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A NOTE ON A FOURIER MULTIPLIER OF TWO VARIABLES

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1. Introduction. Let m be a bounded measurable function on $\boldsymbol{R}^2$. Define a linear operator T_m by

$$(T_m f)\hat{}(\xi, \eta) = m(\xi, \eta)\hat{f}(\xi, \eta) , \qquad f \in L^2(\boldsymbol{R}^2) \cap L^p(\boldsymbol{R}^2) ,$$

where $\hat{f}$ is the Fourier transform of f, and $1 \leqq p \leqq \infty$. We say that m is a multiplier for $L^p(\boldsymbol{R}^2)$ if $T_m \in L^p(\boldsymbol{R}^2)$, and there exists a constant A, independent of f, such that

$$\| T_m f \|_p \leqq A \| f \|_p , \qquad f \in L^2(\boldsymbol{R}^2) \cap L^p(\boldsymbol{R}^2) .$$

Carleson and Sjölin [1] have proved that $(1 - (\xi^2 + \eta^2))_+^\lambda$, $0 < \lambda \leqq 1/2$, is a multiplier for L^p if and only if $4/(3 + 2\lambda) < p < 4/(1 - 2\lambda)$. Here we have used the notation $r_+ = \max(r, 0)$; $r \in \boldsymbol{R}$. Recently Cordoba [2] has proved this two dimensional result by using the Kakeya maximal function and a g-function (see also [3]). On the other hand, the above multiplier theorem has been extended to one for the following more general functions m by Sjölin [5].

THEOREM 1. *Let Γ be a simple and closed C^∞ curve with non-zero curvature in $\boldsymbol{R}^2$ and Ω be the inside of Γ. For $(\xi, \eta) \in \boldsymbol{R}^2$, let $\delta(\xi, \eta)$ denote the distance from (ξ, η) to Γ and let $\lambda > 0$. We assume that m is a bounded function on $\boldsymbol{R}^2$ which has the following properties:*

(A) *The restriction to Ω of m belongs to $C^2(\Omega)$.*

(B) *There exists a neighborhood Ω' of Γ such that*

$$m(\xi, \eta) = \delta(\xi, \eta)^\lambda \quad for \quad (\xi, \eta) \in (\Omega \cap \Omega') .$$

(C) $$m(\xi, \eta) = 0 , \quad for \quad (\xi, \eta) \notin \Omega .$$

Then:

(a) *m is a multiplier for $L^p(\boldsymbol{R}^2)$ for $1 \leqq p \leqq \infty$ if $\lambda > 1/2$.*

(b) *If $0 < \lambda \leqq 1/2$, m is a multiplier for $L^p(\boldsymbol{R}^2)$ if and only if $4/(3 + 2\lambda) < p < 4/(1 - 2\lambda)$.*

Actually Sjölin [5] has proved Theorem 1 for a C^∞ curve Γ which is simple and closed and has a tangent at each point. In this note we shall show that Cordoba's techniques in [2] is applicable to more general cases and we shall give a simpler proof of Theorem 1.

2. A lemma. We begin with the following geometrical observation.

LEMMA 2. *Let $I=[-a, a]$ $(a>0)$ be a compact interval on $\boldsymbol{R}$ and let $\psi\in C^\infty(I)$ be a real valued function such that $\psi''>0$, $\psi<-2$ on I. Furthermore, we assume that $|\psi'(a)|$ and $|\psi'(-a)|$ are less than 1/2.*

For $\delta>0$, and for each integer j, we define a set $E_{\delta,j}$ by

$$E_{\delta,j}=\{(\xi,\eta)\in\boldsymbol{R}^2\,|\,\xi\in I,\ 0\leqq\eta-\psi(\xi)\leqq\delta,\\ -\eta\tan((j-1/2)\delta^{1/2})\leqq\xi\leqq-\eta\tan((j+1/2)\delta^{1/2})\}\ .$$

Then, for each small δ no point of $\boldsymbol{R}^2$ belongs to more than N of the sets $E_{\delta,j}+E_{\delta,j'}$, where N is independent of δ.

PROOF. By changing coordinates, it is sufficient to show that the number of the sets $E_{\delta,j}+E_{\delta,j'}$ that intersect the fixed $E_{\delta,j_0}+E_{\delta,-j_0}$ $(j_0\geqq 1)$ is less than N, assuming $\psi'(0)=0$. Let (ξ_j,η_j) be the point of intersection of the line $\xi=-\eta\tan(j\delta^{1/2})$ with the curve $\eta=\psi(\xi)$. Then, there exist constants c_1 and c_2 not depending on j or δ such that $c_1\delta^{1/2}\leqq\xi_{j+1}-\xi_j\leqq c_2\delta^{1/2}$. Now let $k>0$. By the mean value theorem there exist $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3\in I$ and b_1, b_2 such that $c_1\leqq b_i\leqq c_2$ $(i=1, 2)$,

$$\begin{aligned}\eta_{j_0+k}-\eta_{j_0}&=\psi(b_1j_0\delta^{1/2}+b_2k\delta^{1/2})-\psi(b_1j_0\delta^{1/2})\\ &=b_2k\delta\psi''(\tilde{\xi}_1)\{b_1j_0+b_2k\psi''(\tilde{\xi}_2)/(2\psi''(\tilde{\xi}_3))\}\\ &\geqq Ak(2j_0+k)\delta\end{aligned}$$

for some constant A not depending on k, j_0, or δ.

Let L_j^1 be the length of the projection of $E_{\delta,j}$ on the η-axis. Then, using $\psi'(0)=0$, we have

$$L_{j_0}^1\leqq Bj_0\delta\ ,\qquad L_{j_0+k}^2\leqq B(j_0+k)\delta\ ,$$

for some $B>0$ not depending on j_0, k, or δ. The same argument is also valid for the part of $E_{\delta,-j_0}$, $E_{\delta,-(j_0+k)}$. Therefore if $(E_{\delta,j_0}+E_{\delta,-j_0})\cap(E_{\delta,j_0+k}+E_{\delta,-(j_0+k)})$ is not empty, we have

$$Ak(2j_0+k)\delta\leqq B(2j_0+k)\delta\ ,\quad\text{so}\quad k\leqq B/A\ .$$

The case $k<0$ can be treated similarly.

Next let L_j^2 be the length of the projection of $E_{\delta,j}$ on the ξ-axis. Then $L_j^2\leqq c_2\delta^{1/2}$. If $(E_{\delta,j_0+[B/A]}+E_{\delta,-(j_0+[B/A]+k)})\cap(E_{\delta,j_0}+E_{\delta,-j_0})$ is not empty for $k>0$, we have

$$([B/A]+k)c_1\delta^{1/2}-[B/A]c_2\delta^{1/2}\leqq 4c_2\delta^{1/2}\ ,$$

thus $k\leqq(c_1/c_2)(4+[B/A])-[B/A]$. After the same argument for $E_{\delta,j_0+[B/A]+k}+E_{\delta,-(j_0+[B/A])}$, $E_{\delta,j_0-[B/A]}+E_{\delta,-(j_0-[B/A]-k)}$, and $E_{\delta,j_0-[B/A]-k}+E_{\delta,-(j_0-[B/A])}$, it follows that $(E_{\delta,j_0+j_1}+E_{\delta,-j_0+j_2})\cap(E_{\delta,j_0}+E_{\delta,-j_0})$ is not empty only if $|j_i|\leqq(c_2/c_1)(4+[B/A])$, $i=1, 2$. Thus the lemma is proved.

3. Proof of Theorem 1. Let $\psi\in C^\infty(I)$ be a function given in Lemma 2. It is sufficient to show that $a(\xi,\eta)(\eta-\psi(\xi))_+^\lambda$ is a multiplier, where $a(\xi,\eta)$ is a C^∞-function in $I\times \boldsymbol{R}$ with compact support.

Let $m_\lambda(\xi,\eta)=a(\xi,\eta)(\eta-\psi(\xi))_+^\lambda$. Now we make the first decomposition of $m_\lambda(\xi,\eta)$. Let ϕ be a C^∞-function in $\boldsymbol{R}$ such that $0\leqq\phi\leqq 1$, $\phi\equiv 1$ on $[1/2, 1]$, and $\phi\equiv 0$ outside $[1/4, 2]$. Let

$$\phi_j(r)=\phi(2^jr)\Big/\sum_{k=0}^\infty\phi(2^kr)\,,\qquad j=1,2,3,\cdots .$$

Decompose $m_\lambda(\xi,\eta)$ into

$$m_\lambda(\xi,\eta)=\Big(1-\sum_{j=1}^\infty\phi_j(\eta-\psi(\xi))\Big)m_\lambda(\xi,\eta)+\sum_{j=1}^\infty\phi_j(\eta-\psi(\xi))m_\lambda(\xi,\eta)\,.$$

Then since the first term is a C^∞-function with compact support, it suffices to estimate the second term.

Set $m_j(\xi,\eta)=\phi_j(\eta-\psi(\xi))m_\lambda(\xi,\eta)$, and define T_j by $(T_jf)^\wedge(\xi,\eta)=m_j(\xi,\eta)\hat f(\xi,\eta)$, $j=1,2,3,\cdots$. We shall prove

$$\|T_jf\|_4\leqq C2^{-j\lambda}j^{1/4}\|f\|_4\,, \tag{1}$$

with C independent of j.

For this purpose we make the following decomposition. Let Φ be a C^∞-function such that $0\leqq\Phi\leqq 1$, $\Phi\equiv 1$ on $[-1/2, 1/2]$, and $\Phi\equiv 0$ outside $[-2/3, 2/3]$. Let

$$\Phi_k(\theta)=\Phi(2^{j/2}(\theta-k/2^{j/2}))\Big/\sum_{l=-\infty}^{\infty}\Phi(2^{j/2}(\theta-l/2^{j/2}))$$

for each integer k. Decompose $m_j(\xi,\eta)$ into

$$m_j(\xi,\eta)=\sum_{|k|\leqq C2^{j/2}}m_j(\xi,\eta)\Phi_k(\arctan(-\xi/\eta))\equiv\sum_k m_j^k(\xi,\eta)\,,$$

and define T_j^k by $(T_j^kf)^\wedge(\xi,\eta)=m_j^k(\xi,\eta)\hat f(\xi,\eta)$. Notice $T_j=\sum_k T_j^k$. By Lemma 2 we have

$$\|T_jf\|_4\leqq C\|(\sum_k|T_j^kf|^2)^{1/2}\|_4\,, \tag{2}$$

with C independent of j (see [2], [3]).

Next let (ξ_k,η_k) be the point of intersection of the line $\xi=-\eta\tan(k/2^{j/2})$ with the curve $\eta=\psi(\xi)$, and let θ_k be the angle that the tangent of the curve $\eta=\psi(\xi)$ at (ξ_k,η_k) makes with the ξ-axis. Then define rectangles R_n^0 by

$$R_n^0=\{(x,y)\in\boldsymbol{R}^2\,|\,|x|\leqq 2^{j/2}2^n,\ |y|\leqq 2^j2^n\}\qquad n=0,1,2,\cdots ,$$

and let R_n^k be the rectangle obtained by rotating R_n^0 by θ_k. Let K_j^k be the kernel of T_j^k. We shall show

$$(3) \qquad |K_j^k(x, y)| \leqq C2^{-j\lambda} \sum_{n=0}^{\infty} 2^{-n}\chi_{R_n^k}(x, y)/|R_n^k|$$

with a constant C independent of j and k, where $|R_n^k|$ denotes the Lebesgue measure of R_n^k. To prove (3), define u, v by

$$\xi = u\cos\theta_k - v\sin\theta_k\ , \quad \eta = u\sin\theta_k + v\cos\theta_k\ ,$$

and write $\tilde{m}_j^k(u, v) = m_j^k(\xi, \eta)$. Then

$$\frac{\partial}{\partial u}(\eta - \psi(\xi)) = \sin\theta_k - \psi'(\xi)\cos\theta_k = (\psi'(\xi_k) - \psi'(\xi))\cos\theta_k$$

so that

$$\left|\frac{\partial}{\partial u}(\eta - \psi(\xi))\right| \leqq C2^{-j/2}\ , \quad \text{for} \quad (\xi, \eta) \in \operatorname{supp}(m_j^k)\ .$$

Therefore we have

$$\left|\frac{\partial^{\alpha+\beta}}{\partial u^\alpha \partial v^\beta}\tilde{m}_j^k(u, v)\right| \leqq C_{\alpha,\beta}2^{-j\lambda}2^{j\alpha/2}2^{j\beta}\ , \qquad \alpha \geqq 0\ , \quad \beta \geqq 0\ .$$

Then integration by parts gives

$$|(\tilde{m}_j^k)^\wedge(x, y)| \leqq C_{\alpha,\beta}2^{-j\lambda}2^{-3j/2}2^{j\alpha/2}2^{j\beta}|x|^{-\alpha}|y|^{-\beta}\ .$$

This implies easily

$$|(\tilde{m}_j^k)^\wedge(x, y)|C2^{-j\lambda}\sum_{n=0}^{\infty}2^{-n}\chi_{R_n^0}(x, y)/|R_n^0|\ .$$

Since the Fourier transform commutes with rotations, we have (3).

Having proved (2) and (3), we can now apply the g-function and the maximal theorem in [3] to prove (1), since the ratios of the lengths of the projections of $\{\operatorname{supp}(m_j^k)\}_k$ on the ξ-axis are uniformly bounded. Let K_j be the kernel of T_j. From (3) we have

$$\|K_j\|_1 \leqq C2^{-j\lambda}2^{j/2}\ .$$

Therefore if $\lambda > 1/2$, the Fourier transform of m_λ is integrable and this proves (a) of Theorem 1. If $0 < \lambda \leqq 1/2$, the sufficiency of the condition on p in Theorem 1 follows from interpolation between (1) and the obvious estimate

$$\|T_j f\|_\infty \leqq C2^{-j\lambda}2^{j/2}\|f\|_\infty\ .$$

For the part of necessity, see [5].

Finally we remark that if ψ'' has zeros of finite order in I, the method in [4, p. 8] also applies in our case to improve Theorem 1.

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