 \quad \text{for} \quad t < 0\,. \tag{3.2}$$

Then, for $t \geqq 0$, the function χ_t is the indicator function of the set $[-t, 0]$ in $(-\infty, 0]$. Associated with the Borel prolongation $\tilde{L}$ of L_0, an $n \times n$ matrix function $\eta(\theta)$ for $\theta \leqq 0$ is defined by

$$\eta(\theta) = \begin{cases} 0 & \text{for} \quad \theta = 0 \\ -\tilde{L}(\chi_{-\theta} I) & \text{for} \quad \theta < 0\,, \end{cases} \tag{3.3}$$

where I is the $n \times n$ identity matrix. This function is well defined and continuous to the left at every $\theta < 0$ since the indicator function χ_t lies in $\Gamma((-\infty, 0])$ and converges in Lebesgue to χ_τ as $t \to \tau + 0$, for $\tau \geqq 0$. Also, the function $\eta(\theta)$ has a limit as $\theta \to \sigma + 0$, $\sigma < 0$. But it is possible that this limit does not coincide with $\eta(\sigma)$. On the other hand, it will be verified that η is a function of bounded variation on each compact interval of $(-\infty, 0]$. To estimate the variation of η, the following observation is essential.

LEMMA 3.2. *If ϕ is a function in $\mathscr{C}$ whose support lies in $[-t, -s]$ for some $t > s \geqq 0$, then*

$$|\phi|_{\mathscr{B}} \leqq K(t-s)|\hat{S}(s)||\phi|_{\mathscr{C}},$$

where $K(t)$ is the function arising in Hypothesis (H_2), *and $S(t)$ is defined by Relation* (2.1).

PROOF. From Hypothesis (H_2), if the support of a function ψ in $\mathscr{C}$ is contained in $[-r, 0]$, then $|\psi|_{\mathscr{B}} \leqq K(r)|\psi|_{\mathscr{C}}$. For the function ϕ given in the lemma, define a function ψ in $\mathscr{C}$ by $\psi(\theta) = \phi(\theta - s)$, $\theta \leqq 0$. Then $S(s)\psi = \phi$ and $\operatorname{supp}\psi$ is contained in $[-(t-s), 0]$. Thus one obtains $|\phi|_{\mathscr{B}} = |S(s)\psi|_{\mathscr{B}}$ and $|\psi|_{\mathscr{B}} \leqq K(t-s)|\psi|_{\mathscr{C}}$. These relations and the definition $|\phi|_{\mathscr{B}} = |\hat{\phi}|_{\hat{\mathscr{B}}}$ lead to the inequality in the lemma. q.e.d.

If f is a function of bounded variation on an interval J, let $V(f, J)$ denote the total variation of f on J. A function of bounded variation on each compact interval of an unbounded interval J is called a function locally of bounded variation on J. In case $J = (-\infty, 0]$, such a function f is said to be normalized if $f(0) = 0$ and $f(\theta)$ is continuous to the left for every $\theta < 0$. For a partition P of an interval $[a, b]$ such that $a = \theta(0) < \theta(1) < \cdots < \theta(d) = b$, let $m(P) = \max\{|\theta(i) - \theta(i-1)|: i = 1, \cdots, d\}$.

PROPOSITION 3.3. *The function η defined by Relation* (3.3) *is a normalized function locally of bounded variation on* $(-\infty, 0]$ *such that*

$$V(\eta, [-t, -s]) \leqq c|L|K(t-s)|\hat{S}(s)| \quad \textit{for} \quad t > s \geqq 0,$$

where c is a constant dependent on the norm of C^n.

PROOF. We have already observed that $\eta(\theta)$ is continuous to the left for every $\theta < 0$. To prove the above inequality, it sufficies to show that the similar estimate is valid for each component of η. Thus without restricting the generality one can assume $n = 1$.

For a partition P of $[-t, -s]$ such that $-t = \theta(0) < \theta(1) < \cdots < \theta(d) = -s$, let

$$V^P = \sum_{i=1}^{d} |\eta(\theta(i)) - \eta(\theta(i-1))|.$$

For each $i = 1, \cdots, d$, take a complex number $\sigma(i)$ such that $|\sigma(i)| = 1$ and that $|\eta(\theta(i)) - \eta(\theta(i-1))| = \sigma(i)[\eta(\theta(i)) - \eta(\theta(i-1))]$. In case $\theta(d) = -s < 0$, we set

$$\phi = \sum_{i=1}^{d} [-\chi_{-\theta(i)} + \chi_{-\theta(i-1)}]\sigma(i), \tag{3.4}$$

and in case $\theta(d) = -s = 0$, we set

$$\phi = \sum_{i=1}^{d-1} [-\chi_{-\theta(i)} + \chi_{-\theta(i-1)}]\sigma(i) + \chi_{-\theta(d-1)}\sigma(d) . \tag{3.5}$$

Then the definition of η implies that $V^P = \tilde{L}(\phi)$. Let ϕ^n, $n = 1, 2, \cdots$, be the function defined by Relation (3.4) or (3.5) with χ replaced by χ^n which is given by $\chi^n(t) = 1$ for $t \geqq 0$, $\chi^n(t) = n(t + 1/n)$ for $-1/n < t < 0$ and $\chi^n(t) = 0$ for $t \leqq -1/n$. Obviously, ϕ^n is in $\mathscr{C}$ and $\operatorname{supp} \phi^n$ lies in $[-t - 1/n, -s]$. Moreover, $|\phi^n|_{\mathscr{C}} \leqq 1$ if $1/n < \min\{\theta(i) - \theta(i-1)\}$. Furthermore, $\phi^n(\theta) \to \phi(\theta)$ as $n \to \infty$ for every $\theta \leqq 0$. Thus, by Theorem 3.1 we have $L(\phi^n) \to \tilde{L}(\phi) = V^P$ as $n \to \infty$. Since $V^P \geqq 0$, this implies that $V^P = \lim_{n\to\infty} |L(\phi^n)|$.

On the other hand, applying Lemma 3.2 to the function ϕ^n, we see that, $|\phi^n|_{\mathscr{B}} \leqq K(t - s + 1/n)|\hat{S}(s)|\,|\phi^n|_{\mathscr{C}}$. These relations yield that $V^P \leqq |L|K(t-s)|\hat{S}(s)|$. It is to be noticed that $K(t)$ is continuous. Since the partition P is arbitrary, this concludes the proof of the estimate for $V(\eta, [-t, -s])$ in the lemma. q.e.d.

THEOREM 3.4. *Suppose a function $\phi(\theta)$ is continuous for θ in an interval $(-t, 0]$, continuous to the right for $\theta = -t$ and $\phi(\theta) = 0$ for $\theta < -t$. Then $\tilde{L}(\phi)$ is represented by a Riemann-Stieltjes integral as*

$$\tilde{L}(\phi) = \int_{-t}^{0} d\eta(\theta)\phi(\theta) ,$$

where η is the function defined by Relation (3.3).

PROOF. For a partition P of $[-t, 0]$ such that $-t = \theta(0) < \theta(1) < \cdots < \theta(d) = 0$, let ϕ^P be the function defined by Relation (3.5) with $\sigma(i)$ replaced by $\phi(\tau(i))$, where $\theta(i-1) \leqq \tau(i) \leqq \theta(i)$ for $i = 1, \cdots, d$. Then the definition of η implies $\tilde{L}(\phi^P) = \sum_{i=1}^{d} [\eta(\theta(i)) - \eta(\theta(i-1))]\phi(\tau(i))$. It is obvious that ϕ^P converges in Lebesgue to ϕ as $m(P) \to 0$. This leads to the theorem since $\tilde{L}$ is continuous in Lebesgue. q.e.d.

THEOREM 3.5. *Suppose ϕ is either a function in $\mathscr{C}$ or an exponential function $\omega(\lambda)b$ with $\operatorname{Re}\lambda > \alpha_0$, where α_0 is the type number of $\hat{S}(t)$. Then $L(\phi)$ is represented as*

$$L(\phi) = \int_{-\infty}^{0} d\eta(\theta)\phi(\theta) \equiv \lim_{t\to\infty} \int_{-t}^{0} d\eta(\theta)\phi(\theta) .$$

PROOF. By Theorem 3.4, it is clear that the above formula holds for ϕ in $\mathscr{C}$. Suppose $\phi = \omega(\lambda)b$ for λ with $\operatorname{Re}\lambda > \alpha_0$ and $b \in C^n$. For $t \geqq 0$, we now define $\phi^t = \rho_t\phi$ and $\psi^t = (1 - \rho_t)\phi$, where ρ is χ^1, the first member of the family $\{\chi^n\}$ arising in the proof of Proposition 3.3. Then it is obvious that ϕ^t is in $\mathscr{C}$ and $\phi = \phi^t + \psi^t$, or $\psi^t = \phi - \phi^t$ for $t \geqq 0$. This implies ψ^t also lies in $\mathscr{B}$, and $L(\phi) = L(\phi^t) + L(\psi^t)$ for $t \geqq 0$.

It is easy to see that $L(\psi^t) \to 0$ as $t \to \infty$. Indeed, from the trivial relation $\psi^t(\theta) = e^{-\lambda t}(1 - \rho(t + \theta))e^{\lambda(t+\theta)}b$ for $\theta \leqq 0$, it follows that $\psi^t = \exp(-\lambda t)S(t)\psi^0$ for $t \geqq 0$. Since $\operatorname{Re}\lambda > \alpha_0$, the definition of the type number yields that $\exp(-\lambda t)|\hat{S}(t)| \to 0$ as $t \to \infty$. This implies that $\psi^t \to 0$ as $t \to \infty$, and so $L(\psi^t) \to 0$ as $t \to \infty$. Therefore, we have $L(\phi) = \lim_{t\to\infty} L(\phi^t)$.

Since ϕ^t is in $\mathscr{C}$ and $\phi^t(\theta) = 0$ for $\theta \leqq -(t + 1)$, Theorem 3.4 asserts that

$$L(\phi^t) = \int_{-t}^{0} d\eta(\theta)\phi(\theta) + \int_{-t-1}^{-t} d\eta(\theta)\phi^t(\theta) .$$

Denote the last integral by $a(t)$. Applying Proposition 3.3, one obtains that, for $t \geqq 0$,

$$\begin{aligned} |a(t)| &\leqq V(\eta, [-t-1, -t]) \sup\{|\phi(\theta)|: -t-1 \leqq \theta \leqq -t\} \\ &\leqq \text{const.}\, |L|\, K(1)\, |\hat{S}(t)| \max\{e^{-t\operatorname{Re}\lambda}, e^{-(t+1)\operatorname{Re}\lambda}\} , \end{aligned}$$

which implies $a(t) \to 0$ as $t \to \infty$. Summarizing these results, we have the desired conclusion. q.e.d.

If we set

$$\zeta(t) = -\eta(-t) \quad \text{for} \quad t \geqq 0 , \tag{3.6}$$

then ζ is a function locally of bounded variation on $[0, \infty)$. It is normalized in the sense that $\zeta(0) = 0$ and that $\zeta(t)$ is continuous to the right for $t > 0$. Theorem 3.5 now asserts that

$$L(\omega(\lambda)I) = \int_{0}^{\infty} e^{-\lambda t} d\zeta(t) \quad \text{for} \quad \operatorname{Re}\lambda > \alpha_0 . \tag{3.7}$$

A function defined by such an integral is called the Laplace-Stieltjes transform (of $\zeta(t)$) or the generating function of the Laplace-Stieltjes transform [8]. On the other hand, following Corduneanu [1], we call $L(\omega(\lambda)I)$ the symbol of L. Thus we can say that, for λ with $\operatorname{Re}\lambda > \alpha_0$, the symbol of L coincides with a generating function of some Laplace-Stieltjes transform.

4. Further representation theory for L. We now show that, for λ in the remaining strip $\{\lambda: \beta < \operatorname{Re}\lambda \leqq \alpha_0\}$, the representation of $L(\omega(\lambda)b)$ is still valid. To do this, we impose an additional hypothesis on $\mathscr{B}$:

(H_5) If ϕ and ψ in $\mathscr{B}$ satisfy $|\phi(\theta)| \leqq |\psi(\theta)|$ for all $\theta \leqq 0$, then $|\phi|_{\mathscr{B}} \leqq |\psi|_{\mathscr{B}}$.

The result of this section, however, is not needed in the next section.

We first notice that, if we define $\hat{L}(\hat{\phi}) = L(\phi)$ for $\hat{\phi}$ in $\hat{\mathscr{B}}$, then $\hat{L}$ is clearly a linear and continuous operator on $\hat{\mathscr{B}}$ into C^n, and the symbol of L is identical to $\hat{L}(\hat{\omega}(\lambda)I)$. From Assertion (A_2) in Section 2, the symbol of L therefore is analytic for λ with $\operatorname{Re}\lambda > \beta$. Hence the following question arises: is Relation (3.7) valid for λ with $\operatorname{Re}\lambda > \beta$? Surely, this question has a meaning only if $\beta < \alpha_0$ (cf. (A_1)). At the same time, it is not a trivial question since there exists a generating function which is continued analytically beyond the axis of convergence [8, p. 58]. To answer the question, we need a lemma which is obtained by combining the results in Widder [8, pp. 306–310].

LEMMA 4.1. *For a function $f(x)$ in $0 < x < \infty$, we have*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

with $\alpha(t)$ of bounded variation in $0 \leqq t < \infty$ if and only if $f(x)$ has derivatives of all orders in $0 < x < \infty$ and there exists a constant M such that

$$\sum_{k=0}^\infty |f^{(k)}(x)|(x^k/k!) < M \quad \textit{for} \quad 0 < x < \infty .$$

By Assertion (A_2) and Hypothesis (H_4), one can prove the following lemma without difficulty.

LEMMA 4.2. *If $\operatorname{Re}\lambda > \beta$ and b is in C^n, then, for $k = 0, 1, \cdots$, the k-th derivative $\hat{\omega}^{(k)}(\lambda)b$ of $\hat{\omega}(\lambda)b$ with respect to λ is the equivalence class of the function*

$$[\omega^{(k)}(\lambda)b](\theta) = \theta^k e^{\lambda\theta} b \quad \textit{for} \quad \theta \leqq 0 .$$

Hypothesis (H_5) is used to derive Estimate (4.3) in the following lemma.

LEMMA 4.3. *Let c be a constant such that $c > \beta$ and $\{\sigma(k)\}$ a sequence of C such that $|\sigma(k)| = 1$ for all k. If $(\beta - c)/2 < \lambda < \infty$, then the series*

$$\xi(\lambda)(\theta) = \sum_{k=0}^\infty \sigma(k)((\lambda\theta)^k/k!)e^{(\lambda+c)\theta} b \quad \textit{for} \quad \theta \leqq 0 \tag{4.1}$$

converges absolutely in $\mathscr{B}$, and

$$\hat{\xi}(\lambda) = \sum_{k=0}^\infty \sigma(k)(\lambda^k/k!)[\hat{\omega}^{(k)}(\lambda + c)b] . \tag{4.2}$$

Furthermore, if Hypothesis (H_5) holds for the space $\mathscr{B}$, then

$$|\xi(\lambda)|_{\mathscr{B}} \leqq |\omega(c)b|_{\mathscr{B}} \quad \textit{for} \quad 0 \leqq \lambda < \infty . \tag{4.3}$$

PROOF. Around λ_0 with $\lambda_0 > (\beta - c)/2$, draw a circle C of a radius ρ on the half plane $D = \{\lambda: \operatorname{Re}\lambda > \beta - c\}$. By the assumption $\beta - c < 0$, one can take ρ to satisfy $|\lambda_0|/\rho < 1$. Since $\hat{\omega}(\lambda + c)b$ is an analytic function on D into $\hat{\mathscr{B}}$, Cauchy's estimate implies that

$$|\hat{\omega}^{(k)}(\lambda_0 + c)b|_{\hat{\mathscr{B}}} \leqq k!\, M/\rho k \qquad k = 0, 1, \cdots ,$$

where $M = \sup\{|\hat{\omega}(\lambda + c)b|_{\hat{\mathscr{B}}}: \lambda \in C\}$. This guarantees that Series (4.2) converges absolutely for $\lambda = \lambda_0$. By Lemma 4.2, if $\xi^n(\lambda)(\theta)$ denotes the sum of the first n terms of Series (4.1), then $(\xi^n(\lambda))^\wedge$ coincides with the sum of the corresponding terms of Series (4.2). This implies that $\{(\xi^n(\lambda_0))^\wedge\}$ is a Cauchy sequence of $\hat{\mathscr{B}}$. Since $\xi^n(\lambda)(\theta) \to \xi(\lambda)(\theta)$ as $n \to \infty$ uniformly for θ in every compact set of $(-\infty, 0]$, one has Relation (4.2) by using Hyothesis (H_4).

The assumption $|\sigma(k)| = 1$ for all k leads to the relation $|\sigma(k)(\lambda\theta)^k| = (-\lambda\theta)^k$ for $\lambda \geqq 0$ and $\theta \leqq 0$. Hence, if $\lambda \geqq 0$, the Definition (4.1) of $\xi(\lambda)(\theta)$ immediately gives the inequality $|\xi(\lambda)(\theta)| \leqq |\exp(c\theta)b|$ for all $\theta \leqq 0$. Hypothesis (H_5) therefore implies Relation (4.3). q.e.d.

We now prove the main theorem of this section.

THEOREM 4.4. *Let η be the function defined by Relation* (3.3) *and β the common value of the "essential" type numbers of solution semigroups. If the space $\mathscr{B}$ satisfies Hypotheses* $(H_0), \cdots, (H_4)$ *and* (H_5), *then for b in C^n,*

$$L(\omega(\lambda)b) = \int_{-\infty}^{0} d\eta(\theta)e^{\lambda\theta}b \quad \textit{for} \quad \operatorname{Re}\lambda > \beta \,, \tag{4.4}$$

and, for every $\varepsilon > 0$ there exists a $c(\varepsilon)$ such that, for $t \geqq s \geqq 0$,

$$V(\eta, [-t, -s]) \leqq c(\varepsilon)\max\{e^{(\beta+\varepsilon)t}, e^{(\beta+\varepsilon)s}\} \,.$$

PROOF. Let $\zeta(t)$ be defined by Relation (3.6). To prove Relation (4.4), it sufficies to show that Relation (3.7) holds with α_0 replaced by β, or equivalently, every entry of $L(\omega(\lambda)I)$ is the Laplace-Stieltjes transform of the corresponding entry of $\zeta(t)$ for $\operatorname{Re}\lambda > \beta$. Thus we can assume $n = 1$ without loss of generality.

In the beginning, we set

$$f(\lambda) = \hat{L}(\hat{\omega}(\lambda)) \equiv L(\omega(\lambda)) \quad \text{for} \quad \operatorname{Re}\lambda > \beta \,.$$

Let $c > \beta$ be fixed. Since $f(\lambda)$ is analytic in $\operatorname{Re}\lambda > \beta$, the function $f(\lambda + c)$ is analytic in $\operatorname{Re}\lambda > \beta - c$. We observe that $f(\lambda + c)$ satisfies the condition in Lemma 4.1 with x and $f(x)$ replaced by λ and $f(\lambda + c)$, respectively. In fact, since $\hat{L}$ is linear and continuous, it follows that

$f^{(k)}(\lambda+c)=\hat{L}(\hat{\omega}^{(k)}(\lambda+c))$. For each $k=0,1,\cdots$, take a $\sigma(k)$ in $\boldsymbol{C}$ such that $|\sigma(k)|=1$ and that $|\hat{L}(\hat{\omega}^{(k)}(\lambda+c))|=\sigma(k)\hat{L}(\hat{\omega}^{(k)}(\lambda+c))$. The sequence $\{\sigma(k)\}$ surely depends on λ. Let $\xi(\lambda)(\theta)$ be the function defined by Relation (4.1). Since $\hat{L}$ is linear and continuous, and since Series (4.2) converges in $\hat{\mathscr{B}}$, it follows that $\sum_{k=0}^{\infty}(\lambda^k/k!)\sigma(k)\hat{L}(\hat{\omega}^{(k)}(\lambda+c))=\hat{L}(\hat{\xi}(\lambda))$. Notice that every term of this series is nonnegative, which implies $\hat{L}(\hat{\xi}(\lambda))\geqq 0$. Since the space $\mathscr{B}$ satisfies Hypothesis (H_5), Lemma 4.3 implies that $\hat{L}(\hat{\xi}(\lambda))\leqq|\hat{L}|\,|\hat{\omega}(c)|_{\hat{\mathscr{B}}}$ for $\lambda\geqq 0$. Summarizing these results, we obtain the desired inequality $\sum_{k=0}^{\infty}|f^{(k)}(\lambda+c)|(\lambda^k/k!)\leqq|\hat{L}|\,|\hat{\omega}(c)|_{\hat{\mathscr{B}}}$ for $0\leqq\lambda<\infty$.

From Lemma 4.1, it now follows that, for $\lambda>0$, the function $f(\lambda+c)$ is the Laplace-Stieltjes transform of some function $\mu^c(t)$ of bounded variation in $0\leqq t<\infty$. This relation is obviously rewritten as

$$f(\lambda)=\int_0^{\infty}e^{-\lambda t}e^{ct}d\mu^c(t)\quad\text{for}\quad\lambda>c\,. \tag{4.5}$$

Furthermore, if we set

$$\zeta^c(t)=\int_0^{t}e^{ct}d\mu^c(t)\quad\text{for}\quad t>0\,, \tag{4.6}$$

then Relation (4.5) becomes

$$L(\omega(\lambda))\equiv f(\lambda)=\int_0^{\infty}e^{-\lambda t}d\zeta^c(t)\quad\text{for}\quad\lambda>c\,. \tag{4.7}$$

Combining this with Relation (3.7), we see that

$$\int_0^{\infty}e^{-\lambda t}d\zeta(t)=\int_0^{\infty}e^{-\lambda t}d\zeta^c(t)\,, \tag{4.8}$$

provided $\lambda>\max(\alpha_0,c)$. It is well known that the Laplace-Stieltjes transform of a function μ locally of bounded variation does not change if μ is replaced by its normalized function μ^*, that is, $\mu^*(0)=0$ and $\mu^*(\tau)=\lim_{t\to\tau+0}\mu(t)-\mu(0)$ for $\tau>0$. Thus we can assume that $\zeta^c(t)$ is normalized. Then Relation (4.8) implies that $\zeta(t)=\zeta^c(t)$ for $t\geqq 0$ since ζ is also normalized and "there cannot exist two different normalized functions corresponding to the same generating function" [8, p. 63]. Consequently, we can replace ζ^c in Relation (4.7) by ζ. Since $c>\beta$ is arbitrary, the lower bound c is also replaced by β. Thus we conclude that

$$L(\omega(\lambda))=\int_0^{\infty}e^{-\lambda t}d\zeta(t)\quad\text{for}\quad\operatorname{Re}\lambda>\beta\,.$$

Finally, Relation (4.6) yields that $V(\zeta^c,[s,t])\leqq V(\mu^c,[s,t])\max\{e^{cs},e^{ct}\}$ for $t\geqq s\geqq 0$. This implies the estimate for $V(\eta,[-t,-s])$ in the theorem since μ^c is of bounded variation in $[0,\infty)$. q.e.d.

5. The fundamental matrix. The fundamental matrix $X(t)$ of Equation (1.1) was defined in [6] as follows. Let α_L be the type number of $\hat{T}_L(t)$. A similar number μ is defined for the function $M(t)$ arising in Hypothesis (H_2), that is, $\mu = \lim_{t\to\infty} \log M(t)/t = \inf_{t>0} \log M(t)/t$. The characteristic matrix of Equation (1.1) is a matrix $\Delta(\lambda)$ defined by

$$\Delta(\lambda) = \lambda I - L(\omega(\lambda)I) ,$$

which is well defined and analytic in λ with $\operatorname{Re}\lambda > \beta$. Its determinant does not vanish if $\operatorname{Re}\lambda > \alpha_L$, while $\Delta(\lambda)^{-1} = (\lambda - \alpha_L)^{-1}I + O((\lambda - \alpha_L)^{-2})$ as $\operatorname{Re}\lambda \to \infty$. By this property, the matrix $X(t)$ is defined through the inverse Laplace transform of $\Delta(\lambda)^{-1}$:

$$X(t) = \begin{cases} \lim\limits_{T\to\infty} \dfrac{1}{2\pi i} \displaystyle\int_{c-iT}^{c+iT} e^{\lambda t}\Delta(\lambda)^{-1}d\lambda & \text{for} \quad t > 0 \\ I & \text{for} \quad t = 0 , \end{cases} \tag{5.1}$$

where c is an arbitrary constant such that $c > \max\{\alpha_L, \mu\}$. The following results are proved in [6]:

(A_3) $X(t)$ is continuous for $t \geqq 0$.

(A_4) $|X(t)| = O(\exp(c + \varepsilon)t)$ as $t \to \infty$ for every $\varepsilon > 0$.

(A_5) The matrix $\Delta(\lambda)^{-1}$ is the Laplace transform of $X(t)$ for λ with $\operatorname{Re}\lambda > \max\{\alpha_L, \mu\}$.

(A_6) $X(t)$ gives the variation-of-constants formula for solutions of the nonhomogeneous equation corresponding to Equation (1.1).

For simplicity, we set

$$\mathscr{L}(f)(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt ,$$

whenever this integral converges.

Our next objective is to consider whether $X(t)$ itself satisfies Equation (1.1) or not. This has a meaning only if $X(t)$ is defined for $t < 0$ also. As in the case of finite delay, we set

$$X(t) = 0 \quad \text{for} \quad t < 0 . \tag{5.2}$$

Then every column vector function of X_t lies in Γ for $t \geqq 0$. In short we say that X_t lies in Γ. Similar expressions will be used for matrix functions. Thus $\tilde{L}(X_t)$ is well defined for $t \geqq 0$, while Theorem 3.4 and Relation (3.6) imply that

$$\tilde{L}(X_t) = \int_{-t}^0 d\eta(\theta)X(t+\theta) = \int_0^t d\zeta(s)X(t-s) \quad \text{for} \quad t > 0 . \tag{5.3}$$

Also $\tilde{L}(X_t)$ is continuous to the right for $t \geqq 0$ since, for $\tau \geqq 0$, the function

X_t converges in Lebesgue to X_τ as $t \to \tau + 0$. This observation implies that $\tilde{L}(X_0) = \eta(0-)$. Similarly, $\tilde{L}(X_t)$ has a limit as $t \to \tau - 0$ for $\tau > 0$. Thus $\tilde{L}(X_t)$ has no discontinuity of the second kind. It is well known that such a function is Riemann integrable over compact intervals provided it is bounded there. Since $\tilde{L}(X_t)$ is clearly locally bounded on $[0, \infty)$, it is Riemann integrable over every compact interval of $[0, \infty)$.

To proceed further, let us introduce some results from the theory of Laplace-Stieltjes transform (see [8, pp. 83–91]). The Stieltjes resultant of $f(t)$ and $g(t)$ is the function

$$h(t) = \int_0^t f(t-s)dg(s) = \int_0^t df(s)g(t-s)$$

when these two integrals exist and are equal. Suppose f and g are normalized functions locally of bounded variation in $[0, \infty)$, and denote by P_f the countable set of points where $f(t)$ is discontinuous, with a similar meaning for P_g. Then $h(t)$ exists for every t in $(0, \infty)$ not in the set $P_{f+g} \equiv \{t = u + v \colon u \in P_f \text{ and } v \in P_g\}$, where P_{f+g} is empty if at least one of the sets P_f and P_g is empty. Furthermore, $h(t)$ can be defined in points P_{f+g} so as to become a normalized function locally of bounded variation in $[0, \infty)$.

LEMMA 5.1 [8, Theorem 11.6b, p. 89]. *If $f(t)$, $g(t)$ and $h(t)$ are defined as above, and if the integrals*

$$F(\lambda) = \int_0^\infty e^{-\lambda t} df(t)\ , \quad G(\lambda) = \int_0^\infty e^{-\lambda t} dg(t)$$

converge, one of them absolutely, then

$$F(\lambda)G(\lambda) = \int_0^\infty e^{-\lambda t} dh(t)\ .$$

We can now demonstrate the main result.

THEOREM 5.2. *The fundamental matrix $X(t)$ defined by Relations (5.1) and (5.2) is locally absolutely continuous on $[0, \infty)$. It is a unique solution of the equation*

$$X(t) = \begin{cases} I + \int_0^t \tilde{L}(X_t)dt & \text{for} \quad t \geqq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} \tag{5.4}$$

or

$$dX/dt = \tilde{L}(X_t) \quad \text{a.e. in} \quad t \geqq 0$$
$$X(0) = I \quad \text{and} \quad X(t) = 0 \quad \text{for} \quad t < 0\ .$$

PROOF. By the standard method of successive approximations we can show that Equation (5.4) has a unique solution which is locally absolutely continuous. We are led to consider whether the Laplace transform of this solution coincides with $X(t)$. However it is difficult to follow this line. In fact, suppose U is the solution of Equation (5.4). Then Proposition 3.3 implies that $|\tilde{L}(U_t)| \leqq c|L|K(t)\sup\{|U(s)|: 0 \leqq s \leqq t\}$. Applying Gronwall's lemma, we then obtain

$$|U(t)| \leqq \exp\left\{\int_0^t c|L|K(s)ds\right\} \quad \text{for} \quad t \geqq 0 .$$

Thus, if we impose no other condition on $K(t)$ than continuity, we must estimate $|U(t)|$ in a different manner to consider $\mathscr{L}(U)(\lambda)$.

However, going in the reverse direction, we can easily prove the theorem. We start with the trivial relation $[\lambda I - L(\omega(\lambda)I)]\Delta(\lambda)^{-1} = I$ or

$$(1/\lambda)I = \Delta(\lambda)^{-1} - L(\omega(\lambda)I)(1/\lambda)\Delta(\lambda)^{-1}$$

for $\operatorname{Re}\lambda > \alpha_L$ and $\lambda \neq 0$. It is clear that, for $\operatorname{Re}\lambda > 0$, the function $\lambda^{-1}I$ is the Laplace transform of the constant function I. Also, Assertion (A_5) is already established.

We first show that the function

$$H(\lambda) = L(\omega(\lambda)I)(1/\lambda)\Delta(\lambda)^{-1}$$

is a generating function of a Laplace-Stieltjes transform. Indeed, Relation (3.7) is proved in Section 3. On the other hand, if we set

$$Y(t) = \int_0^t X(t-s)ds = \int_0^t X(s)ds \quad \text{for} \quad t \quad \text{in} \quad \boldsymbol{R} ,$$

then $\lambda^{-1}\Delta(\lambda)^{-1} = \mathscr{L}(Y)(\lambda)$ for $\operatorname{Re}\lambda > \gamma \equiv \max\{\alpha_L, \mu, 0\}$, since $Y(t)$ is the resultant of the constant function I and the function X. We can rewrite this relation as

$$(5.5) \qquad (1/\lambda)\Delta(\lambda)^{-1} = \int_0^\infty e^{-\lambda t}dZ(t) \quad \text{for} \quad \operatorname{Re}\lambda > \gamma ,$$

where

$$Z(t) = \int_0^t Y(s)ds \quad \text{for} \quad t \quad \text{in} \quad \boldsymbol{R} .$$

By Relation (A_4), $|Y(t)|$ satisfies the same order relation as $|X(t)|$ when $t \to \infty$, so Integral (5.5) converges absolutely. Since $Z(t)$ is clearly a continuous and normalized function locally of bounded variation in $[0, \infty)$, the Stieltjes resultant

$$W(t) = \int_0^t d\zeta(s)Z(t-s) = \int_0^t \zeta(t-s)dZ(s)$$

is well defined for every $t \geqq 0$. Therefore, Lemma 5.1 asserts that

$$H(\lambda) = \int_0^\infty e^{-\lambda t} dW(t) \quad \text{for} \quad \operatorname{Re}\lambda > \max\{\gamma, \alpha_0\} . \tag{5.6}$$

We next show that Integral (5.6) is really a Laplace transform. Observe that, for every $t \geqq 0$, the function $Z_t(\theta)$, with $\theta \leqq 0$, satisfies the assumptions in Theorem 3.4. Using Relation (3.6), we have

$$\int_0^t d\zeta(s)Z(t-s) = \int_{-t}^0 d\eta(\theta)Z(t+\theta) = \tilde{L}(Z_t) ,$$

which implies $W(t) = \tilde{L}(Z_t)$ for $t \geqq 0$. Interchanging the order of integration and substituting the integral variable, we obtain

$$Z_t(\theta) = \int_0^{t+\theta}\int_0^u X(s)dsdu = \int_{-\theta}^t (t-s)X(s+\theta)ds .$$

According to Relation (5.2), this becomes

$$Z_t(\theta) = \int_0^t (t-s)X(s+\theta)ds \quad \text{for} \quad \theta \leqq 0 .$$

For a partition P of $[0, t]$ such that $0 = s(0) < s(1) < \cdots < s(d) = t$, we set $\Phi^P = \sum_{i=1}^d (t - \sigma(i))X_{\sigma(i)}(s(i) - s(i-1))$, where $s(i-1) \leqq \sigma(i) \leqq s(i)$, $i = 1, \cdots, d$. Immediately, it follows that Φ^P is in Γ and converges in Lebesgue to Z_t as $m(P) \to 0$. On the other hand, the linearity of $\tilde{L}$ leads to the relation $\tilde{L}(\Phi^P) = \sum_{i=1}^d (t - \sigma(i))\tilde{L}(X_{\sigma(i)})(s(i) - s(i-1))$. Since $\tilde{L}$ is continuous in Lebesgue, it follows that

$$W(t) = \tilde{L}(Z_t) = \int_0^t (t-s)\tilde{L}(X_s)ds \quad \text{for} \quad t \geqq 0 . \tag{5.7}$$

Attention must be paid to the fact that $\tilde{L}(X_t)$ is Riemann integrable. Therefore, Relation (5.6) becomes

$$H(\lambda) = \int_0^\infty e^{-\lambda t}\int_0^t \tilde{L}(X_s)dsdt \quad \text{for} \quad \operatorname{Re}\lambda > \max\{\gamma, \alpha_0\} .$$

Summarizing the above results, we finally obtain

$$\int_0^\infty e^{-\lambda t} I dt = \int_0^\infty e^{-\lambda t} X(t)dt - \int_0^\infty e^{-\lambda t}\int_0^t \tilde{L}(X_s)dsdt$$

provided $\operatorname{Re}\lambda > \max\{\alpha_0, \alpha_L, \mu, 0\}$. By the uniqueness of determining function, we have Relation (5.4). q.e.d.

In case the delay is finite and the phase space is $C([-r, 0], C^n)$, X_t

lies in the phase space for $t \geqq r$ and $dX/dt = L(X_t)$ for all $t \geqq r$. An analogous result holds for our equation. Before stating the theorem, we observe some examples of the space $\mathscr{B}$. Hypotheses $(\mathrm{H}_0), \cdots, (\mathrm{H}_5)$ are satisfied by spaces of functions which are isomorphic to $L^p((-\infty, -r), \mu) \times C([-r, 0])$ for some special measure μ, Also, the space of continuous functions $\phi(\theta)$ which have a limit, $\lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)$, for some γ in $\boldsymbol{R}$. The function X_r lies in the former but, for every $t \geqq 0$, the function X_t does not lie in the latter.

THEOREM 5.3. *Let X be the fundamental matrix, and suppose there exists some $r \geqq 0$ such that X_r lies in $\mathscr{B}$. Then it follows that*

$$dx/dt = L(X_t) \quad \textit{for every} \quad t \geqq r\,.$$

PROOF. Consider the equation

$$dY/dt = L(Y_t) \quad \text{for} \quad t \geqq r\,, \quad \text{and} \quad Y_r = X_r\,. \tag{5.8}$$

Since X_r is in $\mathscr{B}$, this equation has a unique solution $Y(t)$, and Y_t is given by $Y_t = T_L(t-r)X_r$ for $t \geqq r$. Since $|T_L(t)| = O(\exp(\alpha_L + \varepsilon)t)$ as $t \to \infty$ for every $\varepsilon > 0$, the same order relation holds for $|Y_t|_{\mathscr{B}}$ and $|Y(t)|$ (cf. Hypothesis (H_3)). Therefore, the Laplace transform $\mathscr{L}(Y)(\lambda)$ converges for λ with $\operatorname{Re}\lambda > \alpha_L$. Also, by the condition $Y_r = X_r$ or $Y(t) = X(t)$ for $t \leqq r$, Theorem 5.2 says that Y is absolutely continuous in $[0, r]$ and

$$dY/dt = \tilde{L}(Y_t) \quad \text{a.e. in} \quad t \in [0, r]\,. \tag{5.9}$$

Therefore, Y is locally absolutely continuous in $[0, \infty)$, and integration by parts gives

$$\mathscr{L}(dY/dt)(\lambda) = -I + \lambda\mathscr{L}(Y)(\lambda) \quad \text{for} \quad \operatorname{Re}\lambda > \alpha_L\,. \tag{5.10}$$

Combining Relations (5.8) and (5.9), we also obtain

$$\mathscr{L}(dY/dt)(\lambda) = \int_0^r e^{-\lambda t}\tilde{L}(Y_t)dt + \int_r^\infty e^{-\lambda t}L(Y_t)dt \tag{5.11}$$

provided $\operatorname{Re}\lambda > \alpha_L$.

To proceed further, we set

$$\Phi(\theta) = \int_0^r e^{-\lambda t}Y(t+\theta)dt\,, \quad \Psi(\theta) = \int_r^\infty e^{-\lambda t}Y(t+\theta)dt$$

for $\theta \leqq 0$. Following the arguments similar to the proof of Relation (5.7), we know that the first integral in Relation (5.11) coincides with $\tilde{L}(\Phi)$. Since Φ lies in $\mathscr{C}$, Theorem 3.1 implies that $\tilde{L}(\Phi) = L(\Phi)$. Thus we obtain

$$L(\Phi) = \int_0^r e^{-\lambda t}\tilde{L}(Y_t)dt\,.$$

On the other hand, using Hypothesis (H_4) and the relation that $|\hat{Y}_t|_{\hat{\mathscr{B}}} = |Y_t|_{\mathscr{B}} = O(\exp(\alpha_L + \varepsilon)t)$ as $t \to \infty$ for every $\varepsilon > 0$, it is not difficult to show that

$$\hat{\Psi} = \int_r^\infty e^{-\lambda t}\hat{Y}_t dt \quad \text{for} \quad \operatorname{Re}\lambda > \alpha_L .$$

Since $L: \mathscr{B} \to C^n$ is linear and continuous, this implies that

$$L(\Psi) = \hat{L}(\hat{\Psi}) = \int_r^\infty e^{-\lambda t}\hat{L}(\hat{Y}_t)dt = \int_r^\infty e^{-\lambda t}L(Y_t)dt$$

for $\operatorname{Re}\lambda > \alpha_L$.

Thus the right hand side of Relation (5.11) coincides with $L(\Phi + \Psi)$ for $\operatorname{Re}\lambda > \alpha_L$. Since $Y(t) = 0$ for $t < 0$, it follows that $\Phi(\theta) + \Psi(\theta) = \exp(\lambda\theta)\mathscr{L}(Y)(\lambda)$ for $\theta \leqq 0$, that is, $\Phi + \Psi = \omega(\lambda)\mathscr{L}(Y)(\lambda)$. Relation (5.11) now becomes

$$\mathscr{L}(dY/dt)(\lambda) = L(\omega(\lambda)\mathscr{L}(Y)(\lambda)) \quad \text{for} \quad \operatorname{Re}\lambda > \alpha_L .$$

Hence in view of Relation (5.10) we obtain $\Delta(\lambda)\mathscr{L}(Y)(\lambda) = I$ for $\operatorname{Re}\lambda > \alpha_L$. From this result and Assertion (A_5), it follows that $\mathscr{L}(X)(\lambda) = \mathscr{L}(Y)(\lambda)$ provided $\operatorname{Re}\lambda$ is sufficiently large. This implies that $X(t) = Y(t)$ for all t in $(-\infty, +\infty)$. Therefore, Relation (5.8) means that $dX/dt = L(X_t)$ for $t \geqq r$. This is the desired result. q.e.d.

COROLLARY 5.4. *Under the same assumptions as in Theorem* 5.3, *the following conclusions hold*:

(i) $|X(t)| = O(\exp(\alpha_L + \varepsilon)t)$ *as* $t \to \infty$ *for every* $\varepsilon > 0$.

(ii) $\tilde{L}(X_t) = L(X_t)$ *for every* $t \geqq r$.

PROOF. The first statement follows from the estimate for $|Y(t)|$ given in the proof of Theorem 5.3. Theorems 5.2 and 5.3 imply that $\tilde{L}(X_t) = L(X_t)$ a.e. in $t \geqq r$. Since $\tilde{L}(X_t)$ is continuous to the right for $t \geqq 0$, we arrive at the second statement. q.e.d.

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