Tôhoku Math. Journ.
33 (1981), 103-107.

ON HELICES AND MULTIPLE WIENER INTEGRALS OF A GAUSSIAN AUTOMORPHISM

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(Received November 26, 1979)

0. The purpose of this note is to represent helices of a Gaussian automorphism by the multiple Wiener integrals and to calculate the multiplicity of helices.

1. Let $(\Omega, \mathscr{F}, P)$ be a complete separable probability space and $(T, \mathscr{F}_0)$ a system on Ω, that is, a pair of an automorphism of Ω and a complete sub-σ-field of $\mathscr{F}$ such that

(a) $\bigvee_{n=-\infty}^{\infty} T^n\mathscr{F}_0 = \mathscr{F}$,

(b) $T\mathscr{F}_0 \supset \mathscr{F}_0$.

Let $H = L_0^2(\Omega)$ denote the Hilbert space of all squarely integrable real random variables with zero-expectations and H_n the subspace of H consisting of all elements measurable with respect to $T^n\mathscr{F}_0$ for each n.

DEFINITION 1. A process $X = (x_n)$ is called a helix for a system $(T, \mathscr{F}_0)$ if the following conditions are satisfied:

(a) $x_0 = 0$,

(b) $x_n - x_m \in H_n \ominus H_m$ for all m and n with $m < n$,

(c) $(x_n - x_m) \circ T^{-1} = x_{n+1} - x_{m+1}$ for all m and n.

By the condition (b), $(x_n, T^n\mathscr{F}_0)_{n\geqq 0}$ can be regarded as a square-integrable martingale and further by the condition (c), all x_n can be written as

$$x_n = \sum_{k=1}^{n} x \circ T^{-(k-1)}$$

for some $x \in H_1 \ominus H_0$.

DEFINITION 2. For helices $X = (x_n)$ and $X' = (x'_n)$, $\mu_{\langle X,X'\rangle}$ denotes the signed measure on $(\Omega, \mathscr{F}_0)$ such that

$$d\mu_{\langle X,X'\rangle} = E[x_1x'_1 \mid \mathscr{F}_0]dP .$$

If $\mu_{\langle X,X'\rangle}$ is a null measure, we say that X and X' are strictly orthogonal. If $X = X'$, then $\mu_{\langle X,X\rangle}$ is denoted simply by $\mu_{\langle X\rangle}$.

By the martingale property of helices, we can define the following which is similar to the martingale-transform:

DEFINITION 3. For a helix $X=(x_n)$ and a squarely integrable random variable ν on $(\Omega, \mathscr{F}_0, \mu_{\langle X\rangle})$, the helix $Y=(y_n)$ given by

$$y_n=\sum_{k=1}^{n}\nu\circ T^{-(k-1)}(x_k-x_{k-1})$$

is called the helix-transform of X by ν and denoted by $\nu * X$.

Now we state a repesentation theorem of helices for a system, which was proved in [3].

THEOREM 1. *For any system $(T, \mathscr{F}_0)$, there exists a finite or countable sequence of strictly orthogonal helices $\mathscr{X}=(X^{(p)})$ such that*

(a) *$\mu_{\langle X^{(p+1)}\rangle}$ is absolutely continuous with respect to $\mu_{\langle X^{(p)}\rangle}$ for all p,*

(b) *every helix X has the the representation*

$$X=\sum_{p}\nu^{(p)}*X^{(p)}$$

for some $\nu^{(p)}\in L^2(\Omega, \mathscr{F}_0, \mu_{\langle X^{(p)}\rangle})$.

If $\mathscr{Y}=(Y^{(p)})$ is another such sequence, then $\mu_{\langle Y^{(p)}\rangle}$ is equivalent to $\mu_{\langle X^{(p)}\rangle}$ for all p.

We call such a sequence a base of helices for the system. By Theorem 1, we see that the length of a base of helices is determined uniquely by the system.

DEFINITION 4. The length of a base of helices is called the multiplicity of helices for the system $(T, \mathscr{F}_0)$ and denoted by $M(T, \mathscr{F}_0)$.

If T is assumed to be a Bernoulli automorphism, then the following can be said (cf. [3]).

DEFINITION 5. For a sub-σ-field $\mathscr{A}$ of $\mathscr{F}$, the pair $(T, \mathscr{A})$ is called a B-system if

(a) $(T^n\mathscr{A})$ is an independent sequence of sub-σ-fields,

(b) $\bigvee_{n=-\infty}^{\infty} T^n\mathscr{A}=\mathscr{F}$.

If we put $\mathscr{A}_0^-=\bigvee_{n<0} T^n\mathscr{A}$, then $(T, \mathscr{A}_0^-)$ is clearly a system, which is indeed a pair of a K-automorphism and a K-field.

THEOREM 2. *Let $(T, \mathscr{A})$ be a B-system. If $(x^{(p)})$ is a complete orthonormal system of $L_0^2(\mathscr{A})$, then $X=(x_n^{(p)})$ given by*

$$x_0^{(p)}=0\ ,\quad x_n^{(p)}=\sum_{k=1}^{n}x^{(p)}\circ T^{-(k-1)}\quad(n>0)\ ,\quad x_n^{(p)}=-x_{-n}^{(p)}\circ T^{-n}\quad(n<0)$$

is the helix for the system $(T, \mathscr{A}_0^-)$ and $\mathscr{X}=(X^{(p)})$ is a base of helices for $(T, \mathscr{A}_0^-)$. Thus we have $M(T, \mathscr{A}_0^-)=\dim L_0^2(\mathscr{A})$ and further $\mu_{\langle X^{(p)}\rangle}=P$ on $\mathscr{A}_0^-$ for all p.

2. Let $(\Omega, \mathscr{F}, P, (\xi_n))$ be a real Gaussian stationary sequence with the coordinate representation and let m be the non-atomic spectral measure. We can assume that $E[\xi_n] = 0$ for all n without loss of generality. The shift T of Ω defined by

$$\xi_n(T\omega) = \xi_{n-1}(\omega) \qquad \text{for all} \quad n$$

is called a Gaussian automorphism with the spectral measure m. By Kolmogorov's decomposition theorem, we get

$$\xi_n = \int_I e^{-2\pi i n u} dM(u)$$

where $dM(u)$ is the complex normal random measure on $I = [-1/2, 1/2)$ and $dm(u) = \|dM(u)\|^2$. Then the following results are well-known (cf. [1]).

Let $L^2(I^p, m^p)$ be the class of all complex squarely integrable functions on the p-fold direct product measure space of (I, m). Then for every $f \in L^2(I^p, m^p)$, the p-th complex multiple Wiener integral

$$\mathscr{I}_p(f) \equiv \int_{I^p} f(u_1, \cdots, u_p) dM(u_1) \cdots dM(u_p)$$

is defined and has the following properties:

(a) $\mathscr{I}_p$ is linear on $L^2(I^p, m^p)$.

(b) $\mathscr{I}_p(f) = \mathscr{I}_p(\tilde{f})$ for $f \in L^2(I^p, m^p)$, where $\sim$ indicates the symmetrization of f.

(c) $E[\mathscr{I}_p(f)] = 0$ for $p \geqq 1$ and $f \in L^2(I^p, m^p)$.

(d) $(\mathscr{I}_p(f), \mathscr{I}_q(g)) = 0$ for $f \in L^2(I^p, m^p)$ and $g \in L^2(I^q, m^q)$ with $p \neq q$.

When $p = 0$, we let $\mathscr{I}_0(c) = c$ for every complex constant c.

For use in the next section, we recall here the following recurrence formula of multiple Wiener integrals ([1]).

$$\mathscr{I}_p(f)\mathscr{I}_1(g) = \mathscr{I}_{p+1}(f \triangle g) + \mathscr{I}_{p-1}(f \bigtriangledown g) ,$$

where

$$f \triangle g(u_1, \cdots, u_p, u_{p+1}) = f(u_1, \cdots, u_p) g(u_{p+1})$$

$$f \bigtriangledown g(u_1, \cdots, u_{p-1}) = \sum_{k=1}^{p} \int f(u_1, \cdots, u_{k-1}, u, u_k, \cdots, u_{p-1}) g(-u) dm(u) .$$

3. In this section, we set up a system of a Gaussian automorphism and construct the helices for the system by the multiple Wiener integrals. We deal only with a class of Gaussian automorphisms such that

$$dm(u) = \gamma(u)^2 du , \quad \gamma(u) > 0 \quad \text{a.e.}$$

Under this assumption, it is known that the sequence of random variables defined by

$$\eta_n = \int_I \frac{e^{-2\pi i n u}}{\gamma(u)} dM(u)$$

is an innovation of (ξ_n), that is,

(a) η_n, $-\infty < n < \infty$, are independent and the shift T of (ξ_n) is also that of (η_n).

(b) If $\mathscr{F}_n$ denotes the σ-field generated by ξ_k, $k \leqq n$, and $\mathscr{G}_n$ the σ-field generated by η_k, $k \leqq n$, then $\mathscr{F}_n = \mathscr{G}_n$ for all n.

Thus, if $\mathscr{A}_n$ denotes the σ-field generated by η_n for each n, then $(T, \mathscr{A}_1)$ is clearly a B-system.

LEMMA 1. *For every positive integer p,*

$$\eta_n^p = \sum_{2q \leqq p} \frac{2^{-q} p!}{q!(p-2q)!} \mathscr{I}_{p-2q}(e_n^{p-2q}) ,$$

where

$$e_n^p = e^{-2\pi i n(u_1 + \cdots + u_p)} / \gamma(u_1) \cdots \gamma(u_p) .$$

PROOF. If $p = 1$, the formula is just the definition of η_n for all n. If it is valid for some p, then

$$\eta_n^{p+1} = \eta_n^p \cdot \eta_n = \sum_{2q \leqq p} \frac{2^{-q} p!}{q!(p-2q)!} \mathscr{I}_{p-2q}(e_n^{p-2q}) \cdot \mathscr{I}_1(e_n) .$$

By the recurrence formula,

$$\mathscr{I}_{p-2q}(e_n^{p-2q}) \mathscr{I}_1(e_n) = \mathscr{I}_{p+1-2q}(e_n^{p+1-2q}) + (p-2q) \mathscr{I}_{p+1-2(q+1)}(e_n^{p+1-2(q+1)}) .$$

Therefore the coefficient of $\mathscr{I}_{p+1-2q}(e_n^{p+1-2q})$ is as follows.

$$\frac{2^{-q} p!}{q!(p-2q)!} + (p - 2(q-1)) \frac{2^{-(q-1)} p!}{(q-1)!(p-2(q-1))!} = \frac{2^{-q}(p+1)!}{q!(p+1-2q)!} .$$

This completes the induction.

By this lemma, we know that $\mathscr{I}_p(e_n^p)$ can also be expressed as a linear combination of η_n^{p-2q}, $2q \leqq p$, for each p. Therefore, $\mathscr{I}_p(e_n^p)$, $p = 1, 2, \cdots$, are measurable with respect to $\mathscr{A}_n$ for all n and since T is the shift of (η_n),

$$\mathscr{I}_p(e_n^p) \circ T^{-1} = \mathscr{I}_p(e_{n+1}^p) .$$

LEMMA 2. *The sequence $(\mathscr{I}_p(e_n^p))$, $p = 1, 2, \cdots$, is a complete orthonormal system of $L_0^2(\mathscr{A}_n)$ for each n.*

PROOF. The orthogonality is an immediate consequence of the property (d) of multiple Wiener integrals and the completeness is clear by the preceding lemma.

By Lemma 2 and Theorem 2 in Section 1, we can state the following about helices for our system $(T, \mathscr{A}_1)$.

For each positive integer p, define a helix $X^{(p)} = (x_n^{(p)})$ for $(T, \mathscr{F}_0)$ by

$$x_0^{(p)} = 0\ , \qquad x_n^{(p)} = \sum_{k=1}^{n} \mathscr{I}_p(e_k^p) = \sum_{k=1}^{n} \mathscr{I}_p(e_1^p) \circ T^{-(k-1)} \quad (n > 0)\ ,$$

$$x_n = -x_{-n} \circ T^{-n} \quad (n < 0)\ .$$

The sequence $(X^{(p)})$ is denoted by $\mathscr{X}$.

THEOREM 3. *If T is the Gaussian automorphism with the above-mentioned spectral measure, then the sequence $\mathscr{X}$ is a base of helices for the system $(T, \mathscr{F}_0)$ and for all p,*

$$\mu_{\langle X^{(p)} \rangle} = P\ .$$

Hence the multiplicity of helices is infinite.

Furthermore we have the following interesting relationship between the multiplicities and Wiener integrals.

Let W_p denote the totality of all real p-th multiple Wiener integrals. The space W_p is invariant under T by definition.

THEOREM 4. *The finite sequence $(X^{(q)})$, $(q = 1, 2, \cdots, p)$, is a base of helices for the system $(T, \mathscr{F}_0)$ in W_p.*

PROOF. It is obvious, because the sequence $(\mathscr{I}_q(e_n^q))$, $(q = 1, 2, \cdots, p)$, is a complete orthonormal system of $W_p \cap L_0^2(\mathscr{A}_n)$.

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